involve

a journal of mathematics

Roundness properties of graphs

Matthew Horak, Eric LaRose, Jessica Moore,
Michael Rooney and Hannah Rosenthal

mathematical sciences publishers

2010 vol. 3, no. 1

INVOLVE 3:1(2010)

Roundness properties of graphs

Matthew Horak, Eric LaRose, Jessica Moore,
Michael Rooney and Hannah Rosenthal

(Communicated by Scott Chapman)

The notion of the roundness of a metric space was introduced by Per Enflo as a tool to study geometric properties of Banach spaces. Recently, roundness and generalized roundness have been used in the context of group theory to investigate relationships between the geometry of a Cayley graph of a group and the algebraic properties of the group. In this paper, we study roundness properties of connected graphs in general. We explicitly calculate the roundness of members of two classes of graphs and we give results of computer calculations of the roundness of all connected graphs on 7, 8 and 9 vertices. We also show that no connected graph can have roundness between $\log_2 3$ and 2.

1. Introduction

The notions of metric roundness and generalized metric roundness were introduced by Per Enflo [1970a; 1970b] to investigate geometric questions in the theory of Banach spaces. Generalized roundness has also been used in group theory in connection with the coarse Baum–Connes and the Novikov conjectures [LaFont and Passidis 2006]. In the group-theoretic setting, a finitely generated group is viewed as a metric space by viewing elements of the group as vertices of the Cayley graph of the group with respect to a fixed finite generating set and taking the distance between two elements to be the number of edges in a shortest path between them in the Cayley graph.

Recently, more work has been done on the roundness and generalized roundness properties of finitely generated groups, for example in [Jaudon 2008; LaFont and Passidis 2006], relating algebraic properties of a group to the possible values that can be taken by the roundness or generalized roundness of its Cayley graphs with respect to different finite generating sets. However, very little work has been done regarding roundness properties of graphs in general, and a better understanding of

MSC2000: primary 05C99; secondary 46B20, 20F65.

Keywords: roundness, graph, metric invariant.

This work was partially supported by a CURM mini-grant funded by NSF grant DMS-063664.

the roundness of graphs may lead to deeper insight into the connection between the roundness of a Cayley graph and the algebraic properties of the corresponding group. In this paper, we begin to develop a theory of the roundness of general graphs, focusing on the possible values of the roundness of a finite connected graph. Since the roundness of an infinite connected graph is equal to the infimum of the roundnesses of its finite connected metrically embedded subgraphs, this is a first step in understanding roundness for infinite graphs.

This paper is organized as follows. In Section 2, we review the definition of roundness, state and prove several lemmas about roundness in the context of graph theory and work through two concrete examples. In Section 3, we investigate the roundness of the cyclic graphs C_n, finding the roundness of all of these graphs and proving that the roundness of C_n can be made arbitrarily close to 1 by taking values of n sufficiently large. In Section 4, we continue to investigate roundness by working through another class of graphs that we call *triangulated cycles*. We determine the roundness of triangulated cycles in this section and again prove that as the number of vertices in a triangulated cycle goes to infinity, its roundness goes to 1. Finally in Section 5, we summarize some computer-generated data on the distribution of roundness among all 7-, 8- and 9-vertex graphs and make some conjectures on the distribution of roundness based on these data. In this section, we also prove that no graph can have roundness between $\log_2 3$ and 2.

2. Definitions and preliminary lemmas

A *quadrilateral* in a metric space X is an ordered 4-tuple $Q = (A, B, C, D)$ of (not necessarily distinct) points $A, B, C, D \in X$. Informally, we envision Q as the vertices of a quadrilateral embedded in X, and even though there may be no paths in X between the vertices, we talk about the *sides* AB, BC, CD and DA and the *diagonals* AC and BD, as shown in Figure 1. Given four points, $A, B, C, D \in X$, we may form several different quadrilaterals depending on the order in which we take the points. We denote by $Q(A, B, C, D)$ the quadrilateral with sides AB, BC, CD and DA and diagonals AC and BD. If a quadrilateral has two or more of its vertices equal, we call it *degenerate*.

Definition 2.1 (Roundness). If $Q = Q(A, B, C, D)$ is a quadrilateral in the metric space (X, d), then the *roundness* of Q, $\rho(Q)$, is the supremum of all values q such that

$$d(A, C)^q + d(B, D)^q \le d(A, B)^q + d(B, C)^q + d(C, D)^q + d(D, A)^q. \qquad (1)$$

For a metric space X, the *roundness* of X is

$$\rho(X) = \inf\{\rho\,(Q(A, B, C, D)) \mid A, B, C, D \in X\}. \qquad (2)$$

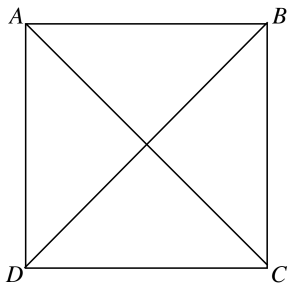

Figure 1. The quadrilateral $Q(A, B, C, D)$.

We remark that this definition of the roundness of a metric space is equivalent to another common formulation of metric roundness below.

Definition 2.2 (Equivalent definition of roundness). The *roundness* of the metric space (X, d) is the supremum of all values q such that for any four points $A, B, C, D \in X$,

$$d(A, C)^q + d(B, D)^q \leq d(A, B)^q + d(B, C)^q + d(C, D)^q + d(D, A)^q. \tag{3}$$

Note that by the triangle inequality, the roundness of any quadrilateral in a metric space is at least 1. This proves:

Lemma 2.3. *The roundness of any metric space X is greater than or equal to* 1.

Observation 2.4. Suppose that A, B, C, D are four distinct points in a metric space. By the symmetry of the inequalities in the definition of roundness, every quadrilateral on A, B, C, D has the same roundness as one of the three quadrilaterals, $Q(A, B, C, D)$, $Q(A, B, D, C)$ or $Q(A, C, B, D)$. Geometrically, this corresponds to the fact that rotating a quadrilateral or reflecting a quadrilateral along a diagonal or middle line preserves its sides and diagonals. Furthermore, at most one of these quadrilaterals can have finite roundness, because a quadrilateral of finite roundness must have its largest distance between vertices as a diagonal. This is true even in the case that the maximal distance between vertices is achieved by two or more pairs of vertices of the quadrilateral.

Throughout this paper we will make generous use of the following lemma that describes how the roundness of a quadrilateral changes if we change the lengths of its diagonals or sides.

Lemma 2.5. *Let Q_1 and Q_2 be quadrilaterals in the metric space X with the same side and diagonal lengths except for exactly one side or diagonal. Further suppose that if the quadrilaterals differ in a diagonal then the diagonal of Q_2 is strictly longer than the diagonal in Q_1 and if they differ in a side then the side in Q_2*

is strictly shorter than the side in Q_1. *If* $\rho(Q_1)$ *is finite then so is* $\rho(Q_2)$, *and* $\rho(Q_2) < \rho(Q_1)$.

Proof. Suppose that Q_1 has finite roundness $q_1 \geq 1$. Suppose that the lengths of the sides of Q_1 are w, x, y, z and the lengths of its diagonals are a, b. Then q_1 satisfies $a^{q_1} + b^{q_1} = w^{q_1} + x^{q_1} +^{q_1} + z^{q_1}$, and if $p > q_1$ then $a^p + b^p > w^p + x^p + y^p + z^p$.

Case 1. Q_1 and Q_2 differ on a diagonal. Let $a_2 > a$ be the length of the diagonal in Q_2 that differs from that of Q_1. Let p be a real number greater than or equal to q_1. Then, $a_2^p + b^p > a^p + b^p \geq w^p + x^p + y^p + z^p$. Therefore, $\rho(Q_2)$, which is the supremum of all values q such that $a_2^q + b^q \leq w^q + x^q + y^q + z^q$, is less than $q_1 = \rho(Q_1)$.

Case 2. Q_1 and Q_2 differ on a side. Let $w_2 < w$ be the length of the side in Q_2 that differs from that of Q_1. Let p be a real number greater than or equal to q_1. Then, $a^p + b^p \geq w^p + x^p + y^p + z^p > w_2^p + x^p + y^p + z^p$. Therefore, $\rho(Q_2)$, which is the supremum of all values q such that $a^q + b^q \leq w_2^q + x^q + y^q + z^q$, is less than $q_1 = \rho(Q_1)$. □

Roundness at it relates to graphs. In this paper, we are concerned with the roundness properties of metric spaces arising from connected graphs. Throughout, we let G denote a finite connected graph with vertex set V and edge set E. We view V as a metric space with the distance, $d(A, B)$, between vertices A and B given by the number of edges in a shortest edge path in G between A and B. We usually abuse notation by referring to G itself as a metric space, but when we do so we are always considering only the vertex set of G. Thus, $\rho(G)$ always denotes the roundness of the metric space consisting of only the vertex set of G. This is important, because if we were to view all of G as a metric space in the usual way by metrically identifying each edge with the unit interval, then any nonsimply connected graph would have roundness equal to 1, which follows from Lemma 2.6 from [LaFont and Passidis 2006]. Another reason this is important is that in the case G is a finite graph, there are only finitely many quadrilaterals in G. Therefore, the infimum of (1) in the definition of roundness is actually a minimum and the roundness of G is actually achieved by some minimum roundness quadrilateral in G.

Lemma 2.6. *Let* X *be a metric space. If* X *contains a metrically embedded circle, then* $\rho(X) = 1$.

Before proceeding with more preliminary lemmas related to graph roundness, we calculate roundness in two examples, the cyclic graph on 5 vertices, C_5, and a graph we call Graph Δ, shown in Figure 2. In the case of a finite graph G since there are only finitely many different quadrilaterals in G, the infimum in (2) is actually a minimum, and we may search for a specific quadrilateral that has

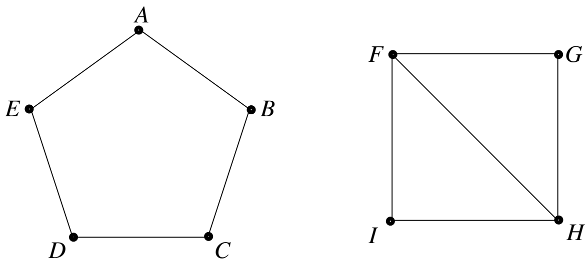

Figure 2. Graphs C_5 (left) and Δ (right).

minimal roundness among all quadrilaterals in G. The roundness of G is then the roundness of this minimal roundness quadrilateral.

Since C_5 and Δ are so small, we can find a minimal roundness quadrilateral by simply determining by hand the roundness of every possible quadrilateral in the graphs. Quadrilaterals $Q_1 = Q(A, B, C, D)$ in C_5 and $Q_2 = Q(F, G, H, I)$ in Δ turn out to be minimal roundness quadrilaterals in C_5 and Δ respectively. In Q_1 and Q_2, we have the distances shown in Figure 3. So, $\rho(Q_1)$ is the supremum over all p values such that

$$2^p + 2^p \leq 1^p + 1^p + 1^p + 2^p.$$

In this case, the supremum is found by solving the equation

$$2^p + 2^p = 1^p + 1^p + 1^p + 2^p$$

for $p = \log_2(3) \approx 1.58$. The roundness of Q_2 is the supremum over all p values such that

$$1^p + 2^p \leq 1^p + 1^p + 1^p + 1^p.$$

Again, the supremum is found by solving the equation

$$1^p + 2^p = 1^p + 1^p + 1^p + 1^p$$

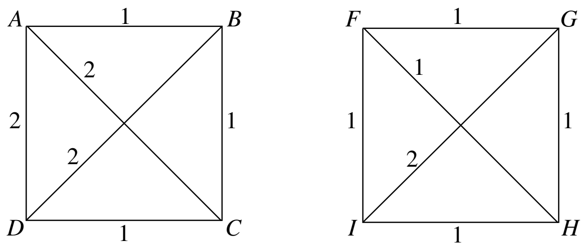

Figure 3. Quadrilaterals Q_1 and Q_2 with diagonal and side lengths indicated.

for $p = \log_2(3) \approx 1.58$. These examples illustrate that two different graphs can have the same roundness and that this roundness may even arise from "different" inequalities.

When calculating roundness of a particular graph G, one often starts by seeking an upper bound for $\rho(G)$ by finding a subgraph of G whose roundness is known or at least not too hard to determine. However, since the distance between vertices through a subgraph may be different than the distance through the whole graph, one must be careful to restrict attention to *metrically embedded subgraphs*, defined below and illustrated in Figure 4.

Definition 2.7. Let G_0 be a subgraph of the graph G. For vertices $A, B \in G$, denote by $d_G(A, B)$ the distance between A and B in G. If A and B happen to belong to G_0, denote by $d_{G_0}(A, B)$ the distance between A and B when viewed as vertices of the graph G_0. The subgraph G_0 is said to be *metrically embedded* in G if $d_{G_0}(A, B) = d_G(A, B)$ for every pair of vertices $A, B \in G_0$. In this case, G_0 is also said to be a *metric subgraph* of G.

The following lemma is easily verified, and it is useful in working through specific examples.

Lemma 2.8. *If G_0 is a metrically embedded subgraph of G, then $\rho(G) \leq \rho(G_0)$.*

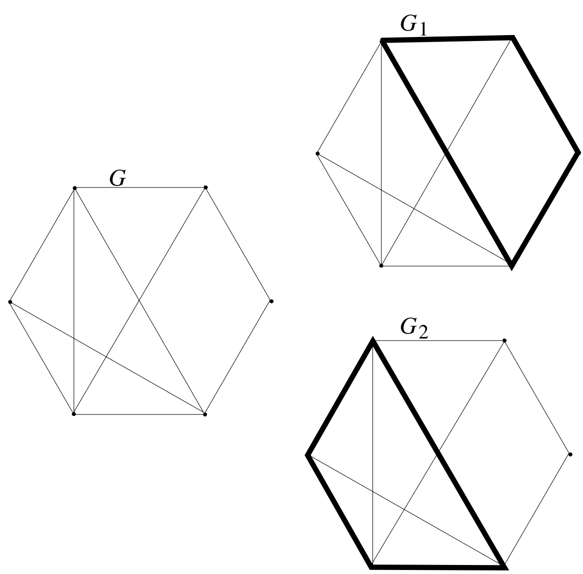

Figure 4. A metrically embedded subgraph, G_1, and a nonmetrically embedded subgraph, G_2 of the graph G.

An immediate application of Lemma 2.8 is that a graph containing a metrically embedded subgraph isomorphic with a cyclic graph with an even number of vertices, C_{2k}, has roundness equal to 1. This follows immediately from Lemma 2.8 and the fact that $\rho(C_{2k}) = 1$. We record this as,

Lemma 2.9. *If G contains a metrically embedded subgraph isomorphic with the cyclic graph C_{2k} for $k \geq 2$ then $\rho(G) = 1$.*

We end this subsection with two lemmas for which we provide short proofs. Together with Lemma 2.3, Lemma 2.10 implies that if a graph G has finite roundness, then $1 \leq \rho(G) \leq 2$. Additionally, Lemmas 2.10 and 2.11 imply that if a graph has finite roundness then its roundness is never given by a degenerate quadrilateral.

Lemma 2.10. *Let G be a finite connected graph. Then $\rho(G) = \infty$ or $\rho(G) \leq 2$.*

Proof. Let G be a finite connected graph such that $\rho(G) \neq \infty$. Since a complete graph has infinite roundness, G is not complete. Choose three vertices $A, B, C \in G$ such that $d(A, B) = d(B, C) = 1$ and $d(A, C) = 2$, which exist because G is not complete. We have $\rho(G) \leq \rho(Q(A, B, C, B)) = 2$. □

Lemma 2.11. *If Q is a quadrilateral in which two or more of the vertices are equal, then $\rho(Q) \geq 2$.*

Proof. If Q is comprised of one or two vertices, it follows immediately after writing down the inequalities that the roundness of Q satisfies that $\rho(Q) = \infty$, so we assume that Q is comprised of three distinct vertices, A, B, C as shown in Figure 5 with distances between vertices indicated. Since A, B and C are distinct, w, x and y are all nonzero. Again, it follows immediately after writing down the equation for roundness and taking into account the symmetries in Observation 2.4 that after possibly renaming the vertices of Q, the only quadrilateral that can possibly have finite roundness has the form $Q(A, B, C, B)$.

Case 1. $y \geq x + w$. In this case $\rho(Q)$ is the supremum of all values of q for which $(x+w)^q \leq 2w^q + 2x^q$. Since $(x+w)^2 \leq 2x^2 + 2w^2$, $\rho(Q) \geq 2$.

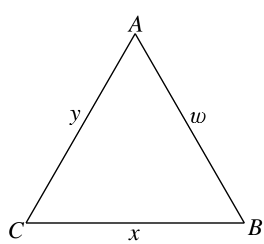

Figure 5. Degenerate quadrilateral of Lemma 2.11.

Case 2. $y < x + w$. In this case, $\rho(Q)$ is the supremum of all values q for which $y^q \le 2w^q + 2x^q$. Note that if $y < w$ and $y < x$ then this inequality holds for all positive q, so $\rho(Q) = \infty$. So we now assume that $y \ge w$ and $y \ge x$. For $q = 2$, we have $y^2 < (w+x)^2 = w^2 + 2wx + x^2 \le 2w^2 + 2x^2$. Therefore, $\rho(Q) > 2$. □

3. Roundness of cyclic graphs

As previously mentioned, in the cyclic graph with an even number of vertices C_{2n} it is not hard to find a quadrilateral whose roundness is equal to 1. Since 1 is the smallest possible value for the roundness of a metric space, this proves that $\rho(C_{2n}) = 1$. For odd cycles, C_{2n+1}, the situation is not as easy because $\rho(C_{2n+1}) \neq 1$ and proving that a candidate for a minimal roundness quadrilateral actually has minimal roundness among all quadrilaterals in C_{2n+1} is more involved. In this section we determine $\rho(C_{2n+1})$ by finding a minimal roundness quadrilateral in C_{2n+1}.

When we talk about the cyclic order of points in C_{2n+1}, we are always referring to the cyclic order given by C_{2n+1} or its reverse. We say that the quadrilateral $Q(A, B, C, D)$ in C_{2n+1} is *in cyclic order* if the vertices are encountered in the order A, B, C, D along a nonrepeating path in C_{2n+1} starting at A. Otherwise, $Q(A, B, C, D)$ is *out of cyclic order*. Depending on the particular way in which C_{2n+1} is represented geometrically by a drawing, the path may appear “clockwise” or “counterclockwise”.

The natural guess for a minimal roundness quadrilateral in C_{2n+1} is one whose vertices are in cyclic order and as evenly spaced as possible. The fact that a quadrilateral of this form has roundness less than 2 proves that $\rho(C_{2n+1}) < 2$. We use this fact during the proof that this guess is in fact a minimal roundness quadrilateral in C_{2n+1}. In this section, we prove that quadrilaterals of this form are of minimal roundness in C_{2n+1}. Calculating the roundness of such a minimal roundness quadrilateral in C_{2n+1} gives the main theorem and corollary of this section.

Theorem 3.1. *Let n be an integer greater than or equal to* 2.

(1) *If $2n+1$ has the form $4k+1$ for an integer k, then $\rho(C_{2n+1})$ is the unique solution to the equation*, $2(2k)^q = 3k^q + (k+1)^q$.

(2) *If $2n+1$ has the form $4k-1$ for an integer k, then $\rho(C_{2n+1})$ is the unique solution to the equation*, $2(2k-1)^q = 3k^q + (k-1)^q$.

Corollary 3.2. *Let C_k be the cyclic graph on k vertices. Then* $\lim_{k\to\infty} \rho(C_k) = 1$.

The first step in the proof of Theorem 3.1 is to prove that a minimal roundness quadrilateral in C_{2n+1} must have its vertices in the cyclic order given by C_{2n+1}. We do this by proving that for any quadrilateral Q' whose vertices are out of order, there is another (possibly out of order) quadrilateral Q'' such that $\rho(Q'') < \rho(Q')$.

The second step in the proof is to show that the vertices used in a minimal roundness quadrilateral must be such that the side lengths are as balanced as possible.

For the rest of the section, we consider a fixed cyclic graph C_{2n+1} and consider four points $A, B, C, D \in C_{2n+1}$ in cyclic order as shown in Figure 6. In this figure, w, x, y, z are the lengths of the paths clockwise around C_{2n+1} from A to B to C to D and back to A. In referring to the figure, we will often refer to A, B, C and D as *points* and the w, x, y, z as the lengths of *sides*, thinking of the quadrilateral $Q(A, B, C, D)$, even if there is another, out of order, quadrilateral $Q(A, B, D, C)$ or $Q(A, C, B, D)$ under consideration.

Every minimal roundness quadrilateral must be in order.

Theorem 3.3. *If Q is a minimal roundness quadrilateral in C_{2n+1} then Q is non-degenerate and its vertices are in the cyclic order given by C_{2n+1}.*

We separate the proof into five cases in which we prove that a degenerate or out of order quadrilateral in C_{2n+1} does not have minimal roundness among all quadrilaterals in C_{2n+1}. The cases are divided according to the lengths of the "sides" w, x, y, z in Figure 6.

- In Lemma 3.4, we deal with the degenerate case.
- In Lemma 3.5, we prove that an out of order quadrilateral on A, B, C, D does not have minimal roundness in the case that none of the side lengths w, x, y, z is greater than the sum of any other two consecutive side lengths.
- In Lemma 3.6, we prove that an out of order quadrilateral on A, B, C, D does not have minimal roundness in the case that the longest side is longer than the sum of any two other consecutive sides, but is shorter than the sum of lengths of the three remaining sides.
- In Lemma 3.7, we prove that an out of order quadrilateral on A, B, C, D does not have minimal roundness in the case that the longest side is longer than the

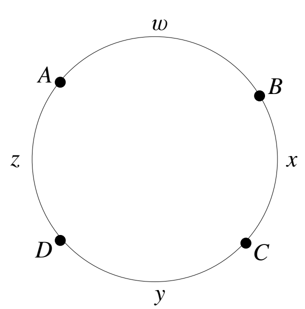

Figure 6. C_{2n+1} with four distinguished points A, B, C, D.

sum of two of the other adjacent sides, but shorter than the sum of the two others.

- In Lemma 3.8, we prove that an out of order quadrilateral on A, B, C, D does not have minimal roundness in the case that the longest side is longer than the other three combined.

Lemma 3.4. *A degenerate quadrilateral in C_{2n+1} is not a minimal roundness quadrilateral for C_{2n+1}.*

Proof. Let Q be a degenerate quadrilateral in C_{2n+1}. By Lemma 2.11, $\rho(Q) \geq 2$, but we have already observed that $\rho(C_{2n+1}) < 2$, so Q cannot be a minimal roundness quadrilateral in C_{2n+1}. □

Lemma 3.5. *Let Q' be an out of order nondegenerate quadrilateral in C_{2n+1} comprised of the vertices A, B, C, D in Figure 6. If no side length w, x, y, z is greater than the sum of the lengths of any remaining pair of adjacent sides, then Q' is not a minimal roundness quadrilateral in C_{2n+1}.*

Proof. Since Q' is nondegenerate, $w, x, y, z \neq 0$. Additionally, by our assumption on side lengths, we have:

$$\begin{array}{llll} w < x+y, & x < w+z, & y < w+x, & z < w+x, \\ w < y+z, & x < y+z, & y < w+z, & z < x+y. \end{array}$$

Consider the in-order quadrilateral $Q = Q(A, B, C, D)$. By Observation 2.4 and the symmetry of the above conditions on the lengths of the sides, we may without loss of generality assume that our out of order quadrilateral is, $Q' = Q(A, B, D, C)$. By our length conditions, these two quadrilaterals have side and diagonal lengths shown in Figure 7. Note that there are two possibilities for the lengths of some of the sides and diagonals, depending on how the two sums in question compare. But, no matter which possibilities are the actual lengths, the diagonals in Q are strictly longer than the diagonals in Q' and the vertical edges in Q are strictly shorter than the vertical edges in Q'. Therefore, by Lemma 2.5, $\rho(Q) < \rho(Q')$, finishing the proof. □

Lemma 3.6. *Let Q' be an out of order nondegenerate quadrilateral in C_{2n+1} comprised of the vertices A, B, C, D as in Figure 6. If the longest side in the in-order quadrilateral $Q = Q(A, B, C, D)$ is at least as long as any remaining pair of adjacent sides but strictly shorter than the other three sides put together, then Q' is not a minimal-roundness quadrilateral in C_{2n+1}.*

Proof. Since Q' is nondegenerate, $w, x, y, z \neq 0$. Without loss of generality, assume that w is the longest length of a side in Q. By our assumptions on lengths of sides, we have

$$w \geq x+y, \quad w \geq y+z, \quad w < x+y+z, \quad w \geq x, y, z.$$

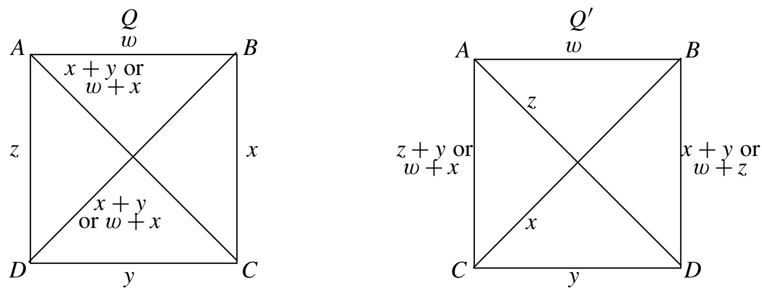

Figure 7. Quadrilaterals in Lemma 3.5. Lengths displayed inside the quadrilateral are lengths of the diagonals. Top ones for the upper-left to lower-right, bottom ones for bottom-left to upper-right.

By Observation 2.4, without loss of generality we may assume that Q' is either $Q(A, B, D, C)$ or $Q(A, C, B, D)$.

Case 1. $Q' = Q(A, B, D, C)$. In this case, we see that the diagonals of Q are longer than the diagonals of Q' and the vertical edges of Q are shorter than the vertical edges of Q' so by Lemma 2.5, $\rho(Q) < \rho(Q')$ so Q' is not a minimal length quadrilateral in C_{2n+1}.

Case 2. $Q' = Q(A, C, B, D)$. Let B' be the vertex of C_{2n+1} reached by moving one edge from B in the direction of C, as shown in Figure 8. Note that we could have $B' = C$. Let $Q'' = Q(A, C, B', D)$ We prove that if $\rho(Q')$ is finite then $\rho(Q'') < \rho(Q')$, which proves that Q' is not a minimal-roundness quadrilateral in C_{2n+1}.

There are two possibilities for the side and diagonal lengths of Q''. These are shown in Figure 9. The lengths in the right hand quadrilateral occur only when $w = x + y + z - 1$. In both possibilities, moving from Q' to Q'' increases or does

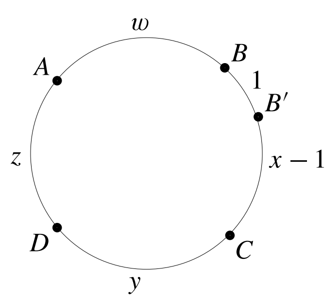

Figure 8. Forming Q'' in case 2 of Lemma 3.6.

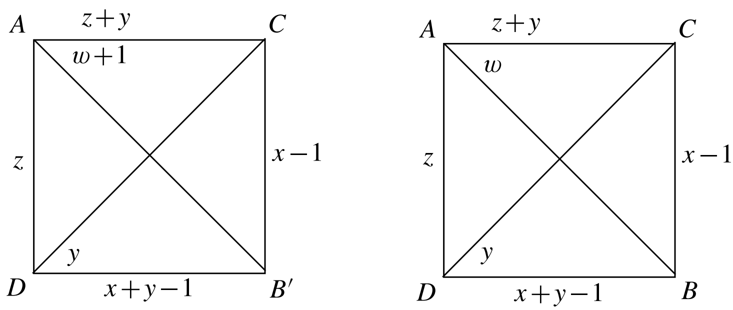

Figure 9. Possible sides and diagonals in Q''.

not change the lengths of diagonals and strictly decreased the lengths of some sides, so if $\rho(Q')$ is finite then by Lemma 2.5 $\rho(Q'') < \rho(Q')$. Since an infinite roundness quadrilateral is not of minimal roundness in C_{2n+1}, which has finite roundness, this proves that Q' is not a minimal-roundness quadrilateral in C_{2n+1}. $\square$

Lemma 3.7. *Let Q' be an out of order nondegenerate quadrilateral in C_{2n+1} comprised of the vertices A, B, C, D as in Figure 6. If the longest side of the in-order quadrilateral $Q = Q(A, B, C, D)$ is at least as long as one of the pairs of remaining adjacent sides but no longer than the other pair of remaining adjacent sides, then Q' is not a minimal-roundness quadrilateral in C_{2n+1}.*

Proof. Without loss of generality, assume that w is the longest length of a side in Q and that $w > x + y$. By our assumptions on lengths of sides, we have

$$w \geq x + y, \quad w \leq y + z, \quad w < x + y + z, \quad w \geq x, y, z.$$

Again by Observation 2.4, without loss of generality we may assume that Q' is either $Q(A, C, B, D)$ or $Q(A, B, D, C)$.

We see the quadrilaterals $Q(A, B, C, D)$, $Q(A, C, B, D)$ and $Q(A, B, D, C)$ in Figure 10 with the lengths of their sides and diagonals. Note that $d(A, C)$ may be either $z+y$ or $w+x$, depending on which is smaller. In either case, moving from $Q(A, C, B, D)$ or $Q(A, B, D, C)$ to $Q(A, B, C, D)$ increases length of diagonals and decreases length of sides, so $\rho(Q) < \rho(Q')$ if $\rho(Q')$ is finite. This proves that Q' is not a minimal-roundness quadrilateral in C_{2n+1}. $\square$

In the proof of the next lemma, we encounter a *linear* graph, l_m, which is a connected graph with exactly two vertices of degree one and the remaining vertices of degree two. Geometrically, a linear graph looks like a line between its two degree one vertices. By case analysis, it is not hard to show that if Q is a quadrilateral in a linear graph, then $\rho(Q) \geq 2$.

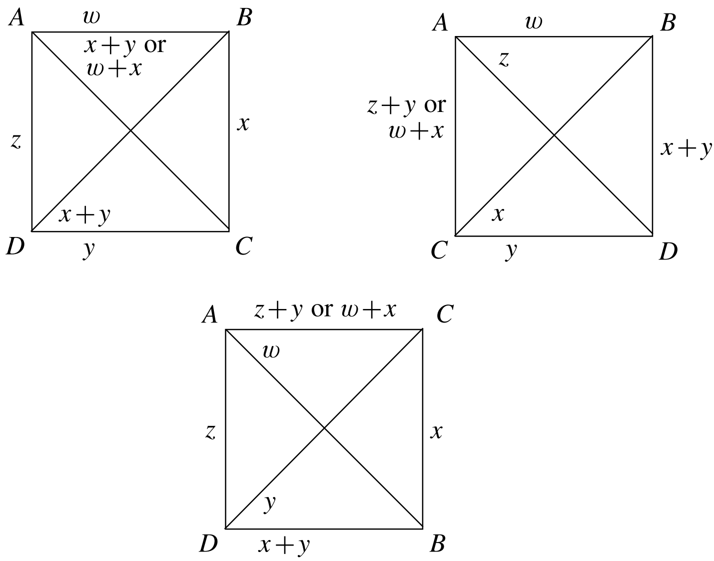

Figure 10. Quadrilaterals in Lemma 3.7.

Lemma 3.8. *Let Q' be an out of order nondegenerate quadrilateral in C_{2n+1} comprised of the vertices A, B, C, D as in Figure 6. If the longest side in the in-order quadrilateral $Q = Q(A, B, C, D)$ is at least as long as the remaining three sides put together, then Q' is not a minimal-roundness quadrilateral in C_{2n+1}.*

Proof. Since $w \geq x + y + z$, Q' actually lies in a metrically embedded linear subgraph l_m of C_{2n+1}. Therefore, $\rho(Q') \geq 2$. Since $\rho(C_{2n+1}) < 2$, Q' is not a minimal roundness quadrilateral in C_{2n+1}. □

Proof of Theorem 3.3. Let Q be a minimal roundness quadrilateral in C_{2n+1} formed from the vertices A, B, C, D as in Figure 6. By Lemma 3.4, Q is nondegenerate. Assume towards a contradiction that Q is out of order. The edge lengths w, x, y, z of Figure 6 satisfy at least one of the conditions of Lemmas 3.5 through 3.8 because these lemmas cover all the possibilities for how long the longest side is in relation to the other sides from being shorter than any pair of adjacent sides to being longer than the three other sides put together. Therefore, by these lemmas, Q is not a minimal roundness quadrilateral, contradicting the fact that it is of minimal roundness. Therefore, the assumption that Q is out of order must be false, proving that Q is in order. □

Balancing sides.

Theorem 3.9. *Let Q be a quadrilateral in C_{2n+1}. Then Q is a minimal roundness quadrilateral in C_{2n+1} if and only if Q is an in-order quadrilateral and the lengths of the longest and shortest sides of Q differ by at most 1.*

We begin with the a lemma that describes the effect of evening out the side lengths of a quadrilateral in the case that the longest side is not too long.

Lemma 3.10. *Let $Q = Q(A, B, C, D)$ be the in-order order quadrilateral in C_{2n+1} comprised of the vertices A, B, C, D as in Figure 6 and suppose that the length longest side of Q is at least two greater than the length of its shortest side. Suppose also that the longest side of Q is shorter than the remaining three sides put together. Then Q has a pair of adjacent sides whose lengths differ by at least two, and the quadrilateral Q' formed by moving the vertex separating these sides into the longer side a distance of one has roundness less than $\rho(Q)$.*

Proof. Without loss of generality, suppose that AB is a longest side, so $w \geq x, y, z$. First we prove that Q must contain a pair of adjacent sides whose lengths differ by at least two. Suppose not and let m be the length of the shortest side. Since no two adjacent side lengths differ by two or more, we must have:

$$\begin{aligned} y = m, \quad & x = w, \quad \text{or} \quad x = w - 1, \\ z = w, \quad & \text{or} \quad z = w - 1. \end{aligned}$$

Since $m \leq w - 2$, and since $w \geq x \geq w - 1$ and $w \geq z \geq w - 1$, we actually have $y = m = w - 2$ and $x = z = w - 1$ because no two adjacent side lengths differ by two or more. This means that C_{2n+1} actually has $4w - 4$ edges, and $2n + 1 = 4w - 4$, which is impossible. Therefore, Q must have two adjacent sides whose lengths differ by at least two.

To prove that evening out the lengths of two adjacent sides whose lengths differ by at least two reduces roundness, we consider two cases.

Case 1. The longest side, AB, is not adjacent to any side of length shorter than itself by at least two. Without loss of generality, suppose that side BC is the longer of the two sides adjacent to AB. Dealing with the four possible combinations for the values of x and z separately, we see that in each case the quadrilateral $Q' = Q(A, B, C', D)$ made from the points A, B, C', D shown in Figure 11 has roundness less than $\rho(Q)$.

Case 2. The longest side AB is adjacent to a side of length shorter than itself by at least two. By our assumptions on lengths, without loss of generality we have

$$w > x + 1, \quad w \geq y, \quad w \geq z, \quad z \geq x, \quad w \leq x + y + z.$$

Consider the quadrilateral $Q' = Q(A, B', C, D)$ constructed from the points A, B', C, D as shown in Figure 11. The possible side and diagonal lengths of Q and Q' are shown in Figure 12. Assume for the moment that the length of the diagonal AC is $w + x$. Let $\rho(Q) = q > 1$. Then, $(w + x)^q + (y + x)^q = w^q + x^q + y^q + z^q$ and $(w + x)^p + (y + x)^p > w^p + x^p + y^p + z^p$ if $p > q$. Additionally, for any

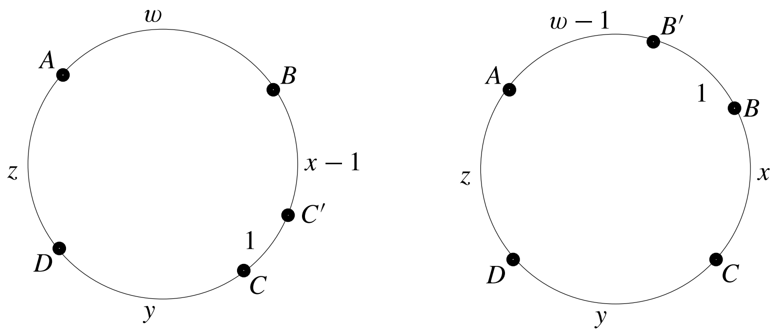

Figure 11. Case 1 (left) and case 2 (right) of Lemma 3.10.

$p > 1$, the function $f(t) = t^p - (t-1)^p$ is increasing for $t \geq 1$, which shows that $w^p + x^p > (w-1)^p + (x+1)^p$ since $w > x+1$. Therefore, if $p \geq q$, we have

$$(w+x)^p + (y+x)^p \geq w^p + x^p + y^p + z^p > (w-1)^p + (x+1)^p + y^p + z^p,$$
$$(w+x)^p + (y+x+1)^p \geq w^p + x^p + y^p + z^p > (w-1)^p + (x+1)^p + y^p + z^p.$$

Since $\rho(Q')$ is the supremum of the values p for which the sum of the p^{th} powers of the diagonals is less than or equal to the sum of the p^{th} power of the sides, we have $\rho(Q') < q = \rho(Q)$ in the case that the length of AC is equal to $w+x$. The proof that $\rho(Q') < \rho(Q)$ in the case that the length of AB is $y+z$ is similar. This finishes the proof of the lemma in case 2. □

Proof of Theorem 3.9. Let Q be a minimal roundness quadrilateral in C_{2n+1}. By Theorem 3.3, we know that Q is in cyclic order. We further know that $\rho(C_{2n+1}) < 2$ and that any quadrilateral in C_{2n+1} whose longest side is at least as long as its other three sides together has roundness greater than 2, so the longest side of Q is shorter than the other three sides together. Therefore, by Lemma 3.10, we know that the lengths of the longest and shortest sides of Q differ by a most 1, for otherwise Q

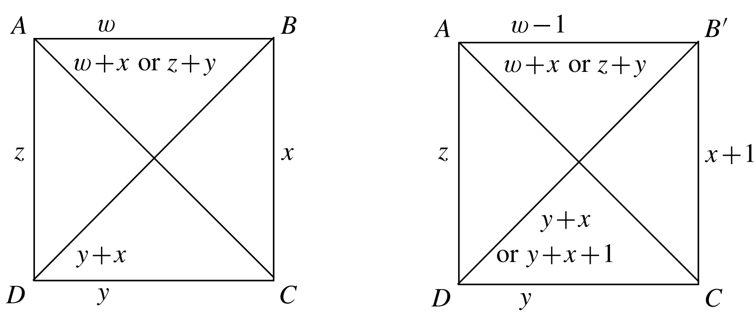

Figure 12. Sides and diagonals in case 2 of Lemma 3.10.

would not have minimal roundness. Therefore, Q is an in-order quadrilateral and the lengths of the longest and shortest sides of Q differ by at most 1.

Conversely, let Q' be an in order quadrilateral with lengths of the longest and shortest sides differing by at most 1. These conditions on Q' uniquely determine the side and diagonal lengths of Q'. Therefore, Q' has the same roundness as the minimal roundness quadrilateral Q from the first half of the proof. Therefore, Q' is itself a minimal-roundness quadrilateral in C_{2n+1}. □

Calculation of $\rho(C_{2n+1})$. Here we prove Theorem 3.1 and Corollary 3.2.

Proof of Theorem 3.1. If $2n+1$ has the form $4k+1$ for integer k then by Theorem 3.9 the diagonals of a minimal roundness quadrilateral Q in C_{2n+1} have length $2k$, one side has length $k+1$ and three sides have length k. Therefore, in this case, $\rho(C_{4k+1})$ is the supremum over all values p such that $2(2k)^p \leq 3k^p + (k+1)^p$. Define the function $f_k(p)$ by $f_k(p) = 2(2k)^p - (k+1)^p - 3k^p$. Then $f_k(1) < 0$ and $f_k(p) > 0$ for sufficiently large p. Therefore $f_k(p)$ has a zero greater than 1. Also, when arranged in decreasing order of the sizes of their bases, the exponential terms in $f_k(p)$ exhibit one "sign change", so $f_k(p)$ has at most one positive zero, (see for example [Langer 1931, p. 128]). Since $f_k(p) > 0$ for sufficiently large p values, $f_k(p)$ is positive for all p values greater than its positive zero. Therefore, $2(2k)^p > 3k^p + (k+1)^p$ for all p values greater than the solution to $2(2k)^p = 3k^p + (k+1)^p$, which proves that $\rho(C_{2n+1})$ is the positive solution of the equation $2(2k)^p = 3k^p + (k+1)^p$ in the case that $2n+1$ has the form $4k+1$. A similar argument shows that $\rho(C_{2n+1})$ is the positive solution of the equation $2(2k-1)^p = 3k^p + (k-1)^p$ if $2n+1$ has the form $4k-1$. □

Since $\rho(C_{2n}) = 1$, to prove Corollary 3.2 it suffices to show that the solutions to the equations (1) and (2) of Theorem 3.1 approach 1 as k goes to infinity. As in the proof of Theorem 3.1, the proofs in each case are similar, so we provide a rigorous proof of only (1), the case that $2n+1$ has the form $4k+1$.

Proof of Corollary 3.2. Restricting our attention to the case $2n+1 = 4k+1$, we have $\rho(C_{4k+1})$ equal to the zero of the function $f_k(p)$ from the proof of Theorem 3.1. We show that this solution can be made arbitrarily close to 1 by choosing k sufficiently large. Since $1 \leq \rho(C_{4k+1}) < \log_2 3$ for k-values greater than 2, we may restrict our attention to p values less than $\log_2 3$. Fix a real number α with $1 < \alpha < \log_2 3$. Consider the function $g(k) = f_k(\alpha) = 2(2k)^\alpha - (k+1)^\alpha - 3k^\alpha$. For all sufficiently large k, $g(k) > 0$. Therefore, for all sufficiently large k, $f_k(\alpha) > 0$. Since $f_k(1) < 0$ for all k, this proves that for all sufficiently large k, the zero of $f_k(p)$ is between 1 and α. It follows that $\lim_{k\to\infty} \rho(C_{4k+1}) = 1$. A similar argument shows that $\lim_{k\to\infty} \rho(C_{4k-1}) = 1$. Since $\rho(C_{2k}) = 1$ for all $k > 2$, this finishes the proof that $\lim_{k\to\infty} \rho(C_k) = 1$. □

4. Triangulated cycles

We now continue our investigation of roundness of finite graphs by investigating the effect of "triangulating" a cycle by connecting various of the vertices in the cycle with edges in a particular way until there are no metrically embedded cycles of length greater than 3. We focus on a particular triangulation of C_k, described below, which we simply denote by T_k.

Definition 4.1. Let C_k be the cyclic graph with vertices $v_1, v_2, \dots, v_k$ in cyclic order. The *triangulated cycle* T_k is formed by connecting with edges the following pairs of vertices

$$(v_2, v_k), (v_k, v_3), (v_3, v_{k-1}), (v_{k-1}, v_4), \dots$$

as shown in Figure 13.

Since the roundness of a circle is equal to 1 and the roundness of $\mathbb{R}^2$ is equal to 2, it seems reasonable that the roundness of the triangulated cycle T_n should be at least a little closer to 2 than the roundness of the nontriangulated cycle of the same length, C_n. We prove this to be true in the main theorem of this section.

Theorem 4.2. *Let T be the triangulated cycle T_{2n} or T_{2n+1}. For $n \geq 2$, $\rho(T)$ is the solution of the equation*

$$n^q = (n-1)^q + 2. \tag{4}$$

Since each T_k for $k \geq 4$ contains an metrically embedded copy of graph Δ, $\rho(T_n) \leq \log_2 3 < 2$, so no minimal roundness quadrilateral in T_n has roundness 2 or greater. This is a fact that we will frequently use without explicitly mentioning it in this section. Another fact we will use throughout is the following lemma whose proof we omit.

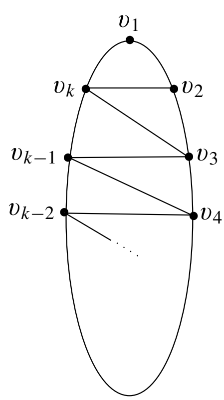

Figure 13. The triangulated cycle T_k.

Lemma 4.3. *If $r \in (0, 1]$ and $x, y > 0$ then $(x + y)^r < x^r + y^r$.*

We now prove our main theorem of this section.

Proof of Theorem 4.2. Let r_n denote the solution of Equation (4). We proceed by induction on n to prove that $\rho(T_{2n}) = \rho(T_{2n+1}) = r_n$. For the base case, $n = 2$, T_4 and T_5 are small enough that one can verify by hand that they have roundness equal to r_2.

Now assume by induction that for all $k < n$, $\rho(T_{2k}) = \rho(T_{2k+1}) = r_k$. We focus first on T_{2n} and prove that $\rho(T_{2n}) = r_n$. Note that $\rho(Q(v_1, v_2, v_{n+1}, v_{n+2})) = r_n$, so $\rho(T_{2n}) \le r_n$. Denote by Q' the quadrilateral $Q' = Q(v_1, v_2, v_{n+1}, v_{n+2})$. Suppose now that Q is a quadrilateral in T_{2n} that does not contain both v_1 and v_{n+1}. In this case, Q is a quadrilateral in a metrically embedded subgraph of T_{2n} isomorphic to T_{2k} or T_{2k+1} for $k < n$. By our induction hypothesis, this subgraph has roundness $r_k > r_n$. So, $\rho(Q) \ge r_k > r_n = \rho(Q')$, and Q is not a minimal roundness quadrilateral in T_{2n}. Therefore, every minimal roundness quadrilateral in T_{2n} contains both v_1 and v_{n+1}. Furthermore, the greatest distance between vertices in T_{2n} is n, and this occurs only between vertices v_1 and v_{n+1}, so any minimal roundness quadrilateral in T_{2n} must be of the form $Q(v_1, u, v_{n+1}, w)$, containing the path from v_1 to v_{n+1} as a diagonal.

Now, let $Q(v_1, u, v_{n+1}, w)$ be a quadrilateral in T_{2n} with the path from v_1 to v_{n+1} on a diagonal. If both u and v occur on the same side of T_{2n} (i.e., either both have subscripts greater than $n + 1$ or both have subscripts less than $n + 1$), then Q lies in a metrically embedded line in T_{2n} and therefore has roundness at least 2 and is not a minimal roundness quadrilateral in T_{2n}. Therefore, assume that $u = v_{b+2}$ and $w = w_{n+2+a}$ for some a, b with $0 \le a \le n-2$ and $4 \le b \le n-2$. We prove that Q is not a minimal roundness quadrilateral in T_{2n} unless $a = b = 0$ or $a = b = n-2$. If $a + b \le n$, then $\rho(Q)$ is the positive solution of

$$n^p + (n - a - b - 1)^p = (n - 1 - a)^p + (a + 1)^p + (n - 1 - b)^p + (b + 1)^p, \quad (5)$$

and if $a + b > n$ then $\rho(Q)$ is the positive solution of

$$n^p + (a + b + 3 - n)^p = (n - 1 - a)^p + (a + 1)^p + (n - 1 - b)^p + (b + 1)^p. \quad (6)$$

We first deal with the case that $a + b \le n$ and prove that the solution to (5) is greater than r_n when $a + b \le n$ and at least one of a and b is strictly greater than 0. Assume now that $a, b \ge 0$, $a + b \le n$ and at least one of a and b is greater than 0. Since

$$n^1 + (n - a - b - 1)^1 < (n - 1 - a)^1 + (a + 1)^1 + (n - 1 - b)^1 + (b + 1)^1,$$

it suffices to prove that

$$n^{r_n} + (n - a - b - 1)^{r_n} < (n - 1 - a)^{r_n} + (a + 1)^{r_n} + (n - 1 - b)^{r_n} + (b + 1)^{r_n}. \quad (7)$$

Consider the function

$$f(a,b)=n^{r_n}+(n-a-b-1)^{r_n}-(n-1-a)^{r_n}-(a+1)^{r_n}-(n-1-b)^{r_n}-(b+1)^{r_n}.$$

We prove inequality (7) by proving that $f(a,b)<0$ for all $a,b\geq 0$ with at least one of a and b greater than 0 and $a+b\leq n$. By the symmetry between a and b, without loss of generality, we may assume that $a\geq b$. Since at least one of a and b is at least 1, we have $a\geq 1$. First consider the function $g(a)=f(a,a)$. Since r_n is the solution to (4), we have $g(0)=0$. Now,

$$g'(a)=-2r_n(n-2a-1)^{r_n-1}+2r_n(n-1-a)^{r_n-1}-2r_n(a+1)^{r_n-1}.$$

By Lemma 4.3, we have

$$(n-1-a)^{r_n-1}<((n-2a-1)+(a+1))^{r_n-1}<(n-2a-1)^{r_n-1}+(a+1)^{r_n-1}.$$

Therefore, $g'(a)<0$ for $a>0$ so $f(a,a)=g(a)<0$ for all $a>0$ and in particular for all $a\geq 1$.

A similar argument proves that $f(a,0)<0$ for all $a\geq 1$, so we are left with proving that $f(a,b)<0$ in the case that $a>b$ and $a,b\geq 1$ and $a+b\leq n$. Since $f(a,a)<0$ it suffices to prove that $f_a(a,b)<0$ for all a and b under consideration. Now, $f_a(a,b)=-r_n(n-a-b-1)^{r_n-1}+r_n(n-1-a)^{r_n-1}-r_n(a+1)^{r_n-1}$. By Lemma 4.3 and the fact that $b<a$ we have

$$(n-1-a)^{r_n-1}<((n-a-b-1)+(a+1))^{r_n-1}<(n-a-b-1)^{r_n-1}+(a+1)^{r_n-1}.$$

Therefore, $f_a(a,b)<0$ for all a,b under consideration, finishing the proof that Q is not a minimal roundness quadrilateral in T_{2n} in the case that $a+b\leq n$. The inequalities in case that $a+b>n$ can be reduced to the inequalities in the case $a+b\leq n$ by making the substitutions $a'=n-2-a$ and $b'=n-2-b$, so the above arguments prove that Q is not a minimal roundness quadrilateral in T_{2n} in this case, either unless $a'=b'=0$, which is the same as $a=b=n-2$.

We have now proved that a minimal roundness quadrilateral in T_{2n} has the form $Q(v_1,u,v_{n+1},w)$ with $u=v_{b+2}$ and $w=w_{n+2+a}$ for some $a,b\geq 0$. But, we have also proved that such a quadrilateral is not a minimal roundness quadrilateral whenever at least one of a and b is greater than 1. Therefore, the quadrilateral given when a and b are equal to 0, $Q(v_1,v_2,v_{n+1},v_{n+2})$, is a minimal roundness quadrilateral in T_{2n}. This finishes the proof that $\rho(T_{2n})=r_n$.

To finish the inductive step the proof, must prove that $\rho(T_{2n+1})=r_n$ also. The details in the argument for this case are very similar to those in the proof that $\rho(T_{2n})=r_n$, but the proof also uses the fact we just proved that $\rho(T_{2n})=r_n$. We therefore omit the proof that $\rho(T_{2n+1})=r_n$. This finishes the proof of the induction step and proves that $\rho(T_{2n})=\rho(T_{2n+1})=r_n$ for all $n\geq 2$. □

We note that it can be proved from our formulas for $\rho(T_{2n})$ and $\rho(T_{2n+1})$ that $\rho(T_{2n}) > 1 = \rho(C_{2n})$ and $\rho(T_{2n+1}) > \rho(C_{2n+1})$, as mentioned in the introduction to this section. The formulas can also be used to prove the following corollary in a way similar to the way that Corollary 3.2 was proved in the previous section.

Corollary 4.4. *Let T_k the triangulated cycle described in Definition 4.1. Then*

$$\lim_{k\to\infty} \rho(T_k) = 1.$$

5. The distribution of roundness for general graphs

As can be seen from the previous two sections, rigorously calculating the roundness of a particular graph or class of graphs can be a daunting task because the number of quadrilaterals in a graph with n vertices grows as n^4. Certainly there is a lot of duplication and some quadrilaterals can be ruled out immediately as not giving the minimal roundness, but the task is still very large. Therefore, we wrote a computer program to aid with example calculations. This program has two forms. In the first form, available at an online calculator, the user enters the adjacency matrix of a graph on 10 or fewer vertices. The program then by brute force enumerates all quadrilaterals in the graph, estimates the roundness of each one and outputs a minimal roundness quadrilateral along with its roundness. In its other form, this program reads in a file containing the adjacency matrices, formatted in a certain way, of a set of graphs on 10 or fewer vertices. The program calculates the roundness of each graph and outputs a list of all the roundness that occurred among the graphs and the number of times each roundness occurred. We ran this program on files containing the adjacency matrices of all nonisomorphic connected graphs on 7, 8 and 9 vertices that we obtained from Gordon Royle's data at the web page *Small Graphs* and found the roundness distributions among these graphs shown in Tables 1–3.

Looking at these data, one notices a number of trends that would be interesting to investigate formally. In particular, most graphs seem to have roundness equal to 1, which makes sense because any graph with a metrically embedded even cycle has roundness equal to 1. Another observation is that, after eliminating the graphs with roundness equal to 1, roundnesses tend to "bunch up" at the upper end around 1.58 and 1.39, with a tail trailing off to roundness equal to 1. It would be interesting to explore and rigorously quantify this phenomenon. One last striking feature of these distributions is that while the gap between the smallest two roundness values gets smaller as the number of vertices gets larger (as it should according to Corollaries 3.2 and 4.4), the gap between the upper two roundness values, $\log_2 3$ and 2 does not seem to shrink. This leads to the question *can any graph have roundness strictly between* $\log_2 3$ *and* 2? The answer is no:

Roundness	Number	Fraction of Total
1.00000	545	0.6389215
1.31091	2	0.0023447
1.39495	26	0.0304807
1.58497	221	0.2590856
2.00000	58	0.0679953
∞	1	0.0011723
Total number of graphs: 853		

Table 1. Roundness distribution: 7 vertices.

Roundness	Number	Fraction of Total
1.00000	9170	0.824862823
1.23336	1	8.99523E-05
1.31091	21	0.001888999
1.32766	4	0.000359809
1.39495	361	0.032472789
1.58497	1395	0.125483494
2.00000	164	0.014752181
∞	1	8.99523E-05
Total number of graphs: 11117		

Table 2. Roundness distribution: 8 vertices.

Roundness	Number	Fraction of Total
1.00000	245324	0.9396507
1.21258	2	7.66E-06
1.23336	16	6.128E-05
1.27156	3	1.149E-05
1.31091	375	0.0014363
1.32766	94	0.00036
1.39495	4844	0.0185537
1.58497	9926	0.038019
2.00000	495	0.001896
∞	1	3.83E-06
Total number of graphs: 261080		

Table 3. Roundness distribution: 9 vertices.

Theorem 5.1. *For any finite graph G, $\rho(G) \notin (\log_2 3, 2)$.*

We call the interval between $\log_2 3$ and 2 a *gap* in the roundness spectrum for finite graphs. Theorem 5.1 and the fact that the data in Tables 1–2 seem to exhibit other gaps suggests the following question:

Are there any other gaps in the roundness spectrum for finite graphs? In particular, does any finite graph have roundness between 1.58497 *and* 1.39495?

We suspect that the answer is yes there are other gaps, including one between 1.58497 and 1.39495, but we do not have a proof at present. For now, we prove Theorem 5.1, beginning with the following lemma. This lemma is the key that allows us to severely restrict the kinds of quadrilaterals that could appear in a graph with roundness between $\log_2 3$ and 2.

Lemma 5.2. *If G is a graph with $\rho(G) > \log_2 3$, every closed nonrepeating path in G is contained in a subgraph of G that is a complete graph.*

Proof. First note that if C_n for $n \geq 4$ or Graph Δ is metrically embedded in a graph, then the graph's roundness is less than or equal to $\log_2 3$. Therefore, G has no metrically embedded subgraph isomorphic to C_n for $n \geq 4$ or Δ. Let γ be a closed nonrepeating path in G of length k. We proceed by induction on k to show that γ is contained in a complete subgraph. The base case $k = 3$ is trivial because in this case, γ itself is a complete graph on 3 vertices. Now assume that every closed nonrepeating path in G of length less at most $n-1$ is contained in a complete subgraph. Consider a closed nonrepeating path γ with length $k = n \geq 4$. If γ is metrically embedded, then γ is a metrically embedded C_n for $n \geq 4$, which is impossible since $\rho(G) < \log_2 3$. Therefore two nonadjacent vertices v and w in γ must be connected by a path in G shorter than the shortest path between them within γ. Let τ be such a path between v and w in G and let γ_1 and γ_2 be the two paths between v and w described by γ. We now have two closed paths, $\gamma' = \tau \cup \gamma_1$ and $\gamma'' = \tau \cup \gamma_2$, both of which have length less than n and which together contain all vertices of γ. The only repetition possible in these paths is in τ. Therefore, by eliminating repetition in τ, or by replacing τ with a segment of τ between two consecutive intersections of τ with γ and choosing new vertices v and w in γ, we may assume that γ' and γ'' are also nonrepeating closed paths of length less that n. By our induction hypothesis both of these paths lie in complete subgraphs of G.

To see that all of γ lies in a single complete subgraph, let G_0 be the subgraph of G consisting of all of the vertices in γ together with all edges between these vertices. Choose vertices s and t in γ. If s and t both lie together in γ' or in γ'', then by fact just proved that both γ' and γ'' lie in complete subgraphs, s and t span an edge in G. If they do not lie together in γ' or γ'' and they do not span an edge in G, then the vertices v, s, w, t span a metrically embedded subgraph isomorphic to

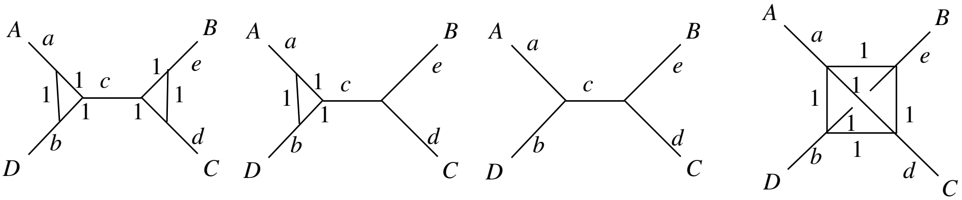

Figure 14. Possible shapes for a quadrilateral in G.

graph Δ, which is impossible since $\rho(G) > \rho(\Delta)$. Therefore, s and t must span an edge in G. This proves that the subgraph G_0 that contains γ is a complete graph, finishing the inductive step. Therefore every closed nonrepeating path in G lies in a complete subgraph of G. □

Using Lemma 5.2, we show that the geodesics comprising any quadrilateral $Q(A, B, C, D)$ in G must fit together into one of the four "shapes" in Figure 14. In this figure, we are considering fixed shortest paths, *geodesics*, between the points A, B, C, D in G. The lines in the figure represent parts of the fixed geodesics, and the lower case letters a, b, c, d and e are the lengths of subpaths of these paths. Paths of length 1 indicate edges connecting nonintersecting subpaths of the geodesics. We note that there may be many geodesics in G between any two points, but for the following arguments, we arbitrarily fix one distinguished geodesic between each pair of vertices that we consider throughout all the proofs.

Lemma 5.3. *If G is a graph with $\rho(G) > \log_2 3$ and if Q is a quadrilateral in G with vertices A, B, C, D then, after possibly renaming A, B, C, D, the geodesics forming Q take on one of the four shapes in Figure 14.*

Proof. First consider the fixed geodesics, X_1 from A to B, X_2 from B to C, and X_3 from A to C. Let A_1 be the vertex at which X_1 and X_3 last agree, A_2 the last vertex at which X_1 and X_2 last agree and let A_3 be the last vertex at which X_2 and X_3 agree. If the A_i are all distinct then these vertices and geodesics must lie as in Figure 15. Otherwise (for example if A_2 were to lie closer along X_1 to A than A_1 lies to A), by making replacements of subpaths, we could shorten at least one of the geodesics X_1, X_2 or X_3. The closed path formed by α followed by β followed by γ in Figure 15 is nonrepeating path, for otherwise we could shorten one of the geodesic paths α, β or γ. By Lemma 5.2 and the fact that each of these is a geodesic, they all must have length 1. Therefore, for any three points in the quadrilateral Q the geodesics between them must form a *degenerate* triangle as in Figure 16, with length ϵ being equal to either 0 or 1. Combining the possibilities for the triangle formed by A, B, C with the possibilities for the triangle formed by A, C, D, and remembering that graph Δ cannot metrically embed in G leads to only the four possible configurations in Figure 14, after possibly renaming the vertices. □

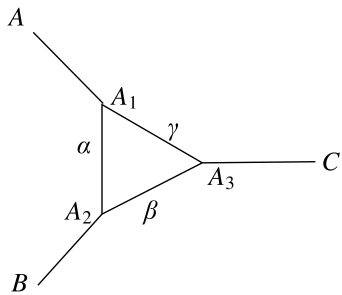

Figure 15. Orientation of vertices in the proof of Lemma 5.3.

Proof of Theorem 5.1. Let G be a finite graph with roundness strictly greater than $\log_2 3$, and let Q be a quadrilateral in G formed with vertices A, B, C, D of G. Fix geodesics in G between each pair of these vertices. By Lemma 5.3, after a possible renaming of the vertices, A, B, C and D and the corresponding geodesics fall into one of the shapes in Figure 14. To prove that $\rho(G) \notin (\log_2 3, 2)$ it suffices to verify that $q = 2$ satisfies the inequality in Definition 2.1 for quadrilateral Q. By Observation 2.4, this amounts to verifying the inequality for quadrilaterals $Q(A,B,C,D)$, $Q(A,B,D,C)$ and $Q(A,C,B,D)$ in all four shapes of Figure 14. All of the verifications are performed similarly, so we show the proof only for $Q(A,B,C,D)$ in the first shape. This amounts to proving that

$$(a+c+d+2)^2 + (b+c+e+2)^2 \\ \leq (a+c+e+2)^2 + (a+b+1)^2 + (d+e+1)^2 + (b+c+d+2)^2.$$

This can be verified through the following sequence of inequalities:

$$0 \leq (a-e)^2 + (b-d)^2 + 2ab + 2ad + 2be + 2de + a + b + d + e + 2,$$

$$0 \leq a^2 + b^2 + d^2 + e^2 + 2ab + 2ad - 2ae - 2bd + 2be + 2de + a + b + d + e + 2,$$

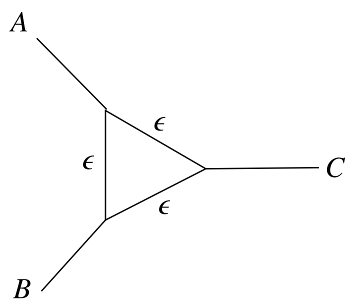

Figure 16. Degenerate triangle in the proof of Lemma 5.3.

$$a^2+b^2+2c^2+d^2+e^2+2ac+2ae+2bc+2bd+2cd+2ce+4a+4b+8c+4d+4e+8 \\ \leq 2a^2+2b^2+2c^2+2d^2+2e^2+2ab+2ac+2ad+2bc+2be \\ +2cd+2ce+2de+5a+5b+8c+5d+5e+10,$$

$$(a+c+d+2)^2+(b+c+e+2)^2 \\ \leq (a+c+e+2)^2+(a+b+1)^2+(d+e+1)^2+(b+c+d+2)^2. \qquad \square$$

We finish this section by noting that the roundness of an infinite connected graph is the infimum of the roundnesses of all of its metrically embedded finite connected subgraphs. Since none of these finite subgraphs can have roundness between $\log_2 3$ and 2, it follows that no graph, finite or infinite, can have roundness between $\log_2 3$ and 2. We record this as our final corollary.

Corollary 5.4. *If G is a connected graph then $\rho(G) \notin (\log_2 3, 2)$.*

References

[Enflo 1970a] P. Enflo, "On a problem of Smirnov", *Ark. Mat.* **8** (1970), 107–109. MR 54 #3661 Zbl 0196.14003

[Enflo 1970b] P. Enflo, "On the nonexistence of uniform homeomorphisms between L_p-spaces", *Ark. Mat.* **8** (1970), 103–105. MR 42 #6600 Zbl 0196.14002

[Jaudon 2008] G. Jaudon, "Some remarks on generalized roundness", *Geom. Dedicata* **135** (2008), 23–27. MR 2009e:20092 Zbl 1152.20037

[LaFont and Passidis 2006] J.-F. LaFont and S. Passidis, "Roundness properties of groups", *Geom. Dedicata* **117** (2006), 137–160.

[Langer 1931] R. E. Langer, "On the zeros of exponential sums and integrals", *Bull. Amer. Math. Soc.* **37**:4 (1931), 213–239. MR 1562129 Zbl 0001.34403

Received: 2009-07-27 Revised: 2009-12-28 Accepted: 2009-12-29

horakm@uwstout.edu — *Mathematics, Statistics and Computer Science Department, University of Wisconsin – Stout, Menomonie, WI 54751, United States*
http://faculty.uwstout.edu/horakm/

larosee@uwstout.edu — *Mathematics, Statistics and Computer Science Department, University of Wisconsin – Stout, Menomonie, WI 54751, United States*

moorej@uwstout.edu — *Mathematics, Statistics and Computer Science Department, University of Wisconsin – Stout, Menomonie, WI 54751, United States*

rooneym@uwstout.edu — *Mathematics, Statistics and Computer Science Department, University of Wisconsin – Stout, Menomonie, WI 54751, United States*

rosenthalh@uwstout.edu — *Mathematics, Statistics and Computer Science Department, University of Wisconsin – Stout, Menomonie, WI 54751, United States*