

involve

a journal of mathematics

The cardinality of the value sets modulo n of $x^2 + x^{-2}$ and
 $x^2 + y^2$

Sara Hanrahan and Mizan Khan

 mathematical sciences publishers

The cardinality of the value sets modulo n of $x^2 + x^{-2}$ and $x^2 + y^2$

Sara Hanrahan and Mizan Khan

(Communicated by Filip Saidak)

Consider the modular circle $\mathcal{C}_{a,n} = \{(x, y) : x^2 + y^2 \equiv a \pmod{n}, 0 \leq x, y \leq n-1\}$ and the modular hyperbola $\mathcal{H}_n = \{(x, y) : xy \equiv 1 \pmod{n}, 0 \leq x, y \leq n-1\}$. We provide explicit formulas for the cardinality of the sets

$$\{a \pmod{n} : \mathcal{C}_{a,n} \cap \mathcal{H}_n \neq \emptyset\} \quad \text{and} \quad \{a \pmod{n} : \mathcal{C}_{a,n} \neq \emptyset\}.$$

Introduction

Let \mathcal{H}_n denote the *modular hyperbola*

$$\{(x, y) : xy \equiv 1 \pmod{n}, 0 \leq x, y \leq n-1\}.$$

This simply defined discrete set of points has connections to a variety of other mathematical topics including Kloosterman sums, consecutive Farey fractions, and quasirandomness. These connections have inspired a closer look at the distribution of the points of \mathcal{H}_n , and many questions remain open. For a discussion of recent results and open problems on modular hyperbolas, see [Shparlinski 2007].

The propensity of the points on \mathcal{H}_n to collect on lines of slope ± 1 was investigated in [Eichhorn et al. 2009]. In the course of that investigation, formulas for the cardinalities of the sets

$$\{(x - y) \pmod{n} : (x, y) \in \mathcal{H}_n\} \quad \text{and} \quad \{(x + y) \pmod{n} : (x, y) \in \mathcal{H}_n\},$$

were derived. The techniques used to determine these formulas are elementary — within the grasp of an undergraduate mathematics major who has had a course in number theory or abstract algebra.

In this article we investigate the intersection of \mathcal{H}_n with the modular circles

$$\mathcal{C}_{a,n} = \{(x, y) : x^2 + y^2 \equiv a \pmod{n}, 0 \leq x, y \leq n-1\},$$

MSC2000: 11A07, 11A25.

Keywords: cardinality, value sets, modular circle, modular hyperbola.

This work was done as part of Hanrahan's undergraduate honors thesis under Khan's supervision.

and in particular we determine the cardinality of the set

$$\{a \bmod n : \mathcal{C}_{a,n} \cap \mathcal{H}_n \neq \emptyset\} = \{(x^2 + y^2) \bmod n : (x, y) \in \mathcal{H}_n\}.$$

Figure 1 contrasts the modular circle $\mathcal{C}_{1,997}$ with the modular hyperbola \mathcal{H}_{997} . Figure 2 shows the two superimposed, and the intersection $\mathcal{C}_{1,997} \cap \mathcal{H}_{997}$.

This short note is a concise version of SH's honors thesis. It is also a natural addendum to [Eichhorn et al. 2009], as we used the formulas found there to prove our results.

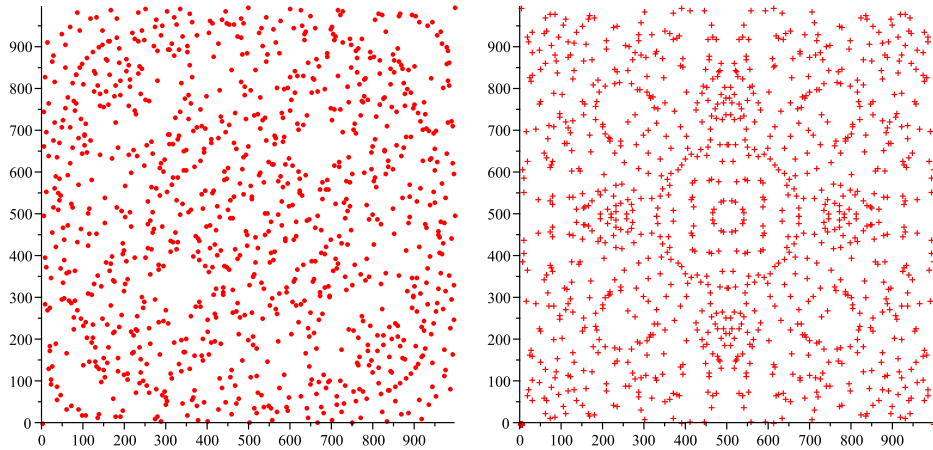


Figure 1. Left: The modular hyperbola \mathcal{H}_{997} . Right: The modular circle $\mathcal{C}_{1,997}$.

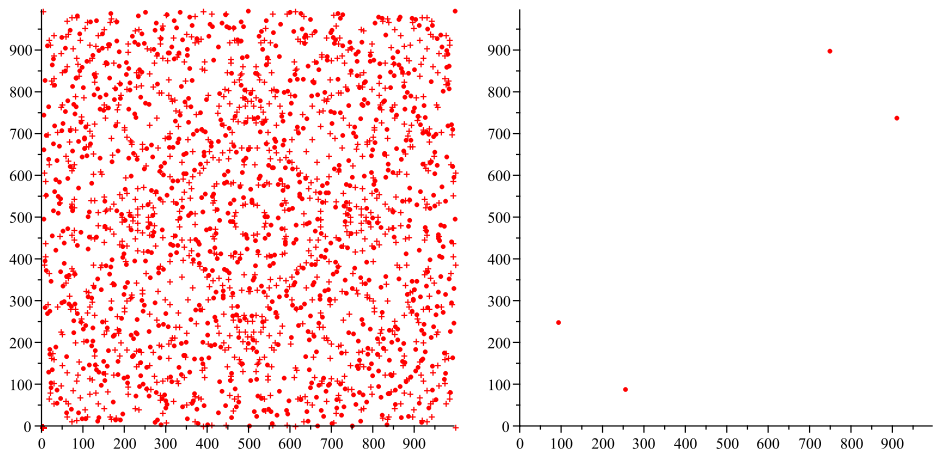


Figure 2. Left: Superposition of the preceding two sets. Points of the modular circle are represented by crosses; those of the modular hyperbola by solid circles. Right: The intersection $\mathcal{C}_{1,997} \cap \mathcal{H}_{997} = \{(91, 252), (252, 91), (745, 906), (906, 745)\}$.

1. Preliminary results

Let $f \in \mathbb{Z}[x_1, \dots, x_k]$ and let $S \subseteq \mathbb{Z}_n^k$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the set of integers modulo n). Then $I(f, S)$ will denote the set

$$I(f, S) = \{f(x_1, \dots, x_k) \bmod n : (x_1, \dots, x_k) \in S\}.$$

We also define two subsets of $I(f, S)$:

$$\begin{aligned} I'(f, S) &= \{a : a \in I(f, S), \gcd(a, n) = 1\}, \\ I''(f, S) &= \{a : a \in I(f, S), \gcd(a, n) \neq 1\}. \end{aligned}$$

Our first result is that the quantity $\#I(f, \mathcal{H}_n)$ is a multiplicative function of n . Furthermore, by replacing each occurrence of \mathcal{H}_n with \mathbb{Z}_n^2 in the statement and proof of the theorem, we get that $\#I(f, \mathbb{Z}_n^2)$ is also a multiplicative function of n .

Proposition 1. *Let $f \in \mathbb{Z}[x, y]$ and define $f_n : \mathcal{H}_n \rightarrow \mathbb{Z}_n$ by*

$$f_n((x, y)) = f(x, y) \bmod n.$$

If $n = a \cdot b$ with $\gcd(a, b) = 1$, then

$$\#I(f, \mathcal{H}_n) = \#I(f, \mathcal{H}_a) \cdot \#I(f, \mathcal{H}_b).$$

It follows that if $n = \prod_{i=1}^m p_i^{e_i}$ is the canonical factorization of n , then

$$\#I(f, \mathcal{H}_n) = \prod_{i=1}^m \#I(f, \mathcal{H}_{p_i^{e_i}}). \tag{1}$$

Proof. The Chinese remainder theorem says that the map $r : \mathbb{Z}_n \rightarrow \mathbb{Z}_a \times \mathbb{Z}_b$ given by

$$r(x) = (x \bmod a, x \bmod b)$$

is an isomorphism of rings. Hence the map $R : \mathcal{H}_n \rightarrow \mathcal{H}_a \times \mathcal{H}_b$ defined by

$$R((x, y)) = ((x \bmod a, y \bmod a), (x \bmod b, y \bmod b))$$

is a bijection. The result now follows from the observation that the diagram

$$\begin{array}{ccc} \mathcal{H}_n & \xrightarrow{R} & \mathcal{H}_a \times \mathcal{H}_b \\ f_n \downarrow & & \downarrow f_a \times f_b \\ \mathbb{Z}_n & \xrightarrow{r} & \mathbb{Z}_a \times \mathbb{Z}_b. \end{array}$$

commutes. □

Thus we have reduced the problem of determining formulas for $\#I(x^2 + y^2, \mathcal{H}_n)$ (or $\#I(x^2 + y^2, \mathbb{Z}_n^2)$) to determining them for prime powers. From this point, we shall refer to the set $I(x^2 + y^2, \mathcal{H}_n)$ as $I(x^2 + x^{-2}, \mathbb{Z}_n)$. All of our formulas were

discovered through extensive numerical experimentation with Maple. Maple was the most valuable research tool at our disposal — only in discovering the formulas, but also in the *proving* stage. In the remainder of this section, we list the mathematical results we need to prove these formulas.

It is more convenient to work with the value set $I((x + x^{-1})^2, \mathbb{Z}_n)$ than with $I(x^2 + x^{-2}, \mathbb{Z}_n)$. The following lemma justifies the change.

Lemma 2. *For any positive integer n ,*

$$\#I(x^2 + x^{-2}, \mathbb{Z}_n) = \#I((x + x^{-1})^2, \mathbb{Z}_n). \tag{2}$$

Proof. The map $z \mapsto (z + 2) \bmod n$ defines a bijection between $I(x^2 + x^{-2}, \mathbb{Z}_n)$ and $I((x + x^{-1})^2, \mathbb{Z}_n)$. □

We next state a basic criterion on the solvability of quadratic congruences modulo prime powers: $x^2 \equiv a \pmod{p^t}$.

Proposition 3 [Ireland and Rosen 1982, Propositions 4.2.3, 4.2.4, p. 46]. *Let p be prime and let a be an integer such that $\gcd(a, p) = 1$.*

- (1) *Suppose $p > 2$. If the congruence $x^2 \equiv a \pmod{p}$ is solvable, then for every $t \geq 2$ the congruence $x^2 \equiv a \pmod{p^t}$ is solvable with precisely 2 distinct solutions.*
- (2) *Suppose $p = 2$. If the congruence $x^2 \equiv a \pmod{2^3}$ is solvable, then for every $t \geq 3$ the congruence $x^2 \equiv a \pmod{2^t}$ is solvable with precisely 4 distinct solutions.*

Proposition 4 [Stangl 1996]. *Let p be an odd prime. Then*

$$\#I(x^2, \mathbb{Z}_{p^t}) = \frac{p^{t+1}}{2(p+1)} + (-1)^{t-1} \frac{p-1}{4(p+1)} + \frac{3}{4}. \tag{3}$$

For the special case $p = 2$ we have

$$\#I(x^2, \mathbb{Z}_{2^t}) = \frac{2^{t-1}}{3} + \frac{(-1)^{t-1}}{6} + \frac{3}{2}, \quad t \geq 2. \tag{4}$$

Proposition 5 [Eichhorn et al. 2009].

$$\#I(x + x^{-1}, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{2} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}. \tag{5}$$

2. The formulas for $\#I((x + x^{-1})^2, \mathbb{Z}_{p^t})$

The central result of this paper is as follows.

Theorem 6. For $p = 2$ and $t \geq 7$,

$$\#I((x + x^{-1})^2, \mathbb{Z}_{2^t}) = \frac{2^{t-7}}{3} + \frac{(-1)^{t-1}}{6} + \frac{3}{2}. \quad (6)$$

If $p \equiv 1 \pmod{4}$ then

$$\#I((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{(p-5)p^{t-1}}{4} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}. \quad (7)$$

If $p \equiv 3 \pmod{4}$ then

$$\#I((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{4} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4}. \quad (8)$$

The proof occupies most of this section.

Proof of Theorem 6, case $p > 2$. We will use the squaring map modulo p^t :

$$Q : I(x + x^{-1}, \mathbb{Z}_{p^t}) \rightarrow I((x + x^{-1})^2, \mathbb{Z}_{p^t}), \quad Q(z) = z^2 \pmod{p^t}.$$

We note that it preserves coprimeness with p :

$$\begin{aligned} Q(I'(x + x^{-1}, \mathbb{Z}_{p^t})) &= I'((x + x^{-1})^2, \mathbb{Z}_{p^t}), \\ Q(I''(x + x^{-1}, \mathbb{Z}_{p^t})) &= I''((x + x^{-1})^2, \mathbb{Z}_{p^t}). \end{aligned}$$

Proposition 7. Let p be an odd prime. For any $a \in I'((x + x^{-1})^2, \mathbb{Z}_{p^t})$, we have $\#Q^{-1}(\{a\}) = 2$, and consequently

$$\#I'((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \#I'(x + x^{-1}, \mathbb{Z}_{p^t})/2. \quad (9)$$

Proof. Let a be an arbitrary element of $I'((x + x^{-1})^2, \mathbb{Z}_{p^t})$. There exists a point $(x_1, y_1) \in \mathcal{H}_{p^t}$ such that

$$(x_1 + y_1)^2 \equiv a \pmod{p^t}.$$

Since $\gcd(x_1 + y_1, p) = 1$,

$$x_1 + y_1 \not\equiv -(x_1 + y_1) \pmod{p^t};$$

hence the two distinct elements of $I'((x + x^{-1})^2, \mathbb{Z}_{p^t})$ that Q maps to a are

$$(x_1 + y_1) \pmod{p^t} \quad \text{and} \quad -(x_1 + y_1) \pmod{p^t}.$$

By Proposition 3, the congruence $x^2 \equiv a \pmod{p^t}$ has at most two solutions and we conclude that $\#Q^{-1}(\{a\}) = 2$. \square

Proposition 8.

$$\#I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \begin{cases} p^{t-1} & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (10)$$

Consequently, when $p \equiv 1 \pmod{4}$,

$$I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \{kp : k = 0, 1, \dots, p^{t-1} - 1\}.$$

Proof. Define $s_{p^t} : \mathcal{H}_{p^t} \rightarrow \mathbb{Z}_{p^t}$ by $s_{p^t}((x, y)) = (x + y) \pmod{p^t}$ and let

$$\mathcal{H}''_{p^t} = \{(x, y) : (x, y) \in \mathcal{H}_{p^t} \text{ with } s_{p^t}((x, y)) \in I''(x + x^{-1}, \mathbb{Z}_{p^t})\}.$$

If $(x, y) \in \mathcal{H}''_{p^t}$, then $x + y = 0 \pmod{p}$ and consequently $x^2 = -1 \pmod{p}$. Since -1 is a quadratic residue modulo p if and only if $p \equiv 1 \pmod{4}$, we obtain the second part of (10).

We now restrict our attention to primes p that are congruent to 1 modulo 4. Since $s_{p^t}(\mathcal{H}''_{p^t}) = I''(x + x^{-1}, \mathbb{Z}_{p^t})$, we prove the first part of (10) by proving the following two assertions:

- (i) $\#s_{p^t}^{-1}(\{a\}) = 2$ for any $a \in I''(x + x^{-1}, \mathbb{Z}_{p^t})$.
- (ii) $\#\mathcal{H}''_{p^t} = 2p^{t-1}$.

The proof of (i) is as follows. Let $(r, s) \in s_{p^t}^{-1}(\{a\})$. Then $(2r - a)$ and $(2s - a)$ are two distinct roots of the congruence

$$x^2 \equiv (a^2 - 4) \pmod{p^t}.$$

Since $p \mid a$, we have $\gcd(a^2 - 4, p) = 1$. Hence by Proposition 3

$$x^2 \equiv (a^2 - 4) \pmod{p^t}$$

cannot have more than two roots. Consequently $s_{p^t}^{-1}(\{a\}) = \{(r, s), (s, r)\}$.

We now prove (ii). Let (r, s) be an arbitrary element of \mathcal{H}''_{p^t} and let

$$r = d_0 + d_1p + d_2p^2 + \dots + d_{t-1}p^{t-1}$$

be the expansion of r in base p . There are only two possible choices for d_0 , specifically, the two roots of $x^2 \equiv -1 \pmod{p}$, and for each of the other d_i 's there are p possible choices: $0, 1, \dots, p - 1$. So there are $2p^{t-1}$ possible r 's. Since s is completely determined by the choice of r , we conclude that $\#\mathcal{H}''_{p^t} = 2p^{t-1}$. \square

Proposition 9. *If $p \equiv 1 \pmod{4}$ then*

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p - 1)}{4(p + 1)} + \frac{3}{4}. \tag{11}$$

Proof. By Proposition 8

$$I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \{kp : 0 \leq k \leq p^{t-1} - 1\}.$$

Consequently,

$$\begin{aligned} I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) &= Q(I''(x + x^{-1}, \mathbb{Z}_{p^t})) \\ &= Q(\{kp : 0 \leq k \leq p^{t-1} - 1\}) = \{j^2 \bmod p^t : p \mid j\}. \end{aligned}$$

Therefore,

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \#\{k^2 \bmod p^t\} - \#\{k^2 \bmod p^t : \gcd(k, p) = 1\}.$$

Combining Stangl's formula (3) with the standard result that the number of quadratic residues modulo p^t is $(p^t - p^{t-1})/2$, we obtain

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4},$$

which proves Proposition 9. □

We are now ready to prove formulas (7) and (8). We have

$$\begin{aligned} \#I((x + x^{-1})^2, \mathbb{Z}_{p^t}) &= \#I'((x + x^{-1})^2, \mathbb{Z}_{p^t}) + \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) \\ &= \frac{\#I'(x + x^{-1}, \mathbb{Z}_{p^t})}{2} + \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) \\ &= \frac{\#I(x + x^{-1}, \mathbb{Z}_{p^t})}{2} - \frac{\#I''(x + x^{-1}, \mathbb{Z}_{p^t})}{2} + \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}). \end{aligned}$$

Formula (5) is

$$\#I(x + x^{-1}, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{2} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}.$$

If $p \equiv 3 \pmod{4}$, then $\#I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = 0$ by (10). If $p \equiv 1 \pmod{4}$, then

$$\#I''(x + x^{-1}, \mathbb{Z}_{p^t}) = p^{t-1}$$

and

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4},$$

by (10) and (11). We complete the proof with simple algebraic computations. □

Proof of Theorem 6, case $p = 2$. Interestingly this was the most difficult and time consuming part. It was only through experimenting with Maple that we discovered the map f (defined below) that allowed us to prove the formula for powers of 2.

Proposition 10. *Let $t \geq 3$. The image of the map*

$$f : I(x^2, \mathbb{Z}_{2^t}) \rightarrow \{0, 1, \dots, 2^{t+6} - 1\}$$

given by

$$f(k^2) = (64k^2 + 4) \pmod{2^{t+6}}$$

is $I((x + x^{-1})^2, \mathbb{Z}_{2^{t+6}})$. Since f is injective we conclude that

$$\#I((x + x^{-1})^2, \mathbb{Z}_{2^{t+6}}) = \#I(x^2, \mathbb{Z}_{2^t}). \tag{12}$$

Proof. First we show that $I((x + x^{-1})^2, \mathbb{Z}_{2^{t+6}}) \subseteq \text{Image}(f)$. Let $(x, y) \in \mathcal{H}_{2^{t+6}}$. We can write

$$x = 8x_1 + a \quad \text{and} \quad y = 8y_1 + a,$$

with $0 \leq x_1, y_1 < 2^{t+3}$ and $a = 1, 3, 5$ or 7 . (We are using the fact that each element in \mathbb{Z}_8^* is its own inverse.) The following calculation now shows that $(x + y)^2 \pmod{2^{t+6}} \in \text{Image}(f)$.

$$\begin{aligned} (x + y)^2 &= (8x_1 + 8y_1 + 2a)^2 \\ &= 64x_1^2 - 128x_1y_1 + 64y_1^2 + 256x_1y_1 + 32x_1a + 32y_1a + 4a^2 \\ &= 64(x_1 - y_1)^2 + 4(64x_1y_1 + 8x_1a + 8y_1a + a^2) \\ &= 64(x_1 - y_1)^2 + 4xy \\ &\equiv (64(x_1 - y_1)^2 + 4) \pmod{2^{t+6}}. \end{aligned}$$

To show the reverse inclusion, let $k^2 \in I(x^2, \mathbb{Z}_{2^t})$. By Proposition 3 the congruence

$$x^2 \equiv 16k^2 + 1 \pmod{2^n}$$

has a solution for all values of n . Let l be any integer such that $l^2 = 16k^2 + 1 \pmod{2^{t+6}}$, and let

$$x = (l - 4k) \pmod{2^{t+6}}, \quad y = (l + 4k) \pmod{2^{t+6}}.$$

The immediate observations that $(x, y) \in \mathcal{H}_{2^{t+6}}$ and

$$(x + y)^2 \equiv 4l^2 \equiv 64k^2 + 4 \pmod{2^{t+6}}$$

complete the proof. □

Now the formula (6) for $\#I((x + x^{-1})^2, \mathbb{Z}_{2^t})$ is obtained by combining (2), (12) and (16). This concludes the proof of Theorem 6. □

We can also derive the formula for $\#I(x^2 + x^{-2}, \mathbb{Z}_p)$ as a special case of an old formula for pairs of quadratic residues.

Theorem 11 [Berndt et al. 1998, Theorem 6.3.1, page 197]. *Let p be an odd prime and let c be an integer relatively prime to p . Let $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$. Then*

$$\begin{aligned} \#\left\{n : 0 \leq n < p, \binom{n}{p} = \epsilon_1, \binom{n+c}{p} = \epsilon_2\right\} \\ = \frac{1}{4} \left\{ p - 2\epsilon_1 \binom{-c}{p} - \epsilon_2 \binom{c}{p} - \epsilon_1 \epsilon_2 \right\}. \tag{13} \end{aligned}$$

The special case of this formula with $\epsilon_1 = \epsilon_2 = c = 1$ was first published by Aladov in 1896. The connection between (13) and $\#I(x^2 + x^{-2}, \mathbb{Z}_p)$ is as follows.

Theorem 12. *Let $a \in \mathbb{Z}$ with $\gcd(a^2 - 4, n) = 1$. Then $\mathcal{C}_{a,n} \cap \mathcal{H}_n \neq \emptyset$ if and only if for every prime, p , in the canonical factorization of n we have*

$$\left(\frac{a-2}{p}\right) = \left(\frac{a+2}{p}\right) = 1. \tag{14}$$

Consequently,

$$\#I(x^2 + x^{-2}, \mathbb{Z}_p) = \#\left\{a : 0 \leq a < p, \left(\frac{a-2}{p}\right) = \left(\frac{a+2}{p}\right) = 1\right\} + 1.$$

Proof. For the “only if” part, let $(r, s) \in \mathcal{C}_{a,n} \cap \mathcal{H}_n$ and let p be an arbitrary prime divisor of n . So, $(r - s)^2 \equiv a - 2 \pmod{p}$ and $(r + s)^2 \equiv a + 2 \pmod{p}$, which leads immediately to (14).

To prove the converse, let $n = \prod_{i=1}^t p_i^{e_i}$ be the canonical factorization of n . By Proposition 3, we can lift the square roots (modulo p) of $(a - 2)$ and $(a + 2)$ to the e_i th power, $p_i^{e_i}$. Let $s_i \equiv \sqrt{a - 2} \pmod{p_i^{e_i}}$, and $r_i \equiv \sqrt{a + 2} \pmod{p_i^{e_i}}$. Then

$$2^{-1} \cdot (r_i + s_i, r_i - s_i) \in \mathcal{C}_{p_i^{e_i}} \cap \mathcal{H}_{p_i^{e_i}},$$

where 2^{-1} denotes the inverse of 2 modulo $p_i^{e_i}$. Now invoke the Chinese remainder theorem to determine integers r and s such that

$$r \equiv r_i \pmod{p_i^{e_i}} \text{ and } s \equiv s_i \pmod{p_i^{e_i}} \quad \text{for } i = 1, \dots, t.$$

Clearly $(r, s) \in \mathcal{C}_n \cap \mathcal{H}_n$. □

3. The formulas for $\#I(x^2 + y^2, \mathbb{Z}_{p^t}^2)$

We now determine the formulas for $\#I(x^2 + y^2, \mathbb{Z}_{p^t}^2)$ to contrast them to

$$\#I(x^2 + x^{-2}, \mathbb{Z}_{p^t}).$$

Theorem 13. *Let p be an odd prime. Then*

$$\#I(x^2 + y^2, \mathbb{Z}_{p^t}^2) = \begin{cases} p^t & \text{if } p \equiv 1 \pmod{4}, \\ p & \text{if } p \equiv 3 \pmod{4} \text{ and } t = 1, \\ p^t - \sum_{j=0}^{\lfloor t/2 \rfloor - 1} \varphi(p^{t-1-2j}) & \text{if } p \equiv 3 \pmod{4} \text{ and } t > 1, \end{cases} \tag{15}$$

When $p = 2$ we have

$$\#I(x^2 + y^2, \mathbb{Z}_{2^t}^2) = \varphi(2^t) + 1. \tag{16}$$

As is typically the case, the formula for powers of two, 2^t , will require a separate argument. We first prove (15).

Proof of formula (15). We treat each case separately.

- $p \equiv 1 \pmod{4}$. Let $a \in \{0, 1, \dots, p^t - 1\}$. The simultaneous congruences

$$x - y \equiv 1 \pmod{p^t} \quad \text{and} \quad x + y \equiv a \pmod{p^t}$$

have the solutions

$$\begin{aligned} x &= ((a + 1) \cdot (2^{-1} \pmod{p^t})) \pmod{p^t}, \\ y &= ((a - 1) \cdot (2^{-1} \pmod{p^t})) \pmod{p^t}. \end{aligned}$$

It immediately follows that $x^2 + (i_{p^t} y)^2 \equiv a \pmod{p^t}$, where

$$i_{p^t}^2 \equiv -1 \pmod{p^t}.$$

- $p \equiv 3 \pmod{4}$, $t = 1$. Let $a \in \{0, 1, \dots, p - 1\}$. By (3), $\#I(x^2, \mathbb{Z}_p) = (p + 1)/2$ and therefore $\#(a - I(x^2, \mathbb{Z}_p)) = (p + 1)/2$. Since

$$\#I(x^2, \mathbb{Z}_p) + \#(a - I(x^2, \mathbb{Z}_p)) = p + 1,$$

it follows that there is an element $(a - x_1^2) \in (a - I(x^2, \mathbb{Z}_p))$ and an element $x_2^2 \in I(x^2, \mathbb{Z}_p)$ such that $(a - x_1^2) \equiv x_2^2 \pmod{p}$.

- $p \equiv 3 \pmod{4}$, $t \geq 2$. The key is to prove that an element $a \in \{0, 1, 2, \dots, p^t - 1\}$ satisfies $a \equiv x^2 + y^2 \pmod{p^t}$ if and only if $a = p^k b$, with $\gcd(p, b) = 1$ and k even.

(\Leftarrow) Since p^k is a square in \mathbb{Z} , it is sufficient to prove this for integers a that are relatively prime to p . We argue by induction. The previous case shows that the result holds for $t = 1$. Let us assume it is true for t . So

$$a \equiv (x^2 + y^2) \pmod{p^t}.$$

If $p^{t+1} \mid (a - x^2 - y^2)$, there is nothing to prove. So let us assume that $(a - x^2 - y^2) = p^t l$, with $\gcd(l, p) = 1$. Since $\gcd(a, p) = 1$ either $\gcd(x, p) = 1$ or $\gcd(y, p) = 1$. Without loss of generality we assume the former. We now define $s \in \mathbb{Z}$, with $1 \leq s < p$, to be the solution of the congruence

$$2xs \equiv l \pmod{p}.$$

An immediate calculation shows that

$$a \equiv (x + sp^t)^2 + y^2 \pmod{p^{t+1}}.$$

(\Rightarrow) We argue by contradiction. Suppose $a = p^k b$, with $a < p^t$, $\gcd(b, p) = 1$, and k odd, be the sum of two squares modulo p^t . So there are integers $x = p^{e_1} x_1$, $y = p^{e_2} y_1$, with $\gcd(x_1 y_1, p) = 1$, such that

$$p^k b \equiv (x^2 + y^2) \pmod{p^t},$$

that is,

$$p^k b \equiv (p^{2e_1} x_1^2 + p^{2e_2} y_1^2) \pmod{p^t}.$$

Since $b \not\equiv 0 \pmod{p}$ and k is odd we have $\min\{2e_1, 2e_2\} < k$. Without loss of generality we may assume that $e_1 \leq e_2$. We can reduce the congruence

$$p^k b \equiv (x^2 + y^2) \pmod{p^t}$$

to $p^{k-2e_1} b \equiv x_1^2 + p^{2(e_2-e_1)} y_1^2 \pmod{p^{k-2e_1}}$, which in turns reduces to

$$x_1^2 + p^{2(e_2-e_2)} y_1^2 \equiv 0 \pmod{p}.$$

Since $x_1 \not\equiv 0 \pmod{p}$ we must have $p^{2(e_2-e_2)} y_1^2 \not\equiv 0 \pmod{p}$, that is $e_2 = e_1$, and consequently $(x_1^2 + y_1^2) \equiv 0 \pmod{p}$, with $\gcd(x_1 y_1, p) = 1$. But this gives us the contradiction that $x^2 \equiv -1 \pmod{p}$ is solvable for a prime p with $p \equiv 3 \pmod{4}$. This concludes the proof of (15). \square

Proposition 14. *Let $t \geq 3$ and $0 < m < 2^t$. Then $m \in I(x^2 + y^2, \mathbb{Z}_{2^t}^2)$ if and only if $m = 2^j \cdot a$, with $j < t$ and $a \equiv 1 \pmod{4}$.*

Proof. (\Leftarrow) Let $a \equiv 1 \pmod{4}$. Since 2^j is a sum of squares (in \mathbb{Z}) we only need to show that a is a sum of two squares modulo 2^t . If $a \equiv 1 \pmod{8}$ then a is a square modulo 2^t by Proposition 3. If $a \equiv 5 \pmod{8}$, then $a - 4 \equiv 1 \pmod{8}$ and is therefore a square modulo 2^t . Consequently a is a sum of two squares modulo 2^t .

(\Rightarrow) We now assume that $a \equiv 3 \pmod{4}$ and argue by contradiction. Let

$$x^2 + y^2 \equiv m \pmod{2^t}.$$

We look at four possible cases.

(1) $j = 0$: We obtain the contradiction that

$$x^2 + y^2 \equiv 3 \pmod{4}.$$

(2) $j = 1$: We obtain the contradiction that

$$x^2 + y^2 \equiv 6 \pmod{8}.$$

(3) $j \geq 2, j \leq (t - 2)$: We have $x = 2^{e_1} \cdot x_1$ and $y = 2^{e_2} \cdot y_1$, with x_1, y_1 odd and $j = \min\{2e_1, 2e_2\}$. Without loss of generality we may assume that $e_1 \leq e_2$. We now obtain the contradiction

$$x_1^2 + 4^{e_2-e_1} y_1^2 \equiv a \equiv 3 \pmod{4}.$$

(4) $j = t - 1$: Then

$$m = 2^{t-1} \cdot a \geq 2^{t-1} \cdot 3 > 2^t,$$

contradicting the fact that the elements of $I(x^2 + y^2, \mathbb{Z}_{2^t}^2)$ are less than 2^t . \square

Proof of formula (16). Let M_t denote the set

$$M_t = \{m : 0 < m < 2^t, m = 2^j \cdot a, j < t, a \equiv 1 \pmod{4}\}.$$

In our previous proposition we proved that

$$I(x^2 + y^2, \mathbb{Z}_{2^t}^2) \setminus \{0\} = M_t.$$

We now make the following two observations about elements in M_t :

- (i) If $m \in M_t$, then $(m + 2^t) \in M_{t+1}$ provided $m \neq 2^{t-1}$.
- (ii) If $m \in M_{t+1}$ with $m > 2^t$, then $(m - 2^t) \in M_t$.

From these two observations we conclude that

$$M_{t+1} \setminus \{2^t\} = M_t \cup \{m + 2^t : m \in M_t \setminus \{2^{t-1}\}\},$$

and consequently $\#M_{t+1} = 2 \cdot \#M_t$. An inductive argument now proves that $\#M_t = \varphi(2^t)$ and therefore $\#I(x^2 + y^2, \mathbb{Z}_{2^t}^2) = \varphi(2^t) + 1$. \square

References

- [Berndt et al. 1998] B. C. Berndt, R. J. Evans, and K. S. Williams, *Gauss and Jacobi sums*, CMS Monographs and Advanced Texts **21**, Wiley, New York, 1998. MR 99d:11092 Zbl 0906.11001
- [Eichhorn et al. 2009] D. Eichhorn, M. R. Khan, A. H. Stein, and C. L. Yankov, “Sums and differences of the coordinates of points on modular hyperbolas”, pp. 17–39 in *Combinatorial number theory* (Carrollton, GA, 2007), edited by B. Landman et al., de Gruyter, Berlin, 2009. Also published in *Integers: Electronic J. Combin. Number Theory*, **9** supplement (2009), article 3. MR 2010i:11149 Zbl 1178.11004
- [Ireland and Rosen 1982] K. F. Ireland and M. I. Rosen, *A classical introduction to modern number theory*, Grad. Texts in Math. **84**, Springer, New York, 1982. MR 83g:12001 Zbl 0482.10001
- [Shparlinski 2007] I. E. Shparlinski, “Distribution of points on modular hyperbolas”, pp. 155–189 in *Sailing on the sea of number theory: Proc. 4th China-Japan Seminar on Number Theory* (Weihai, 2006), edited by S. Kanemitsu and J.-Y. Liu, Ser. Number Theory Appl. **2**, World Sci. Publ., Hackensack, NJ, 2007. MR 2008m:11162 Zbl 1175.11044
- [Stangl 1996] W. D. Stangl, “Counting squares in Z_n ”, *Math. Mag.* **69**:4 (1996), 285–289. MR 1424442 Zbl 1055.11500

Received: 2009-02-08 Accepted: 2010-06-21

hanrahans@stu.easternct.edu *Department of Mathematics and Computer Science,
Eastern Connecticut State University,
Willimantic, CT 06226, United States*

khanm@easternct.edu *Department of Mathematics and Computer Science,
Eastern Connecticut State University,
Willimantic, CT 06226, United States*

involve

pjm.math.berkeley.edu/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Ken Ono	University of Wisconsin, USA ono@math.wisc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	Robert J. Plemmons	Wake Forest University, USA plemmons@wfu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Sat Gupta	U of North Carolina, Greensboro, USA sgupta@uncg.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Filip Saidak	U of North Carolina, Greensboro, USA f.saidak@uncg.edu
Karen Kafadar	University of Colorado, USA karen.kafadar@cudenver.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
David Larson	Texas A&M University, USA larson@math.tamu.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu

PRODUCTION

Silvio Levy, Scientific Editor

Sheila Newbery, Senior Production Editor

Cover design: ©2008 Alex Scorpan


See inside back cover or <http://pjm.math.berkeley.edu/involve> for submission instructions.

The subscription price for 2010 is US \$100/year for the electronic version, and \$120/year (+\$20 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**

<http://www.mathscipub.org>

A NON-PROFIT CORPORATION

Typeset in L^AT_EX

Copyright ©2010 by Mathematical Sciences Publishers

involve

2010

vol. 3

no. 2

Recursive sequences and polynomial congruences J. LARRY LEHMAN AND CHRISTOPHER TRIOLA	129
The Gram determinant for plane curves JÓZEF H. PRZYTYCKI AND XIAOQI ZHU	149
The cardinality of the value sets modulo n of $x^2 + x^{-2}$ and $x^2 + y^2$ SARA HANRAHAN AND MIZAN KHAN	171
Minimal k -rankings for prism graphs JUAN ORTIZ, ANDREW ZEMKE, HALA KING, DARREN NARAYAN AND MIRKO HORŇÁK	183
An unresolved analogue of the Littlewood Conjecture CLARICE FEROLITO	191
Mapping the discrete logarithm DANIEL CLOUTIER AND JOSHUA HOLDEN	197
Linear dependency for the difference in exponential regression INDIKA SATHISH AND DIAWARA NOROU	215
The probability of relatively prime polynomials in $\mathbb{Z}_{p^k}[x]$ THOMAS R. HAGEDORN AND JEFFREY HATLEY	223
\mathbb{G} -planar abelian groups ANDREA DEWITT, JILLIAN HAMILTON, ALYS RODRIGUEZ AND JENNIFER DANIEL	233



1944-4176(2010)3:2;1-F