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Nontrivial solutions to a checkerboard problem

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The squares of an $m \times n$ checkerboard are alternately colored black and red. It has been shown that for every pair m, n of positive integers, it is possible to place coins on some of the squares of the checkerboard (at most one coin per square) in such a way that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity. All solutions to this problem have been what is referred to as trivial solutions, namely, for either black or red, no coins are placed on any square of that color. A nontrivial solution then requires at least one coin to be placed on a square of each color. For some pairs m, n of positive integers, however, nontrivial solutions do not exist. All pairs m, n of positive integers are determined for which there is a nontrivial solution.

1. Introduction

Suppose that the squares of an $m \times n$ checkerboard (m rows and n columns), where $1 \leq m \leq n$ and $n \geq 2$, are alternately colored black and red. Figure 1 shows a 4×5 checkerboard (where a shaded square represents a black square). Two squares are said to be *neighboring* if they belong to the same row or to the same column and there is no square between them. Thus every two neighboring squares are of different colors.

The checkerboard conjecture [Okamoto et al. 2010]. *For every pair m, n of positive integers, it is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) in such a way that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.*

MSC2000: 05C15.
Keywords: $m \times n$ checkerboard, coin placement, trivial solution, nontrivial solution.

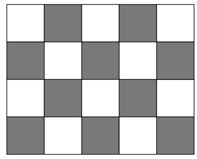

Figure 1. A 4×5 checkerboard.

Figure 2 shows a placement of 5 coins on the 4×5 checkerboard such that the number of coins on neighboring squares of every black square is even and the number of coins on neighboring squares of every red square is odd. Thus for every two squares of different colors, the numbers of coins on neighboring squares are of opposite parity. Consequently, the checkerboard conjecture is true for a 4×5 checkerboard. Observe that each of the five coins on the 4×5 checkerboard of Figure 2 is placed on a black square. Thus the number of coins on neighboring squares of each black square is 0, while the number of coins on neighboring squares of each red square is either 1 or 3. For a given checkerboard, if it is possible to place all coins on squares of one of the two colors, say black, in such a way that the number of coins on neighboring squares of a square is even if and only if that square is black; such a coin placement is called a *trivial solution*. Hence, the coin placement for the 4×5 checkerboard in Figure 2 is a trivial solution. In [Okamoto et al. $\geq$ 2010] it is shown that every $m \times n$ checkerboard has a trivial solution, through the analysis of a vertex coloring of graphs called the *modular coloring*.

The checkerboard theorem. *Every $m \times n$ checkerboard has a trivial solution.*

A *nontrivial solution* to the $m \times n$ checkerboard problem requires at least one coin to be placed on a square of each color. As we will see in this work, some $m \times n$ checkerboards have no nontrivial solution. For an $m \times n$ checkerboard C, we consider the following two related problems.

Problem 1.1. Place coins on some of the red squares of C (at most one coin per square) in such a way that the number of coins on neighboring squares of every black square is even.

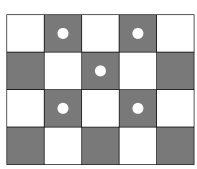

Figure 2. A trivial solution for a 4×5 checkerboard.

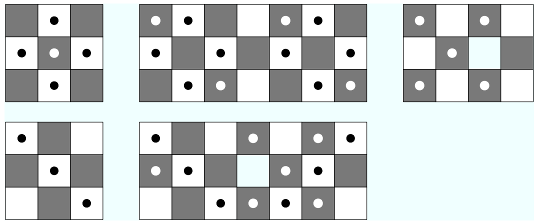

Figure 3. Some $3 \times n$ checkerboards for $n = 3, 4, 7$.

Problem 1.2. Place coins on some of the black squares of C (at most one coin per square) in such a way that the number of coins on neighboring squares of every red square is odd.

Note that in Problem 1.1 there is no restriction on whether or not there are coins on black squares, since coins on black squares do not affect the number of coins on neighboring squares of each black square. Similarly, in Problem 1.2 placing coins on red squares is allowed. Of course, every solution to Problem 1.2 must place at least one coin on a black square of C; while Problem 1.1 has a trivial solution of placing no coins at all on the red squares of C.

In Figure 3, the checkerboards of size 3×3 and 3×7 are shown with coins placed on some of the squares. Since both m and n are odd, every square on the four corners must be of the same color. Observe that the 3×3 checkerboard whose four corner squares are all black has a nontrivial solution to each of Problems 1.1 and 1.2, while the 3×3 checkerboard having four red corner squares has no solution to Problem 1.2. On the other hand, each of the two 3×7 checkerboards has a nontrivial solution to each of Problems 1.1 and 1.2 regardless of the color of the corner squares. As another example, consider the 3×4 checkerboard, which must have two black corner squares and two red corner squares. In this case, the only possible solution to Problem 1.1 is the trivial solution, while there is a solution to Problem 1.2 as shown in Figure 3.

Next consider a checkerboard of size $1 \times n$ with $n \geq 2$. The following is easy to verify.

Observation 1.3. *A* $1 \times n$ *checkerboard* ($n \geq 2$) *has a nontrivial solution to* Problem 1.1 *if and only if the two corner squares are both red. Also, there is a solution to* Problem 1.2 *if and only if* $n \not\equiv 1 \pmod 4$ *or at least one of the two corner squares is black.*

As a result, every $1 \times n$ checkerboard belongs to one of the three categories described in the next corollary.

Corollary 1.4. *For every integer* $n \geq 2$, *a* $1 \times n$ *checkerboard has a nontrivial solution to at least one of* Problems 1.1 *and* 1.2. *In particular*:

(A) *There is a nontrivial solution to each of* Problems 1.1 *and* 1.2 *if and only if* $n \equiv 3 \pmod 4$ *and the two corner squares are both red.*

(B) *There is a nontrivial solution to* Problem 1.1 *but not to* Problem 1.2 *if and only if* $n \equiv 1 \pmod 4$ *and the two corner squares are both red.*

(C) *There is a nontrivial solution to* Problem 1.2 *but not to* Problem 1.1 *if and only if at most one of the two corner squares is red.*

Let $\mathscr{C} = \mathscr{C}_R \cup \mathscr{C}_B$ be the set of $m \times n$ checkerboards with $mn \geq 2$, where $C \in \mathscr{C}_R$ if C contains a red corner square; while $C \in \mathscr{C}_B$ if C contains a black corner square. Hence, a checkerboard belongs to $\mathscr{C}_R \cap \mathscr{C}_B$ if and only if the number of squares is even. Our goal in this paper is to classify all checkerboards (size and color configuration) for which (A) there is a nontrivial solution to each of Problems 1.1 and 1.2; (B) there is a nontrivial solution to Problem 1.1 but not to Problem 1.2; (C) there is a nontrivial solution to Problem 1.2 but not to Problem 1.1; and (D) there is a nontrivial solution to neither Problem 1.1 nor Problem 1.2.

2. Even and odd extensions

Definitions and notation. Before considering checkerboards having multiple rows and multiple columns, we give additional definitions and notation. For an $m \times n$ checkerboard C, let $S = B \cup R$ be the set of mn squares in C, where B and R are the sets of black squares and red squares, respectively, and let $s_{i,j} \in S$ be the square in the i-th row and j-th column for $1 \leq i \leq m$ and $1 \leq j \leq n$.

We express a coin placement for C using a coin placement function $f : S \to \{0, 1\}$ defined by $f(s) = 1$ if and only if there is a coin placed on the square s. The corresponding *neighbor sum* of a square s, denoted by $\sigma_f(s)$ (or simply $\sigma(s)$), is the number of coins placed on the neighboring squares of s. For simplicity, we further assume that $\sigma(s)$ is expressed as one of 0 and 1 modulo 2.

An *even placement* f is a coin placement such that $f(s) = \sigma(s) = 0$ for every $s \in B$. Hence, a checkerboard C has a solution to Problem 1.1 if and only if C has an even placement. In particular, C has a nontrivial solution to Problem 1.1 if and only if C has a nontrivial even placement. An *odd placement* g is a coin placement such that $g(s) = 0$ and $\sigma(s) = 1$ for every $s \in R$. Then a checkerboard C has a solution to Problem 1.2 if and only if C has an odd placement. Recall that every solution to Problem 1.2 must place at least one coin on a black square, implying that there is no trivial odd placement.

To achieve the goal described in the first section, therefore, we investigate the conditions on the size and color configuration of checkerboards under which (A) there are a nontrivial even placement and an odd placement; (B) there is a nontrivial even placement but no odd placement; (C) there is an odd placement but no nontrivial even placement; and (D) there is neither an odd placement nor a nontrivial even placement.

Let $S = S_1 \cup S_2 \cup \cdots \cup S_n$, where $S_j = \{s_{i,j} : 1 \le i \le m\}$ for $1 \le j \le n$. Hence, S_j is the set of the m squares in the j-th column. Further, let $S'_i = S_1 \cup S_2 \cup \cdots \cup S_i$ for $1 \le i \le n$. (Hence $S'_1 = S_1$ and $S'_n = S$.) Let $f_1 : S'_1 \to \{0, 1\}$ be an arbitrary coin placement for the squares in S'_1 such that $f_1(s) = 0$ for every $s \in B \cap S'_1$. Observe then that there exists a unique coin placement $f_2 : S'_2 \to \{0, 1\}$ such that $f_2(s) = 0$ for every $s \in B \cap S'_2$; f_2 restricted to S'_1 equals f_1; and $\sigma(s) = 0$ for every $s \in S'_1$. After finding such a coin placement f_2, observe further that there exists a unique coin placement $f_3 : S'_3 \to \{0, 1\}$ such that $f_3(s) = 0$ for every $s \in B \cap S'_3$; f_3 restricted to S'_2 equals f_2; and $\sigma(s) = 0$ for every $s \in S'_2$. In general, for every integer j $(1 \le j \le n-1)$, suppose that $f_j : S'_j \to \{0, 1\}$ is a coin placement such that $f_j(s) = 0$ for every $s \in B \cap S'_j$ and $\sigma(s) = 0$ for every $s \in S'_{j-1}$ (if $j \ge 2$). Then there exists a unique coin placement $f_{j+1} : S'_{j+1} \to \{0, 1\}$ such that $f_{j+1}(s) = 0$ for every $s \in B \cap S'_{j+1}$; f_{j+1} restricted to S'_j equals f_j; and $\sigma(s) = 0$ for every $s \in S'_j$.

Similarly, let $g_1 : S'_1 \to \{0, 1\}$ be an arbitrary coin placement for the squares of S'_1 such that $g_1(s) = 0$ for every $s \in R \cap S'_1$. Then there exists a unique coin placement $g_2 : S'_2 \to \{0, 1\}$ such that $g_2(s) = 0$ for every $s \in R \cap S'_2$; g_2 restricted to S'_1 equals g_1; and $\sigma(s) = 1$ for every $s \in R \cap S'_1$. In general, for every integer j $(1 \le j \le n-1)$, suppose that $g_j : S'_j \to \{0, 1\}$ is a coin placement such that $g_j(s) = 0$ for every $s \in R \cap S'_j$ and $\sigma(s) = 1$ for every $s \in R \cap S'_{j-1}$ (if $j \ge 2$). Then there exists a unique coin placement $g_{j+1} : S'_{j+1} \to \{0, 1\}$ such that $g_{j+1}(s) = 0$ for every $s \in R \cap S'_{j+1}$; g_{j+1} restricted to S'_j equals g_j; and $\sigma(s) = 1$ for every $s \in R \cap S'_j$.

These observations yield the following two lemmas.

Lemma 2.1. *For each coin placement $f_1 : S_1 \to \{0, 1\}$ such that $f_1(s) = 0$ for every $s \in B \cap S_1$, there exists a unique coin placement $F : S \to \{0, 1\}$ such that* (i) *$F(s) = 0$ for every $s \in B$,* (ii) *F restricted to S_1 equals f_1, and* (iii) *$\sigma(s) = 0$ for every $s \in S'_{n-1} (= S - S_n)$. Furthermore, F is nontrivial if and only if f_1 is nontrivial.*

Lemma 2.2. *For each coin placement $g_1 : S_1 \to \{0, 1\}$ such that $g_1(s) = 0$ for every $s \in R \cap S_1$, there exists a unique coin placement $G : S \to \{0, 1\}$ such that* (i) *$G(s) = 0$ for every $s \in R$,* (ii) *G restricted to S_1 equals g_1, and* (iii) *$\sigma(s) = 1$ for every $s \in R \cap S'_{n-1}$.*

For the coin placements f_1 and F for a checkerboard C described in Lemma 2.1, we say that F is the *even extension* of f_1; while for the coin placements g_1 and G of C described in Lemma 2.2, G is said to be the *odd extension* of g_1. We also say that F and G are the even and odd extensions for C, respectively.

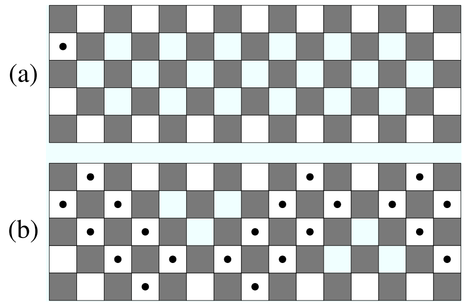

Figure 4. An even extension for a 5×15 checkerboard.

Properties of even extensions. Consider an $m \times n$ checkerboard, where n is sufficiently large. Let $f_1 : S_1 \to \{0, 1\}$ be an arbitrary coin placement such that $f_1(s) = 0$ for every $s \in B \cap S_1$. Obtain the unique even extension F of f_1. We will next show that $F(s) = 0$ for every $s \in S_j$ whenever $j \equiv 0 \pmod{m+1}$. Before we verify this, let us consider an example. In Figure 4(a), there is a 5×15 checkerboard in $\mathscr{C}_B - \mathscr{C}_R$ with coins placed on some of the red squares in S_1. Figure 4(b) shows its even extension and observe that there are no coins placed on the squares in $S_6 \cup S_{12}$.

Proposition 2.3. *For an $m \times n$ checkerboard with $2 \le m \le n$, let $f_1 : S_1 \to \{0, 1\}$ be an arbitrary coin placement with $f_1(s) = 0$ for every $s \in B \cap S_1$. Then for every $j \equiv 0 \pmod{m+1}$, the unique even extension of f_1 assigns 0 to every square in S_j.*

Proof. We begin by assuming that m is even. Furthermore, we may assume that $s_{1,1} \in R$. Therefore, $s_{i,j} \in R$ if and only if $i + j$ is even. Let $f_1 : S_1 \to \{0, 1\}$ be given by

$$f_1(s_{i,1}) = \begin{cases} a_{(i+1)/2} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

for $1 \le i \le m$. Furthermore, for each integer j $(1 \le j \le m/2)$ let $A_j = \sum_{i=1}^{j} a_i$ while $A_0 = 0$. Now define a coin placement $F_1 : S \to \{0, 1\}$ by $F_1(s) = 0$ if $s \in B$ and

$$F_1(s_{i,j}) = \begin{cases} A_{(i+j)/2} + A_{(i-j)/2} & \text{if } j \le i \text{ and } i + j \le m, \\ F_1(s_{(m+1)-j,(m+1)-i}) & \text{if } j \le i \text{ and } i + j \ge m + 2, \\ F_1(s_{j,i}) & \text{if } i + 2 \le j \le m, \\ 0 & \text{if } j = m + 1, \\ F_1(s_{(m+1)-i,j-(m+1)}) & \text{if } m + 2 \le j \le 2m + 2, \\ F_1(s_{i,j-(2m+2)}) & \text{if } j \ge 2m + 3, \end{cases}$$

if $s_{i,j} \in R$ (and so $i + j$ is even), where addition is performed modulo 2 except on the subscripts. Note that $F_1(s_{i,2m+2}) = F_1(s_{(m+1)-i,m+1}) = 0$ and so $F_1(s_{i,j}) = 0$

whenever $j \equiv 0 \pmod{m+1}$. Also, $F_1(s_{i,1}) = A_{(i+1)/2} + A_{(i-1)/2} = a_{(i+1)/2} = f_1(s_{i,1})$ for $i = 1, 3, \ldots, m-1$, so F_1 restricted to S_1 equals f_1.

We now show that F_1 is the even extension of f_1, that is, $F_1 = F$. To do this, we need only verify that $\sigma(s) = 0$ for every $s \in B \cap S'_{n-1}$. Hence, we show that $\sigma(s_{i,j}) = 0$ for integers i and j with $1 \le i \le m$ and $1 \le j \le n-1$ such that $i + j$ is odd. By symmetry, we may further suppose that either $1 \le j < i \le m$ or $j = m+1$.

***Case 1*: $\mathbf{1 \le j < i \le m}$.** First suppose that $j = 1$. If $2 \le i \le m-2$, then

$$\begin{aligned}\sigma(s_{i,1}) &= F_1(s_{i,2}) + F_1(s_{i-1,1}) + F_1(s_{i+1,1})\\ &= \big(A_{(i+2)/2} + A_{(i-2)/2}\big) + \big(A_{i/2} + A_{(i-2)/2}\big) + \big(A_{(i+2)/2} + A_{i/2}\big) = 0,\end{aligned}$$

while

$$\sigma(s_{m,1}) = F_1(s_{m,2}) + F_1(s_{m-1,1}) = F_1(s_{m-1,1}) + F_1(s_{m-1,1}) = 0.$$

Next suppose that $i = m$ and $3 \le j \le m-1$. Then

$$\begin{aligned}\sigma(s_{m,j}) &= F_1(s_{m,j-1}) + F_1(s_{m,j+1}) + F_1(s_{m-1,j})\\ &= F_1(s_{m-j+2,1}) + F_1(s_{m-j,1}) + F_1(s_{m-j+1,2})\\ &= (A_{(m-j+3)/2} + A_{(m-j+1)/2}) + (A_{(m-j+1)/2} + A_{(m-j-1)/2})\\ &\qquad + (A_{(m-j+3)/2} + A_{(m-j-1)/2}) = 0.\end{aligned}$$

Hence, suppose next that $2 \le j < i \le m-1$. If $i + j \le m-1$, then

$$\begin{aligned}\sigma(s_{i,j}) &= F_1(s_{i,j-1}) + F_1(s_{i,j+1}) + F_1(s_{i-1,j}) + F_1(s_{i+1,j})\\ &= (A_{(i+j-1)/2} + A_{(i-j+1)/2}) + (A_{(i+j+1)/2} + A_{(i-j-1)/2})\\ &\quad + (A_{(i+j-1)/2} + A_{(i-j-1)/2}) + (A_{(i+j+1)/2} + A_{(i-j+1)/2}) = 0.\end{aligned}$$

For $i + j = m+1$,

$$\begin{aligned}\sigma(s_{i,j}) &= F_1(s_{i,j-1}) + F_1(s_{i,j+1}) + F_1(s_{i-1,j}) + F_1(s_{i+1,j})\\ &= F_1(s_{i,j-1}) + F_1(s_{m-j,m-i+1}) + F_1(s_{i-1,j}) + F_1(s_{m-j+1,m-i})\\ &= (A_{m/2} + A_{(i-j+1)/2}) + (A_{m/2} + A_{(i-j-1)/2})\\ &\qquad + (A_{m/2} + A_{(i-j-1)/2}) + (A_{m/2} + A_{(i-j+1)/2}) = 0.\end{aligned}$$

Similarly, if $i + j \ge m+3$, then

$$\begin{aligned}\sigma(s_{i,j}) &= F_1(s_{i,j-1}) + F_1(s_{i,j+1}) + F_1(s_{i-1,j}) + F_1(s_{i+1,j})\\ &= F_1(s_{m-j+2,m-i+1}) + F_1(s_{m-j,m-i+1}) + F_1(s_{m-j+1,m-i+2}) + F_1(s_{m-j+1,m-i})\\ &= (A_{(2m-i-j+3)/2} + A_{(i-j+1)/2}) + (A_{(2m-i-j+1)/2} + A_{(i-j-1)/2})\\ &\quad + (A_{(2m-i-j+3)/2} + A_{(i-j-1)/2}) + (A_{(2m-i-j+1)/2} + A_{(i-j+1)/2}) = 0.\end{aligned}$$

Case 2: $\boldsymbol{j = m+1}$. Then i is even and $2 \le i \le m$. If $2 \le i \le m-2$, then

$$\begin{aligned}\sigma(s_{i,m+1}) &= F_1(s_{i,m}) + F_1(s_{i,m+2}) + F_1(s_{i-1,m+1}) + F_1(s_{i+1,m+1}) \\ &= F_1(s_{m,i}) + F_1(s_{m-i+1,1}) + 0 + 0 \\ &= F_1(s_{m-i+1,1}) + F_1(s_{m-i+1,1}) = 0.\end{aligned}$$

Finally,

$$\begin{aligned}\sigma(s_{m,m+1}) &= F_1(s_{m,m}) + F_1(s_{m,m+2}) + F_1(s_{m-1,m+1}) \\ &= F_1(s_{1,1}) + F_1(s_{1,1}) + 0 = 0.\end{aligned}$$

Therefore, F_1 is the even extension of f_1 as claimed.

Next we assume that m is odd. If $\{s_{1,1}, s_{m,1}\} \subseteq R$, then let $f_1 : S_1 \to \{0, 1\}$ be given by

$$f_1(s_{i,1}) = \begin{cases} a_{(i+1)/2} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

for $1 \le i \le m$. For each integer j ($1 \le j \le (m+1)/2$) let $A_j = \sum_{i=1}^{j} a_i$, while $A_0 = 0$. Then define a coin placement $F_2 : S \to \{0, 1\}$ by $F_2(s) = 0$ if $s \in B$ and

$$F_2(s_{i,j}) = \begin{cases} A_{(i+j)/2} + A_{(i-j)/2} & \text{if } j \le i \text{ and } i+j \le m+1, \\ F_2(s_{(m+1)-j,(m+1)-i}) & \text{if } j \le i \text{ and } i+j \ge m+3, \\ F_2(s_{j,i}) & \text{if } i+2 \le j \le m, \\ 0 & \text{if } j = m+1, \\ F_2(s_{(m+1)-i,j-(m+1)}) & \text{if } m+2 \le j \le 2m+2, \\ F_2(s_{i,j-(2m+2)}) & \text{if } j \ge 2m+3, \end{cases}$$

if $s_{i,j} \in R$ (and so $i+j$ is even), where addition is performed modulo 2 except on the subscripts. Then it can be verified that $F_2 = F$ in a manner similar to that used to show that $F_1 = F$ when m is even.

Similarly, if $\{s_{1,1}, s_{m,1}\} \subseteq B$, then let $f_1 : S_1 \to \{0, 1\}$ be given by

$$f_1(s_{i,1}) = \begin{cases} a_{i/2} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd,} \end{cases}$$

for $1 \le i \le m$. For each integer j ($1 \le j \le (m-1)/2$) let $A_j = \sum_{i=1}^{j} a_i$ while $A_0 = 0$. Then define a coin placement $F_3 : S \to \{0, 1\}$ by $F_3(s) = 0$ if $s \in B$ and

$$F_3(s_{i,j}) = \begin{cases} A_{(i+j-1)/2} + A_{(i-j-1)/2} & \text{if } j \le i-1 \text{ and } i+j \le m, \\ F_3(s_{(m+1)-j,(m+1)-i}) & \text{if } j \le i-1 \text{ and } i+j \ge m+2, \\ F_3(s_{j,i}) & \text{if } i+1 \le j \le m, \\ 0 & \text{if } j = m+1, \\ F_3(s_{(m+1)-i,j-(m+1)}) & \text{if } m+2 \le j \le 2m+2, \\ F_3(s_{i,j-(2m+2)}) & \text{if } j \ge 2m+3, \end{cases}$$

if $s_{i,j} \in R$ (and so $i + j$ is odd), where addition is performed modulo 2 except on the subscripts. Then again it can be verified that $F_3 = F$. $\square$

Properties of odd extensions. Next we present some properties of odd extensions. Again consider an $m \times n$ checkerboard, where n is sufficiently large, and let $g_1 : S_1 \to \{0, 1\}$ be an arbitrary coin placement such that $g_1(s) = 0$ for every $s \in R \cap S_1$. Obtain the unique odd extension G of g_1. It turns out that the behavior of odd extensions can be sometimes different from that of even extensions, depending on the size and color configuration of checkerboards. Before continuing our discussion, we first define a special odd extension.

Definition 2.4. For the trivial coin placement $g_1 : S_1 \to \{0, 1\}$, its unique odd extension is called the trivial odd extension and denoted by G_0.

See Figure 5 for examples of the trivial odd extensions. We state the following observation without a proof.

Observation 2.5. *If G_0 is the trivial odd extension for an $m \times n$ checkerboard, then $G_0(s) = 0$ for every $s \in S_j$, where $j \equiv 0 \pmod{2m + 2}$. Furthermore, if $j \equiv m + 1 \pmod{2m + 2}$, then* (i) *$G_0(s) = 0$ for every $s \in S_j$ if $\{s_{1,1}, s_{m,1}\} \not\subseteq R$, while* (ii) *$G_0(s) = 1$ for every $s \in B \cap S_j$ if $\{s_{1,1}, s_{m,1}\} \subseteq R$.*

In fact, *every* odd extension has the property described in Observation 2.5, as shown in the next result.

Proposition 2.6. *For an $m \times n$ checkerboard with $2 \leq m \leq n$, let $g_1 : S_1 \to \{0, 1\}$ be an arbitrary coin placement with $g_1(s) = 0$ for every $s \in R \cap S_1$. Then for every $j \equiv 0 \pmod{2m + 2}$, the unique odd extension G of g_1 assigns* 0 *to every square in S_j. Furthermore, if $j \equiv m + 1 \pmod{2m + 2}$, then* (i) *$G(s) = 0$ for every $s \in S_j$ if $\{s_{1,1}, s_{m,1}\} \not\subseteq R$, while* (ii) *$G(s) = 1$ for every $s \in B \cap S_j$ if $\{s_{1,1}, s_{m,1}\} \subseteq R$.*

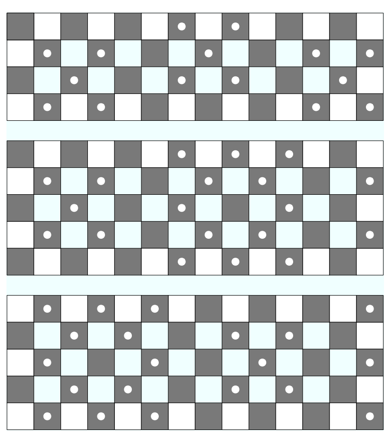

Figure 5. Trivial odd extensions.

Proof. First suppose that m is even. We may assume that $s_{1,1} \in B$. Therefore, $s_{i,j} \in B$ if and only if $i + j$ is even. Let $g_1 : S_1 \to \{0, 1\}$ be given by

$$g_1(s_{i,1}) = \begin{cases} a_{(i+1)/2} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

for $1 \le i \le m$. Also, for each integer j $(1 \le j \le m/2)$ let $A_j = \sum_{i=1}^{j} a_i$ while $A_0 = 0$. Then define the coin placement $G_1 : S \to \{0, 1\}$ by $G_1 = F_1 + G_0$, where F_1 is the even extension defined in the proof of Proposition 2.3 and G_0 is the trivial odd extension. (See Figure 6 for an example.)

If m is odd and $\{s_{1,1}, s_{m,1}\} \subseteq B$, then let $g_1 : S_1 \to \{0, 1\}$ be given by

$$g_1(s_{i,1}) = \begin{cases} a_{(i+1)/2} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

for $1 \le i \le m$. For each integer j $(1 \le j \le (m+1)/2)$ let $A_j = \sum_{i=1}^{j} a_i$ while $A_0 = 0$. Then let $G_2 : S \to \{0, 1\}$ be a coin placement such that $G_2 = F_2 + G_0$, where F_2 is the even extension defined in the proof of Proposition 2.3.

Finally, if m is odd and $\{s_{1,1}, s_{m,1}\} \subseteq R$, then let $g_1 : S_1 \to \{0, 1\}$ be given by

$$g_1(s_{i,1}) = \begin{cases} a_{i/2} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd,} \end{cases}$$

for $1 \le i \le m$. For each integer j $(1 \le j \le (m-1)/2)$ let $A_j = \sum_{i=1}^{j} a_i$ while $A_0 = 0$. Then consider the coin placement $G_3 = F_3 + G_0$, where again F_3 is the even extension defined in the proof of Proposition 2.3.

Observe that G_1, G_2, and G_3 are the odd extensions of g_1 depending on the parity of m and color configuration of the checkerboard. Furthermore, each G_i $(1 \le i \le 3)$ has the desired properties by Proposition 2.3 and Observation 2.5. $\square$

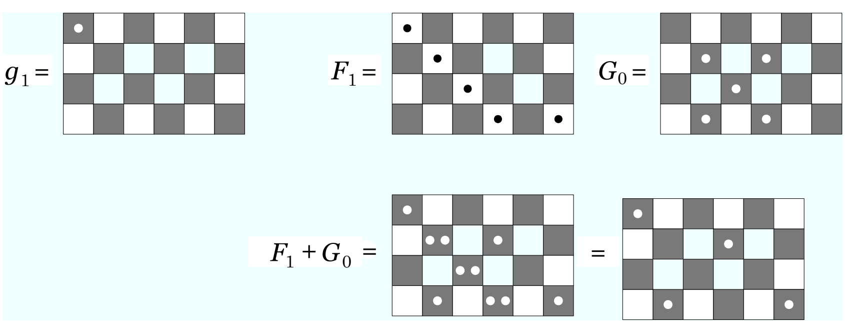

Figure 6. Illustrating F_1, G_0, and $G_1 = F_1 + G_0$.

3. Extensions and reductions of checkerboards

In this section we explore possibilities of obtaining an even (odd) placement for a checkerboard of a certain size and color configuration from an even (odd) placement for another checkerboard of a different size and color configuration.

For example, a 5×8 checkerboard with an even placement is shown in Figure 7(a). Extending this coin placement, we are able to obtain an even placement for a 5×11 checkerboard as well as an even placement for an 8×11 checkerboard, as shown in Figure 7(b). Therefore, the even placement for the 5×8 checkerboard in Figure 7(a) can be *extended* to even placements for a 5×11 checkerboard and an 8×11 checkerboard. On the other hand, we may also say that the even placement for the 5×11 checkerboard shown in Figure 7(b) can be *reduced* to an even placement for a 5×8 checkerboard.

The following observation describes a fact on this process of extending and reducing even (odd) placements. Recall that if f is an even placement for an $m \times n$ checkerboard, then $\sigma_f(s) = 0$ for every $s \in B$; while if F is an even extension for an $m \times n$ checkerboard, then $\sigma_F(s) = 0$ for every $s \in B$ except possibly for those in $B \cap S_n$. Similarly, if g is an odd placement for an $m \times n$ checkerboard, then $\sigma_g(s) = 1$ for every $s \in R$; while if G is an odd extension for an $m \times n$ checkerboard, then $\sigma_G(s) = 1$ for every $s \in R$ except possibly for those in $R \cap S_n$.

Observation 3.1. *Suppose that ℓ, m, and n are positive integers such that $m \leq n$ and $\ell < n$. Then there exists an even* (*odd*) *placement for an $m \times \ell$ checkerboard in $\mathscr{C}_R$ if and only if there exists an even* (*odd*) *extension F for an $m \times n$ checkerboard in $\mathscr{C}_R$ such that $F(s) = 0$ for every $s \in S_{\ell+1}$. Similarly, there exists an even* (*odd*) *placement for an $m \times \ell$ checkerboard in $\mathscr{C}_B$ if and only if there exists an even* (*odd*) *extension F for an $m \times n$ checkerboard in $\mathscr{C}_B$ such that $F(s) = 0$ for every $s \in S_{\ell+1}$.*

As a consequence of Propositions 2.3 and 2.6 and Observation 3.1, we obtain a result for $m \times m$ checkerboards.

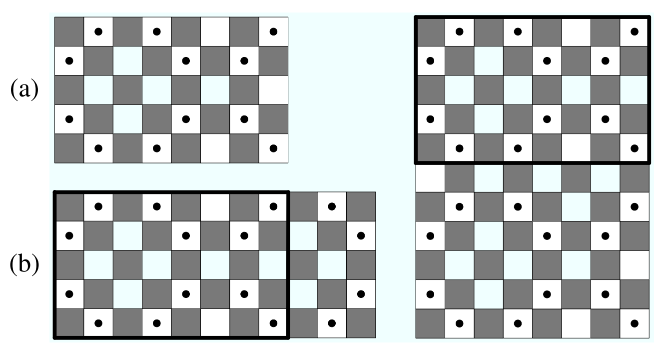

Figure 7. Examples of extension and reduction of checkerboards.

Corollary 3.2. *For every integer $m \geq 2$, an $m \times m$ checkerboard C has a nontrivial even placement. Furthermore, C has an odd placement if and only if $C \in \mathscr{C}_B$.*

In the following subsections we discuss in more detail a way to extend or to reduce a given even (odd) placement for a checkerboard to obtain even (odd) placements for checkerboards of different sizes and color configurations.

Determining the existence of even placements. By Observation 3.1, we take a closer look at the even extensions for checkerboards. Recall the construction of an even extension $F \in \{F_1, F_2, F_3\}$ described in Proposition 2.3. We make the following observation on F.

Observation 3.3. *Let F be an even extension for an $m \times n$ checkerboard with $2 \leq m \leq n$.*

(a) *Suppose that $F(s) = 0$ for every $s \in S_k$ for some k. Then $F(s) = 0$ for every $s \in S_\ell$ whenever $\ell \equiv \pm k \pmod{m+1}$. Also, $F(s) = 0$ for every $s \in S_\ell$ whenever ℓ is a multiple of k.*

(b) *Suppose that $F(s) = 1$ for some $s \in S_k$ for some k. Then $F(s) = 1$ for some $s \in S_\ell$ whenever $\ell \equiv \pm k \pmod{m+1}$. In particular, if F is nontrivial, then $F(s) = 1$ for some $s \in S_\ell$ whenever $\ell \equiv \pm 1 \pmod{m+1}$.*

The following is a consequence of Observations 3.1 and 3.3 and Corollary 3.2.

Corollary 3.4. *Let C be an $m \times n$ checkerboard with $1 \leq m \leq n$ and $n \geq 2$.*

(a) *If $n \equiv m \pmod{m+1}$, then C has a nontrivial even placement.*

(b) *Suppose that ℓ is a positive integer with $\ell \equiv n \pmod{\ell+1}$. If there exists an $\ell \times m$ checkerboard C' such that either $\{C, C'\} \subseteq \mathscr{C}_R$ or $\{C, C'\} \subseteq \mathscr{C}_B$ and there exists a nontrivial even placement for C', then there exists a nontrivial even placement for C.*

(c) *If $n \equiv 0 \pmod{m+1}$ or $n \equiv m-1 \pmod{m+1}$, then C has no nontrivial even placement.*

We are now prepared to present a complete result on even placements.

Theorem 3.5. *Let C be an $m \times n$ checkerboard with $1 \leq m \leq n$ and $n \geq 2$. Then C has a nontrivial even placement if and only if either* (i) $m \equiv n \equiv \ell \pmod{\ell+1}$ *for some integer $\ell \geq 2$, or* (ii) $C \in \mathscr{C}_R - \mathscr{C}_B$.

Proof. We may assume that $m \geq 2$ since the result holds for $m = 1$ by Corollary 1.4. If (i) occurs, then first suppose that $C \in \mathscr{C}_R$ and consider an $\ell \times \ell$ checkerboard $C' \in \mathscr{C}_R$. By Corollary 3.2, there exists a nontrivial even placement for C'. Then by Corollary 3.4(a) there exists a nontrivial even placement for an $\ell \times m$ checkerboard $C'' \in \mathscr{C}_R$, which in turn implies that there exists a nontrivial even placement for C by Corollary 3.4(b). Observe also that the same argument holds if $C \in \mathscr{C}_B$. If (ii)

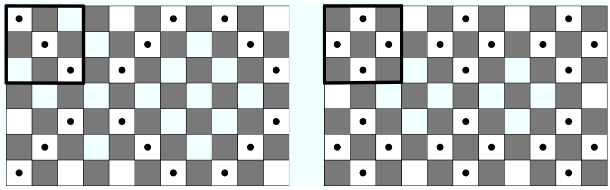

Figure 8. Two 7×11 checkerboards.

occurs, then observe that the coin placement $f : S \to \{0, 1\}$ defined by $f(s_{i,j}) = 1$ if and only if both i and j are odd is an even placement for C.

For the converse, assume, to the contrary, that there exists an $m \times n$ checkerboard in $\mathscr{C}_B$ having a nontrivial even placement with no integer $\ell \geq 2$ such that $m \equiv n \equiv \ell \pmod{\ell + 1}$. In particular, $n \not\equiv m \pmod{m + 1}$. Suppose that C is such a checkerboard with the smallest number of squares. Since $C \in \mathscr{C}_B$, we may assume that $s_{1,n} \in B$. Let f be a nontrivial even placement for C. Hence, $n \not\equiv 0, m - 1 \pmod{m + 1}$ by Corollary 3.4(c). This implies that there exists an integer k with $1 \leq k \leq m-2$ such that $n \equiv k \pmod{m+1}$. However then, f restricted to $S - S'_{n-k}$ induces an even placement for an $k \times m$ checkerboard belonging to $\mathscr{C}_B$, which is impossible since $km < mn$. $\square$

Figure 8 shows even placements for the two 7×11 checkerboards. Note that $7 \equiv 11 \equiv 3 \pmod 4$, so we use an even placement for a 3×3 checkerboard as a building block.

Determining the existence of odd placements for checkerboards in $\mathscr{C}_B$. We now show that every checkerboard in $\mathscr{C}_B$ has an odd placement. We start with:

Proposition 3.6. *An $m \times \ell$ checkerboard in $\mathscr{C}_B$, where $\ell \in \{1, m, m \pm 1\}$, has an odd placement.*

Proof. Since the result holds for $\ell \in \{1, m\}$ by Corollaries 1.4 and 3.2, we first assume that $\ell = m - 1$. Let C be an $m \times m$ checkerboard, where $s_{1,1} \in B$. If m is odd, then the trivial odd extension assigns 0 to every $s \in S_m$. If m is even, then define $g_1 : S_1 \to \{0, 1\}$ by $g_1(s) = 0$ if and only if $s \in R \cap S_1$ and observe that the odd extension of g_1 assigns 0 to every $s \in S_m$. Therefore, an $m \times (m - 1)$ checkerboard in $\mathscr{C}_B$ has an odd placement by Observation 3.1. This also implies that an $m \times (m + 1)$ checkerboard in $\mathscr{C}_B$ has an odd placement. $\square$

Let us also recall the construction of an odd extension $G \in \{G_1, G_2\}$ for checkerboards in $\mathscr{C}_B$ described in Proposition 2.6. We saw that $G(s) = 0$ for every $s \in S_j$ whenever $j \equiv 0 \pmod{m + 1}$. This together with Observation 3.1 leads to:

Proposition 3.7. *Let ℓ, m, and n be integers with $1 \leq \ell \leq m + 1$ and $n \equiv \ell \pmod{m + 1}$. If an $m \times n$ checkerboard in $\mathscr{C}_B$ has an odd placement, then so does* *an $m \times \ell$ checkerboard in $\mathscr{C}_B$. Conversely, if an $m \times \ell$ checkerboard in $\mathscr{C}_B$ has an odd placement, then so does an $m \times n$ checkerboard in $\mathscr{C}_B$.*

Proof. First suppose that g is an odd placement for an $m \times n$ checkerboard with $s_{1,n} \in B$. Then $g(s) = 0$ for every $s \in S_j$ whenever $j \equiv 0 \pmod{m+1}$ by Proposition 2.6. This implies that g restricted to the set $S - S'_{n-\ell}$ induces an odd placement for an $m \times \ell$ checkerboard in $\mathscr{C}_B$.

Conversely, suppose that g is an odd placement for an $m \times \ell$ checkerboard with $s_{1,\ell} \in B$. Then extending g to the right, we obtain an odd placement for an $m \times n$ checkerboard C. Furthermore, at least one of $s_{1,n}$ and $s_{m,n}$ belongs to B. Therefore, $C \in \mathscr{C}_B$. □

We state the following as a corollary of Propositions 3.6 and 3.7.

Corollary 3.8. *Let C be an $m \times n$ checkerboard in $\mathscr{C}_B$. If $n \equiv \ell \pmod{m+1}$, where $\ell \in \{1, m, m+1\}$, then C has an odd placement.*

We are now prepared to show that every checkerboard in $\mathscr{C}_B$ has an odd placement. We first introduce the following algorithm that allows us to obtain a sequence $X = \langle \ell_0, \ell_1, \dots, \ell_k \rangle$ of positive integers for each $m \times n$ checkerboard ($2 \le m < n$) in $\mathscr{C}_B$, where $\ell_0 = n$ and $\ell_1 = m$. For a sequence X, we denote the sequence X followed by ℓ by $\langle X, \ell \rangle$.

Algorithm 3.9.

Input: Two integers m and n with $2 \le m < n$.

Output: A sequence $X = \langle \ell_0, \ell_1, \dots, \ell_k \rangle$ of positive integers with $\ell_0 = n$ and $\ell_1 = m$.

Step 1. Let $\ell_0 \leftarrow n$, $\ell_1 \leftarrow m$, and $X_1 \leftarrow \langle \ell_0, \ell_1 \rangle$. Let $i \leftarrow 1$.

Step 2. Let ℓ_{i+1} be the integer with $1 \le \ell_{i+1} \le \ell_i + 1$ such that

$$\ell_{i+1} \equiv \ell_{i-1} \pmod{\ell_i + 1}.$$

Step 3. If $\ell_{i+1} \in \{1, \ell_i, \ell_i \pm 1\}$, then go to Step 4. Otherwise, let $i \leftarrow i + 1$ and $X_i \leftarrow \langle X_{i-1}, \ell_i \rangle$. Return to Step 2.

Step 4. Output $X = \langle X_i, \ell_{i+1} \rangle$.

As an example, consider a 12×23 checkerboard in $C \in \mathscr{C}_B$. Then $X = \langle 23, 12, 10, 1 \rangle$. Let C_1, C_2, and C_3 be checkerboards in $\mathscr{C}_B$, whose sizes are 1×10, 10×12, and 12×23, respectively. We saw that C_1 has an odd placement by Corollary 3.8 (or by Corollary 1.4). Then by Proposition 3.7, so does C_2, which in turn implies that so does $C_3 = C$. We illustrate how we obtain an odd placement for each of C_1, C_2, and C_3 in Figure 9.

We have proved the result for checkerboards having some black corner squares:

Theorem 3.10. *Every checkerboard in $\mathscr{C}_B$ has an odd placement.*

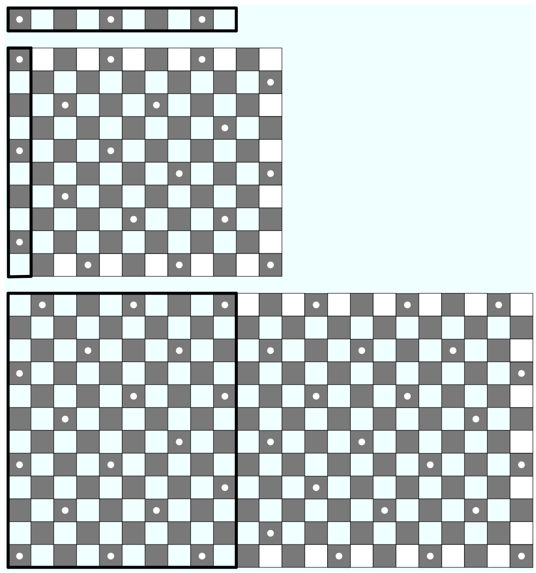

Figure 9. Obtaining odd placements for C_1, C_2, and C_3.

Determining the existence of odd placements for checkerboards in $\mathscr{C}_R - \mathscr{C}_B$. Finally, we consider those checkerboards in $\mathscr{C}_R - \mathscr{C}_B$, whose corner squares are all red. Note that if C is an $m \times n$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$, then both m and n are odd. We have seen in Figure 3 that the 3×7 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has an odd placement while the 3×3 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ does not. Hence, our goal here is to characterize the checkerboards in $\mathscr{C}_R - \mathscr{C}_B$ for which there are odd placements.

Recall the construction of the odd extension G_3 described in Proposition 2.6. We saw that

$$G(s) = 0 \text{ for every } s \in S_j \quad \text{whenever } j \equiv 0 \pmod{2m+2}. \tag{1}$$

As a consequence of (1) with Observation 3.1, we state the following. Note also that the result holds for $m = 1$ by Corollary 1.4.

Corollary 3.11. *An $m \times (2m+1)$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has an odd placement for every $m \geq 1$.*

Here is another useful fact.

Proposition 3.12. *Let ℓ and m be odd integers with $m \geq 3$ and $1 \leq \ell \leq 2m-1$. Then an $m \times \ell$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has an odd placement if and only if an $m \times (2m-\ell)$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has an odd placement.*

Proof. Let C be an $m \times \ell$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ and let g be an odd placement for C. Let G be the odd extension of g for an $m \times (2m+3)$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ and observe that $\{s_{1,\ell+2}, s_{m,\ell+2}, s_{1,2m+1}, s_{m,2m+1}\} \subseteq R$. By Proposition 2.6 and Observation 3.1, $G(s) = 0$ for every $s \in S_{\ell+1} \cup S_{2m+2}$. This implies that G restricted to the set $S'_{2m+1} - S'_{\ell+1}$ induces an odd placement for an $m \times (2m - \ell)$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$. The converse can be verified in the same manner. □

The following is another consequence of (1) and Observation 3.1, stated without a proof since it will be almost identical to that of Proposition 3.7.

Proposition 3.13. *Let ℓ, m, and n be odd integers with $1 \le \ell \le 2m+1$ and $n \equiv \ell \pmod{2m+2}$. Then an $m \times n$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has an odd placement if and only if an $\ell \times m$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has an odd placement.*

By Proposition 3.13 with Corollaries 3.2 and 3.11, we have the following. Note again that the result holds for $m = 1$ as well.

Corollary 3.14. *Let C be an $m \times n$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$.*

(a) *If $n \equiv 2m+1 \pmod{2m+2}$, then C has an odd placement.*

(b) *If $n \equiv m \pmod{2m+2}$, then C does not have an odd placement.*

We now present an algorithm that finds a sequence Y of positive integers for each $m \times n$ checkerboard C in $\mathscr{C}_R - \mathscr{C}_B$, where m and n are positive odd integers with $m \le n$. We then use Y to determine whether or not C has an odd placement.

Algorithm 3.15.

Input: Two odd integers m and n with $1 \le m \le n$ and $n \ge 3$.

Output: A sequence $Y = \langle \ell_0, \ell_1, \dots, \ell_k \rangle$ of positive integers with $\ell_0 = n$ and $\ell_1 = m$.

Step 1. Let $\ell_0 \leftarrow n$, $\ell_1 \leftarrow m$, and $Y_1 = \langle \ell_0, \ell_1 \rangle$. Let $i \leftarrow 1$.

Step 2. Let ℓ'_{i+1} be the odd integer with $1 \le \ell'_{i+1} \le 2\ell_i + 1$ such that $\ell'_{i+1} \equiv \ell_{i-1} \pmod{2\ell_i + 2}$.

Step 3. If $\ell'_{i+1} \in \{\ell_i, 2\ell_i + 1\}$, then let $\ell_{i+1} = \ell'_{i+1}$ and go to Step 4. Otherwise, let ℓ_{i+1} be the odd integer with $1 \le \ell_{i+1} \le \ell_i - 2$ such that either $\ell_{i+1} \equiv \ell'_{i+1} \pmod{2\ell_i}$ or $\ell_{i+1} \equiv -\ell'_{i+1} \pmod{2\ell_i}$. Let $i \leftarrow i+1$ and $Y_i \leftarrow \langle Y_{i-1}, \ell_i \rangle$. Return to Step 2.

Step 4. Output $Y = \langle Y_i, \ell_{i+1} \rangle$.

We present the result on odd placements for checkerboards in $\mathscr{C}_R - \mathscr{C}_B$ as a consequence of Propositions 3.12 and 3.13 and Corollary 3.14.

Theorem 3.16. *Let $C \in \mathscr{C}_R - \mathscr{C}_B$ and obtain the sequence $Y = \langle \ell_0, \ell_1, \dots, \ell_k \rangle$ for C. Then C has an odd placement if and only if $\ell_{k-1} \ne \ell_k$.*

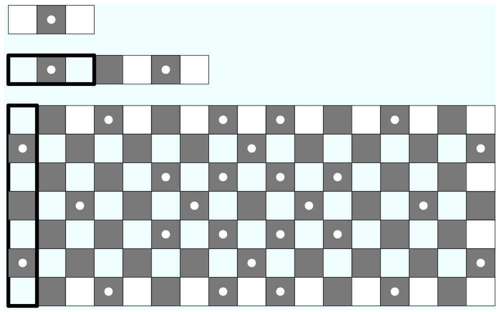

Figure 10. An odd placement for the 7×17 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$.

For example, the 1×9 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has no odd placement, as verified in Corollary 1.4, since $Y = \langle 9, 1, 1\rangle$. Also, the 5×29 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ has no odd placement since $Y = \langle 29, 5, 5\rangle$.

On the other hand, the 7×17 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$, whose associated sequence is $Y = \langle 17, 7, 1, 3\rangle$, has an odd placement. To actually build an odd placement using Y, we start with the 1×3 checkerboard C_1 in $\mathscr{C}_R - \mathscr{C}_B$ with an odd placement g_1. (Note also that this is the unique odd placement for C_1.) Since $7 \equiv 3 \pmod{(2 \cdot 1 + 2)}$, obtain an odd placement g_2 for the 1×7 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ by extending g_1. Since $17 \equiv 1 \pmod{(2 \cdot 7 + 2)}$, we can extend g_2 to obtain an odd placement for the 7×17 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$, as shown in Figure 10.

As another example, let us consider the 7×21 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$. Then $Y = \langle 21, 7, 5, 3, 1, 3\rangle$. We again start with the 1×3 checkerboard C_1 in $\mathscr{C}_R - \mathscr{C}_B$ with the unique odd placement g_1 for C_1. Since $5 \equiv -1 \pmod{(2 \cdot 3)}$, we can obtain an odd placement g_2 for the 3×5 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ from the odd placement for the $3 \times (2 \cdot 3 + 1)$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ obtained by extending g_1 (see Figure 11 on the next page). Similarly, since $7 \equiv -3 \pmod{(2 \cdot 5)}$, obtain an odd placement g_3 for the 5×7 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ from the odd placement for the $5 \times (2 \cdot 5 + 1)$ checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ obtained by extending g_2 (again see Figure 11). Finally, since $21 \equiv 5 \pmod{(2 \cdot 7 + 2)}$, we obtain an odd placement for the 7×21 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$ by simply extending g_3.

4. Conclusion

The following two theorems summarize the results obtained in the previous sections.

Theorem 4.1. *Every checkerboard in $\mathscr{C}$ has a nontrivial solution to at least one of* Problems 1.1 *and* 1.2.

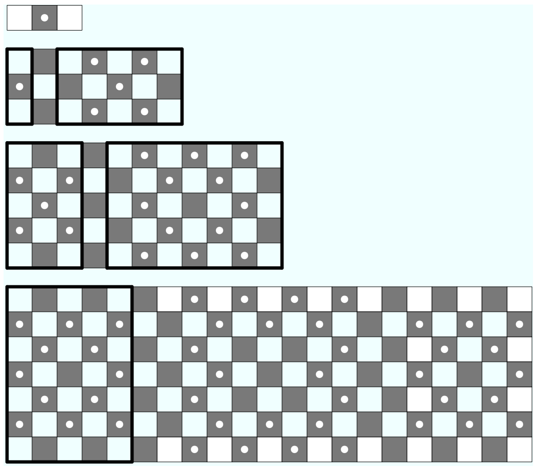

Figure 11. An odd placement for the 7×21 checkerboard in $\mathscr{C}_R - \mathscr{C}_B$.

Proof. If $C \in \mathscr{C}_R - \mathscr{C}_B$, then C has a nontrivial even placement. If $C \in \mathscr{C}_B$, then C has an odd placement. □

Theorem 4.2. *Let C be an $m \times n$ checkerboard, where $1 \leq m \leq n$ and $n \geq 2$. If $C \in \mathscr{C}_R - \mathscr{C}_B$, then let Y be the associated sequence obtained by Algorithm 3.15.*

(A) *C has a nontrivial solution to each of* Problems 1.1 *and* 1.2 *if and only if either* (i) *$C \in \mathscr{C}_R - \mathscr{C}_B$ and the last two terms in Y are not equal or* (ii) *$C \in \mathscr{C}_B$ and $m \equiv n \equiv \ell \pmod{\ell + 1}$ for some integer $\ell \geq 2$.*

(B) *C has a nontrivial solution to* Problem 1.1 *but not to* Problem 1.2 *if and only if $C \in \mathscr{C}_R - \mathscr{C}_B$ and the last two terms in Y are equal.*

(C) *C has a nontrivial solution to* Problem 1.2 *but not to* Problem 1.1 *if and only if $C \in \mathscr{C}_B$ and there is no integer $\ell \geq 2$ such that $m \equiv n \equiv \ell \pmod{\ell + 1}$.*

We conclude this paper with related open questions.

Problem 4.3. If a checkerboard has a nontrivial solution to Problem 1.1 (or 1.2), how many solutions (up to symmetry) are there? Which checkerboards have unique nontrivial solutions?

Problem 4.4. If a checkerboard has a nontrivial solution to Problem 1.1 (or 1.2), what is the minimum number of coins necessary to construct such a solution? Also, what is the maximum number of coins that can be used in a solution?

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meaghan.m.heires@wmich.edu *Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, United States*

ryan.c47.jones@wmich.edu *Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, United States*

okamoto.futa@uwlax.edu *Mathematics Department, University of Wisconsin, La Crosse, WI 54601, United States*

willem.a.renzema@wmich.edu *Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, United States*

matthew.j.roberts@wmich.edu *Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, United States*