involve
a journal of mathematics

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Joseph N. Krenicky and Jan Rychtář

mathematical sciences publishers

2010 vol. 3, no. 1

INVOLVE 3:1(2010)

On the relationship between volume and surface area

Joseph N. Krenicky and Jan Rychtář

(Communicated by Kenneth S. Berenhaut)

We show that the statement that the surface area is the derivative of the volume, which is well known for a ball, can be generalized and stated in a proper way for any set with finite volume and surface area. We also provide a specific statement for star-shaped sets.

1. Introduction

The well known connection between the area of a disk $A = \pi r^2$ and its circumference $C = 2\pi r$ is

$$\frac{\mathrm{d}A}{\mathrm{d}r} = C.$$

The same type of formula,

$$\frac{\mathrm{d}V}{\mathrm{d}r} = S, \tag{1}$$

holds for a volume V of a ball and its surface area S. In fact, it holds for Euclidean balls in any dimension. Indeed, as derived in [Kendall 1961], the n-dimensional volume of an n-dimensional ball of radius r is

$$V_n(r) = \frac{r^n \pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}, \tag{2}$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$ is the gamma function [Abramowitz and Stegun 1972, Chapter 6], while the $(n-1)$-dimensional volume of a surface of the ball is [Coxeter 1963, p. 125]

$$S = \frac{2r^{n-1}\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} = \frac{nr^{n-1}\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\mathrm{d}V_n(r)}{\mathrm{d}r}.$$

MSC2000: 28A75, 51M25.

Keywords: volume, surface, star-shaped sets.

J. N. Krenicky was an undergraduate research assistant supported by the Office of Undergraduate Research of The University of North Carolina at Greensboro.

Emert and Nelson [1997] generalized Equation (1) for regular n-dimensional polytopes. First they showed that

$$\frac{\mathrm{d}}{\mathrm{d}r}\lambda_n(P_r) = \lambda_{n-1}(\partial P_r), \tag{3}$$

where r is the inner radius of the polytope, that is, the minimal distance from a center to the boundary ∂P_r. Theorem 2 of their paper is a generalization of the formula in (3) to any polytope P_r that circumscribes a ball of radius r.

The main aim of this paper is to generalize (3) to a larger family of sets. We show that when formulated properly, (3) holds for any set with finite volume and surface area.

2. Definitions and preliminaries

Let $n \geq 2$ be a fixed natural number. All sets considered will be subsets of $\mathbb{R}^n$. The n-dimensional Lebesgue measure on $\mathbb{R}^n$ will be denoted by λ_n.

We recall the notion of *similarity* between sets in $\mathbb{R}^n$, which is an equivalence relation. Two subsets S_1 and S_2 of $\mathbb{R}^n$ are *similar*, and we write $S_1 \sim S_2$, if there exist $c \in \mathbb{R}^n$ and $\alpha > 0$ such that the image of S_1 under the map defined by

$$f_{c,\alpha}(x) = c + \alpha(x - c), \quad x \in \mathbb{R}^n, \tag{4}$$

is congruent to S_2 — that is, there is an isometry of $\mathbb{R}^n$ taking $f_{c,\alpha}(S_1)$ to S_2. The map $f_{c,\alpha}$ is the *homothety* or *scaling* of center c and ratio α. It preserves the point c and dilates or contracts distances between any two points by a factor of α.

An equivalence class of $\sim$ will be called a *shape*. A *ball* is an example of a shape. One can shift, rotate, or resize it, and always get a ball.

Let $d > 0$ be any positive real number. The d-dimensional *Hausdorff measure* [Federer 1969; Morgan 2000] of a set E is defined by

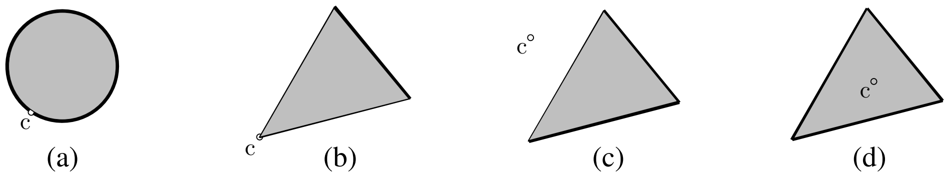
$$H^d(E) = \limsup_{\delta \to 0+} H^d_\delta(E),$$

where $H^d_\delta(E)$ is the infimum, over all countable covers of E by sets A_i of diameter at most δ, of a measure of volume associated with the cover:

$$H^d_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} V_d\left(\frac{\operatorname{diam} A_i}{2}\right) : E \subset \bigcup_{i=1}^{\infty} A_i,\ \operatorname{diam} A_i < \delta\right\}.$$

Here the summand is the Lebesgue measure of a ball of radius $\frac{1}{2}\operatorname{diam} A_i$; see (2). On Borel sets of $\mathbb{R}^n$, $H^n = \lambda_n$ [Morgan 2000, Corollary 2.8]. For any set S and any point c,

$$H^d\big(f_{c,\alpha}(S)\big) = \alpha^d H^d(S). \tag{5}$$



(a) (b) (c) (d)

Figure 1. Horizons of c visibility (in bold) for different sets (in gray) and different positions of point c.

The *Hausdorff dimension* [Morgan 2000] of a nonempty set E is defined by

$$\dim_H E = \inf\{\, d \geq 0 : H^d(E) < \infty\}.$$

For $c \in \mathbb{R}^n$ we will call a map ∂_c the *generalized boundary* if it maps subsets of $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$, assigns measurable sets to measurable sets, and satisfies

$$\partial_c(f_{c,\alpha}(S)) = f_{c,\alpha}(\partial_c(S)), \tag{6}$$

for all $\alpha > 0$ and all $S \subset \mathbb{R}^n$. It means that the boundary grows and shrinks together with the set S, but it is not necessarily invariant under translations or other isometries, nor connected to S in any sense. For example, the topological boundary is a generalized boundary.

If S is a set and $c \in \mathbb{R}^n$ any point, we define the *horizon of c-visibility* $\partial_c^* S$ by

$$\partial_c^* S = (\mu_{S,c})^{-1}(1),$$

where $\mu_{S,c} : \mathbb{R}^n \mapsto [0, \infty]$ is the *Minkowski functional* [Fabian et al. 2001, p. 42] given by

$$\mu_{S,c}(x) = \begin{cases} \inf\{r > 0,\ x \in f_{c,r}(S)\}, & \text{if } x \in f_{c,r}(S) \text{ for some } r < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

It follows directly from the definition that $\partial_c^* S$ is measurable when S is. Yet $\partial_c^* S$ does not have to be closed (Figure 1a); it does not coincide with the topological boundary ∂ even if it is closed (Figure 1a–c), and ∂_c^* is not preserved by shifts (Figure 1b–d). On the other hand, it satisfies (6). Thus ∂_c^* is a generalized boundary.

A set S is called *star-shaped* if there is a point $c \in S$ such that for every point $p \in S$ the line segment $\overline{cp}$ is contained in S. Such a point c is called a *center* of S. A star-shaped set can have many centers; for example, every convex set C is star-shaped and every point $c \in C$ is its center. However, not all star-shaped sets are convex; see, for instance, the drawing at the end of this section.

A set S is called *flat* if S is contained in an affine subspace $p + \mathbb{R}^{\lceil \dim_H S \rceil}$ for some point $p \in \mathbb{R}^n$, where $\lceil\ \rceil$ denotes the ceiling function (least integer not less than). If c is a point and S a flat set, we define $d_f(c, S)$ to be the distance from c

to the affine space $p+\mathbb{R}^{\lceil \dim_H S \rceil}$ that witnesses the flatness of S. Here we see a flat and a nonflat subset of $\mathbb{R}^2$ of dimension 1:

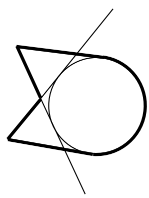
We say that a star-shaped set S *circumscribes a ball of radius r in a generalized sense* if there is a center c of S and the decomposition of $\partial_c^* S$ into countably many pairwise disjoint measurable sets F_i, $i \geq 0$, such that

(a) $\operatorname{dist}(f, c) = r$, for any $f \in F_0$,

(b) the sets F_i, $i \geq 1$, are flat, and

(c) $d_f(c, F_i) = r$, for all $i \geq 1$.

By the definition, the center of the circumscribed ball is a center of the set S. Here is a nontrivial set S circumscribing a ball in a generalized sense:



3. Generalization of the volume-area relationship

We now state the key lemma that is in fact a direct consequence of (5).

Lemma 1. *Let S and B be* any *measurable sets, fix $c \in \mathbb{R}^n$ and let $d \geq 1$ be such that $H^d(B) \in (0, \infty)$ and $H^{d-1}(S) \in (0, \infty)$. Set $S_r = f_{c,r}(S)$ and $B_r = f_{c,r}(B)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}h} H^d(B_r) = H^{d-1}(S_r),$$

where

$$h = d\frac{H^d(B)}{H^{d-1}(S)} r.$$

Also

$$H^d(B_r) = \frac{H^d(B)}{H^{d-1}(S)} H^{d-1}(S_r) r. \tag{7}$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}h} H^d(B_r) = \frac{\mathrm{d}}{\mathrm{d}r} H^d(B_r) \cdot \frac{\mathrm{d}r}{\mathrm{d}h} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r^d H^d(B)\right) \cdot \left(d\frac{H^d(B)}{H^{d-1}(S)}\right)^{-1}$$
$$= r^{d-1} H^{d-1}(S) = H^{d-1}(S_r).$$

Equation (7) follows directly from (5). □

It follows from Lemma 1 that there is always a relationship in the spirit of (3) between any pairs of families $\{S_r\}, \{B_r\}$ that are being "inflated" together (but otherwise may have nothing in common). In particular, S does not have to be a boundary of B in any sense, B does not have to be convex or of any particular shape, and the center of inflation c can be anywhere. However, the price for such general assumptions is the need to differentiate with respect to h, the multiple of the inflation factor r, not with respect to r itself.

The parameter $n(\lambda_n(C)/(\lambda_{n-1})(\partial C))$ for convex polytopes in $\mathbb{R}^n$ was studied by Fjelstad and Ginchev [2003]. They called h the *harmonic parameter* of C and showed that it is a weighted average of distances from a central point to the faces (the weight being proportional to the size of the face), and for some objects like boxes, it is the harmonic mean of distances from a central point to the faces of the object, thus providing certain geometrical intuition when Lemma 1 is applied to B and $S = \partial B$.

The next theorem shows that, for reasonable shapes, there is always an appropriate representative of the shape that makes the parameter h to be exactly r, that is, (3) holds for that shape.

Theorem 2. *Let $\mathfrak{S}$ be a shape, fix $d \geq 1$, $c \in R^n$, and let ∂_c be a generalized boundary such that, for some $B \in \mathfrak{S}$,*

(i) $H^d(B) \in (0, \infty)$, *and*

(ii) $H^{d-1}(\partial_c B) \in (0, \infty)$.

Then there is a $B_1 \in \mathfrak{S}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}r} H^d\big(f_{c,r}(B_1)\big) = H^{d-1}\big(f_{c,r}(\partial_c B_1)\big).$$

Proof. By Lemma 1 we need to find $B_1 \in \mathfrak{S}$ such that $h = r$, that is,

$$\frac{H^{d-1}(\partial_c B_1)}{H^d(B_1)} = d. \tag{8}$$

For that, by (7), it is enough to take

$$B_1 = f_{c,\alpha}(B), \quad \text{where } \alpha = \frac{H^{d-1}(\partial_c B)}{d H^d(B)}. \qquad \square$$

The statements of Theorem 2 may seem too abstract. However, in general, we cannot do any better, since a shape is a purely geometrical object. For example, without our measuring the distance, all balls in $\mathbb{R}^3$ are alike. If we can measure a distance, we can pick a ball and say this is the ball with radius 1. If we pick the wrong ball, say with radius $\varrho \neq 1$, its r-inflation would have volume $\frac{4}{3}\pi(\varrho r)^3$ and surface area $4\pi(\varrho r)^2$ — losing the relationship (3). Hence choosing the right representative for balls is equivalent to choosing the length unit.

We can pick the proper representative for cubes as well. Picking the cube with side length 1 is not good, since its r-inflation has volume $V = r^3$ and surface area $S = 6r^2$, that is, $\mathrm{d}V/\mathrm{d}r \neq S$. For cubes the right representative is a cube with side length 2, because then its r inflation has volume $V = 8r^3$ and surface area $S = 24r^2$, thus recovering (3). It was observed by Emert and Nelson [1997] that this right cube circumscribes the ball of radius 1 (which we already know is a special ball).

As another example, consider a torus — which is not a star-shaped set — with radii R and r (where r is a radius of the tube and R is a distance from a center of the tube to the center of the torus). Note that the shape is determined by the fraction R/r. The volume of such a torus is $V = 2\pi^2 Rr^2$ and the surface area is $A = 4\pi^2 Rr$. The right representative for a torus shape is a torus T_1 that satisfies $A/V = 3$, that is, the one that is inflated to have $r = 2/3$. Observe that there is apparently nothing significant about that particular torus. However, in order to know which representative to pick, we had to know how to calculate the volume and surface area of a torus in general. In the next section, we will show how to avoid this problem for certain star-shaped sets.

4. Star-shaped sets

The following lemma is an easy consequence of the definition of a star-shaped set.

Lemma 3. *A closed set S is star-shaped if and only if there is a point $c \in S$ such that*

$$S = \bigcup_{\alpha \in [0,1]} f_{c,\alpha}(\partial_c^* S). \tag{9}$$

The next theorem shows how to pick a representative S_1, whose existence is guaranteed by Theorem 2, from among certain star-shaped sets.

Theorem 4. *Let $d \geq 1$ and S_1 be a closed star-shaped set that circumscribes a ball of radius 1 centered at c in a generalized sense. Then*

$$H^d(S_1) = \frac{1}{d} H^{d-1}(\partial_c^* S_1).$$

In particular, if $H^d(S_1) \in (0, \infty)$, then

$$\frac{\mathrm{d}}{\mathrm{d}r} H^d\big(f_{c,r}(S_1)\big) = H^{d-1}\big(f_{c,r}(\partial_c^* S_1)\big).$$

Proof. Let F_i, $i \geq 0$, be the decomposition of $\partial_c^* S$ that witnesses that S circumscribes a ball of radius 1 centered at c in a generalized sense. Set

$$C_i = \bigcup_{\alpha \in [0,1]} f_{c,\alpha}(F_i), \quad i \geq 0.$$

Namely, C_i is the cone corresponding to the face F_i. By definition of F_i and ∂_c^*, $C_i \cap C_j = \varnothing$ for all $i \neq j$, and by (9),

$$S_1 = \bigcup_{i=0}^{\infty} C_i. \tag{10}$$

Note that C_i for $i \geq 0$ is a star-shaped set and c is its center. Moreover, $\partial_c^* C_i = F_i$. Thus

$$\begin{aligned} H^d(C_i) &= (\lambda_1 \times H^{d-1})(C_i) = \int_0^1 H^{d-1}\big(f_{c,\varrho}(\partial_c^* C_i)\big)\,\mathrm{d}\varrho \\ &= H^{d-1}(\partial_c^* C_i) \int_0^1 \varrho^{d-1}\,\mathrm{d}\varrho = \frac{1}{d} H^{d-1}(\partial_c^* C_i). \end{aligned}$$

The first part of the theorem then follows from (10). The second part is a consequence of Lemma 1. □

Corollary 5 [Emert and Nelson 1997, Theorem 1 and 2]. *If P_r is any regular n-dimensional polytope with the inner radius r or more generally a polytope that circumscribes a ball of radius r, then*

$$\frac{\mathrm{d}}{\mathrm{d}r}\lambda_n(P_r) = \lambda_{n-1}(\partial P_r).$$

Corollary 6. *If S_r is any closed star-shaped n-dimensional polytope that circumscribes a ball of radius r in a generalized sense, then*

$$\frac{\mathrm{d}}{\mathrm{d}r}\lambda_n(S_r) = \lambda_{n-1}(\partial S_r).$$

5. Discussion

Equation (3) is in principle an integral relationship

$$\lambda_n(P_r) = \int_0^r \lambda_{n-1}(\partial P_\varrho)\,\mathrm{d}\varrho,$$

which implicitly assumes

$$P_r = \bigcup_{\varrho=0}^{r} \partial P_\varrho. \tag{11}$$

By Lemma 3, this implies that P_r is star-shaped.

Moreover, if a star-shaped set P does not circumscribe any ball in the generalized sense, then for any center c of P, the faces of $\partial_c^* P$ have different distances from c. In other words, as the set P is inflated from a center c, the volume of corresponding cones grows by a different rate (this was observed in [Emert and

Nelson 1997, p. 368] and also in [Fjelstad and Ginchev 2003]). Consequently, one needs the faces to be equidistant to the center.

Therefore, we argue that Theorem 4 generalizes Equation (3) as much as possible while still keeping the geometrical intuition that provides a natural interpretation of the parameter r. Theorem 2 is much more general, but without any specific intuition behind it.

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Received: 2008-12-30 Accepted: 2010-01-07

jnkrenic@uncg.edu
Department of Physics,
The University of North Carolina at Greensboro,
Greensboro, NC 27403, United States

rychtar@uncg.edu
Department of Mathematics and Statistics,
The University of North Carolina at Greensboro,
Greensboro, NC 27403, United States