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# The probability of relatively prime polynomials in $\mathbb{Z}_{p^k}[x]$

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Let  $P_R(m, n)$  denote the probability that two randomly chosen monic polynomials  $f, g \in R[x]$  of degrees  $m$  and  $n$ , respectively, are relatively prime. Let  $q = p^k$  be a prime power. We establish an explicit formula for  $P_R(m, 2)$  when  $R = \mathbb{Z}_q$ , the ring of integers mod  $q$ .

## 1. Introduction

Given two polynomials  $f(x), g(x)$  chosen at random, what is the probability that they are relatively prime? For a ring  $R$ , we say that two polynomials  $f, g \in R[x]$  are relatively prime if there is no monic polynomial of positive degree that divides both  $f$  and  $g$ . Let  $P_R(m, n)$  denote the probability that two randomly chosen monic polynomials  $f, g \in R[x]$  of degrees  $m$  and  $n$ , respectively, are relatively prime. If  $R$  has an infinite number of elements, then  $P_R(m, n) = 1$ , so we restrict our attention to finite rings  $R$ . Let  $R = \mathbb{F}_q$ , the finite field with  $q$  elements. The formula,  $P_{\mathbb{F}_q}(m, m) = 1 - 1/q$  was proved in [Cortee et al. 1998]. When  $q = p = 2$ , Reifergerste [2000] gave a combinatorial proof that  $P_{\mathbb{F}_2}(m, m) = 1/2$ . Benjamin and Bennett subsequently found a beautifully simple proof generalizing these results:

**Theorem 1.1** [Benjamin and Bennett 2007]. *If  $m, n \geq 1$ , then  $P_{\mathbb{F}_q}(m, n) = 1 - \frac{1}{q}$ .*

This can be generalized in at least two ways. Hou and Mullen [2009] have generalized Theorem 1.1 by considering the problem of relatively prime polynomials in several variables over a finite field. In earlier work, Gao and Panario [2006] considered the probability distribution of the greatest common divisor of  $l$  randomly chosen monic single-variable polynomials in  $\mathbb{F}_q[x]$  with degrees  $n_1, \dots, n_l$  as the  $n_i \rightarrow \infty$ . In this paper, we restrict ourselves to single-variable polynomials and explore a different perspective.

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As the formula in Theorem 1.1 only depends on the number of elements in the field  $\mathbb{F}_q$ , one can ask whether the same formula holds when  $R$  is another ring with  $q$  elements. For example, if  $R = \mathbb{Z}_q$ , the integers mod  $q$ , does the same formula hold? It does not, but the formula for  $P_{\mathbb{F}_q}(m, n)$  can be viewed as a first approximation to the formula for  $P_{\mathbb{Z}_q}(m, n)$ . In this paper, we prove an explicit formula for  $P_{\mathbb{Z}_{p^k}}(m, 2)$  for  $p$  odd.

For each positive integer  $k$ , we define a monic polynomial  $f_k(x) \in \frac{1}{2}\mathbb{Z}[x]$  by

$$f_k(x) = x^{2k} + (1-x) \sum_{i=0}^{(k-3)/2} x^{(k+3)/2+3i} + \frac{1}{2} \sum_{i=0}^{k-1} (-x)^i + \frac{1}{2}x^{(k-1)/2} - 1,$$

for  $k$  odd, and

$$f_k(x) = x^{2k} + (1-x) \sum_{i=1}^{k/2-1} x^{2k-3i} - \frac{1}{2} \sum_{i=1}^{k-1} (-x)^i - x^{k/2+1} + \frac{3}{2}x^{k/2} - 1,$$

for  $k$  even. The polynomial  $f_k(x)$  has degree  $2k$  and its coefficients have absolute value at most 2.

**Theorem 1.2.** *Let  $p$  be an odd prime and let  $m, k \geq 1$  be integers. The probability that two randomly chosen monic polynomials in  $\mathbb{Z}_{p^k}[x]$  of degrees  $m$  and  $2$ , respectively, are relatively prime is*

$$P_{\mathbb{Z}_{p^k}}(m, 2) = 1 - \frac{1}{p^{3k}} f_k(p).$$

When  $k = 1$ , we rediscover  $P_{\mathbb{F}_p}(m, 2) = 1 - 1/p$ . For small values of  $k$ , we have

$$\begin{aligned} P_{\mathbb{Z}_{p^2}}(m, 2) &= 1 - \frac{1}{p^2} + \frac{1}{p^4} - \frac{2}{p^5} + \frac{1}{p^6}, \\ P_{\mathbb{Z}_{p^3}}(m, 2) &= 1 - \frac{1}{p^3} + \frac{1}{p^5} - \frac{1}{p^6} - \frac{1}{2p^7} + \frac{1}{2p^9}, \\ P_{\mathbb{Z}_{p^4}}(m, 2) &= 1 - \frac{1}{p^4} + \frac{1}{p^6} - \frac{1}{p^7} + \frac{1}{2p^9} - \frac{1}{p^{10}} - \frac{1}{2p^{11}} + \frac{1}{p^{12}}. \end{aligned}$$

As an immediate corollary to Theorem 1.2, we obtain:

**Corollary 1.3.** *Given  $k \geq 1$ , there exists a monic polynomial*

$$g_k(x) = \sum a_i x^i \in \frac{1}{2}\mathbb{Z}[x]$$

*with degree  $2k - 2$  and  $|a_i| \leq 2$ , such that*

$$P_{\mathbb{Z}_{p^k}}(m, 2) = 1 - \frac{1}{p^k} + \frac{1}{p^{3k}} g_k(p) \quad \text{for all odd primes } p \text{ and all } m \geq 1.$$

We obtain Theorem 1.2 and its corollary by adapting the arguments of Benjamin and Bennett [2007], who proved Theorem 1.1 by a clever use of the Euclidean algorithm in  $\mathbb{F}_q[x]$ . While  $\mathbb{Z}_{p^k}[x]$  does not have the Euclidean algorithm, due to the existence of noninvertible elements in  $\mathbb{Z}_{p^k}$ , it does have a division algorithm for monic polynomials. This division algorithm, together with some facts about polynomial factorization of quadratics in  $\mathbb{Z}_{p^k}[x]$ , suffices to prove Theorem 1.2 for odd primes  $p$ . It appears that our arguments can also be used to prove the formula for  $P_{\mathbb{Z}_{p^k}}(m, 2)$  when  $p = 2$ , and also a formula for  $P_{\mathbb{Z}_{p^k}}(m, 3)$ , but the details are much more involved and have not yet been fully worked through. However, the present approach does not seem able to establish a formula for  $P_{\mathbb{Z}_{p^k}}(m, n)$  for general  $m, n \geq 4$  as the number of cases to consider in the proof grows as a function of  $\min(m, n)$ .

## 2. Arithmetic in $\mathbb{Z}_{p^k}[x]$

In this section, we establish some basic results on the rings  $\mathbb{Z}_{p^k}$  and  $\mathbb{Z}_{p^k}[x]$ . Recall that  $\mathbb{Z}_n$  denotes the ring of integers mod  $n$ . We will make use of Hensel's lemma [Gouvêa 1997, page 70] in the following form:

**Lemma 2.1** (Hensel's lemma). *Let  $f(x) \in \mathbb{Z}_{p^k}[x]$  be a polynomial and denote its reduction mod  $p$  by  $\bar{f}(x) \in \mathbb{Z}_p[x]$ . Suppose there exists  $u_0 \in \mathbb{Z}_p$  with  $\bar{f}(u_0) = 0$  in  $\mathbb{Z}_p$  and  $\bar{f}'(u_0) \neq 0$  in  $\mathbb{Z}_p$ . Then there exists a unique  $u \in \mathbb{Z}_{p^k}$ , with  $f(u) = 0$  in  $\mathbb{Z}_{p^k}$  and  $u \equiv u_0 \pmod{p}$ .*

We start by counting the squares in  $\mathbb{Z}_{p^k}$  and its unit subgroup  $\mathbb{Z}_{p^k}^*$ .

**Lemma 2.2.** *Let  $p$  be an odd prime and  $k \geq 1$ .*

- (a)  $\mathbb{Z}_{p^k}^*$  has  $\frac{1}{2}p^{k-1}(p-1)$  squares.
- (b) Let  $d$  be even, with  $0 \leq d < k$ . There are  $\frac{1}{2}(p-1)p^{k-1-d}$  nonzero squares  $x \in \mathbb{Z}_{p^k}$  with  $x \in p^d\mathbb{Z}_{p^k} \setminus p^{d+1}\mathbb{Z}_{p^k}$ .
- (c) There are  $1 + \frac{1}{2(p+1)}(p^{k+1} - p^{1-k+2\lfloor k/2 \rfloor})$  squares in  $\mathbb{Z}_{p^k}$ .

*Proof.* (a) We first note that the  $(p-1)/2$  squares  $x = 1^2, \dots, (\frac{p-1}{2})^2$  are distinct nonzero squares in both  $\mathbb{Z}_p$  and  $\mathbb{Z}_{p^k}$ . Now consider a unit  $u \in \mathbb{Z}_{p^k}$  satisfying  $u \equiv 1 \pmod{p}$ . Letting  $f(x) = x^2 - u \in \mathbb{Z}_{p^k}[x]$ , and  $u_0 = 1$ , by Lemma 2.1,  $u$  is a square in  $\mathbb{Z}_{p^k}$ . Thus the  $p^{k-1}$  units  $u \in \mathbb{Z}_{p^k}$  with  $u \equiv 1 \pmod{p}$  are squares. Hence, the  $\frac{1}{2}p^{k-1}(p-1)$  distinct units  $xu$  are all squares and every unit square can be seen to be of this form.

(b) Let  $x \in \mathbb{Z}_{p^k}$  satisfy  $x \in p^d\mathbb{Z}_{p^k} \setminus p^{d+1}\mathbb{Z}_{p^k}$ . Let  $x = (p^t u)^2 = p^{2t} u^2$ , where  $u$  is a unit. To satisfy the given conditions,  $t = d/2$ ,  $u^2$  is a unit square in  $\mathbb{Z}_{p^k}^*$ ,

and  $u^2 \equiv u_1^2 \pmod{p^{k-d}}$ . Hence, the number of distinct  $x$  equals the number of unit squares in  $\mathbb{Z}_{p^{k-d}}$ , which is given by (a).

(c) Every nonzero square can be written as  $p^{2d}u$ , where  $u$  is a unit square and  $0 \leq 2d < n$ . Counting the square 0, the total sum is, thanks to (b),

$$1 + \frac{1}{2}(p-1) \sum_{d=0}^{\lfloor (k-1)/2 \rfloor} p^{k-1-2d}.$$

This expression simplifies to the claimed formula. □

For  $g(x) = x^2 + bx + c \in \mathbb{Z}_{p^k}[x]$ , define the discriminant  $\Delta_g = b^2 - 4c$ . As when  $k = 1$ , we can describe the number of roots of  $g(x) \in \mathbb{Z}_{p^k}[x]$  using  $\Delta_g$ .

**Lemma 2.3.** *Let  $p$  be an odd prime,  $k \geq 1$ , and  $g(x) = x^2 + bx + c \in \mathbb{Z}_{p^k}[x]$ .*

- (a)  $\Delta$  is a square mod  $p^k$  if and only if  $g$  is reducible.
- (b) If  $\Delta \equiv 0 \pmod{p^k}$ , then  $g$  has the  $p^{\lfloor k/2 \rfloor}$  roots given by  $\frac{-b}{2} + p^{\lfloor (k+1)/2 \rfloor}t \pmod{p^k}$ , where  $t = 1, \dots, p^{\lfloor k/2 \rfloor}$ .
- (c) Suppose  $\Delta \equiv p^d u \pmod{p^k}$  is a nonzero square with  $0 \leq d < k$ ,  $d$  even,  $u \in \mathbb{Z}_{p^k}^*$  a square. Choose  $a$  such that  $u \equiv a^2 \pmod{p^k}$ . Then  $g$  has the  $2p^{d/2}$  roots

$$-\frac{1}{2}b \pm \frac{1}{2}ap^{d/2} + tp^{k-d/2} \pmod{p^k}, \quad \text{where } t = 1, \dots, p^{d/2}.$$

*Proof.* Since  $p$  is odd, we have  $g(x) = (x + b/2)^2 - \Delta/4$ . Hence  $r = -(b+z)/2$  is a root of  $g(x)$  if and only if  $z$  is a solution of the equation  $z^2 \equiv \Delta \pmod{p^k}$ . Condition (a) is thus proved. Condition (b) follows as well as the roots of the equation  $z^2 \equiv 0 \pmod{p^k}$  are  $z \equiv p^{\lfloor (k+1)/2 \rfloor}t \pmod{p^k}$ , for  $t = 1, \dots, p^{\lfloor k/2 \rfloor}$ , or equivalently,  $z \equiv 2p^{\lfloor (k+1)/2 \rfloor}t \pmod{p^k}$ , for  $t = 1, \dots, p^{\lfloor k/2 \rfloor}$ . (c) By the hypothesis,  $d$  is even and  $a \not\equiv 0 \pmod{p}$ . The solutions to the equation  $z^2 \equiv p^d a^2 \pmod{p^k}$  have the form  $z \equiv p^{d/2}w \pmod{p^k}$ , where  $w \in \mathbb{Z}_{p^k}$  is a solution of  $x^2 \equiv a^2 \pmod{p^{k-d}}$ . Hensel's lemma (using the polynomial  $f(x) = x^2 - a^2$ ), shows that the solutions to this latter equation are the  $w \in \mathbb{Z}_{p^k}$  satisfying  $w \equiv \pm a \pmod{p^{k-d}}$ . Thus  $w = \pm a + tp^{k-d}$ , for  $t = 1, \dots, p^d$ , or equivalently, as 2 is a unit mod  $p^d$ ,  $w = \pm a + 2tp^{k-d}$  for  $t = 1, \dots, p^d$ . Now two roots  $z = p^{d/2}w$  and  $z_1 = p^{d/2}w_1$  are equal precisely when the signs in the expressions for  $w$  and  $w_1$  agree and the respective parameters  $t$  and  $t_1$  satisfy  $t \equiv t_1 \pmod{p^{d/2}}$ . Hence we have shown that the original equation  $z^2 \equiv p^d a^2 \pmod{p^k}$  has the  $2p^{d/2}$  distinct roots given by  $z = \pm ap^{d/2} + 2tp^{k-d/2}$ , for  $t = 1, \dots, p^{d/2}$ . □

**Lemma 2.4.** *Let  $p$  be an odd prime and  $k \geq 1$ .*

- (a) Given  $\Delta \in \mathbb{Z}_{p^k}$ , there are  $p^k$  monic, quadratic polynomials  $g \in \mathbb{Z}_{p^k}[x]$  with  $\Delta_g \equiv \Delta \pmod{p^k}$ .

(b) *There are*

$$\frac{p^k}{2(p+1)}(p^{k+1} + 2p^k - p - p^{k-2\lfloor k/2\rfloor} - 1)$$

*monic, irreducible, quadratic polynomials  $g \in \mathbb{Z}_{p^k}[x]$ .*

*Proof.* If  $g = x^2 + bx + c$ , then  $\Delta_g = b^2 - 4c$ . Since 4 is invertible mod  $p^k$ , for every  $\Delta$ ,  $b \in \mathbb{Z}_{p^k}$ , there is a unique choice of  $c$  such that  $\Delta_g \equiv \Delta \pmod{p^k}$ . Since there are  $p^k$  choices for  $b$ , (a) is proved. Now  $g$  is irreducible precisely when  $\Delta_g$  is not a square. Let  $S$  be the number of squares in  $\mathbb{Z}_{p^k}$ . Then for each  $b \in \mathbb{Z}_{p^k}$ , there are  $p^k - S$  choices for  $c$  such that  $b^2 - 4c$  is not a square. Thus, using the formula for  $S$  given by Lemma 2.2(c), there are

$$p^k(p^k - S) = \frac{p^k}{2(p+1)}(p^{k+1} + 2p^k - 2p + p^{1-k+2\lfloor k/2\rfloor} - 2)$$

irreducible polynomials  $g$ . Simplification gives (b). □

Given a monic, quadratic polynomial  $g \in \mathbb{Z}_{p^k}[x]$ , we define the set

$$A_g = \{h \in \mathbb{Z}_{p^k}[x] : \deg h \leq 1 \text{ and } g, h \text{ are not relatively prime}\},$$

and let  $|A_g|$  denote its cardinality. We note that in the definition of  $A_g$ , we allow nonmonic polynomials  $h$ .

**Lemma 2.5.** *Let  $p$  be an odd prime and  $g(x)$  be a monic quadratic polynomial in  $\mathbb{Z}_{p^k}[x]$ .*

(a) *If  $\Delta_g \equiv 0 \pmod{p^k}$ , then*

$$|A_g| = p^{k-\lfloor k/2\rfloor} \left( \frac{p^{2\lfloor k/2\rfloor+1} + 1}{p+1} \right).$$

(b) *Assume  $\Delta_g \in \mathbb{Z}_{p^k}$  is a nonzero square. Let  $\Delta_g \equiv p^d v \pmod{p^k}$ , where  $d$  is even,  $0 \leq d < k$ , and  $v \in (\mathbb{Z}_{p^k}^*)^2$ . Then*

$$|A_g| = 2p^{k-d/2} \left( \frac{p^{d+1} + 1}{p+1} \right) - p^{d/2}.$$

*Proof.* We first note that a linear factor of  $g(x)$  must have the form  $u(x-r)$ , where  $u, r \in \mathbb{Z}_{p^k}$ ,  $u$  is a unit, and  $r$  is a root of  $g$ . Therefore, the elements  $h(x) \in A_g$  are exactly the polynomials  $h(x) = \alpha(x-r)$ , for some  $\alpha \in \mathbb{Z}_{p^k}$  and some root  $r \in \mathbb{Z}_{p^k}$  of  $g$ . Hence, to calculate  $|A_g|$ , we need to count the number of distinct  $h(x)$  of this form.

Suppose  $r_1$  and  $r_2$  are two roots of  $g$  and  $\alpha(x-r_1) \equiv \beta(x-r_2) \pmod{p^k}$ . Then  $\beta \equiv \alpha \pmod{p^k}$  and  $\alpha(r_1-r_2) \equiv 0 \pmod{p^k}$ . Let  $\alpha = p^s u$ , with  $u \in \mathbb{Z}_{p^k}^*$ . If  $s = k$ , then  $\alpha = 0$  is the only choice. Now suppose  $s < k$ . Then there are  $p^{k-s-1}(p-1)$  distinct choices for  $u$  giving rise to distinct  $\alpha$ . For each such  $\alpha$ , we need to calculate the

number of roots of  $g$  in  $\mathbb{Z}_{p^{k-s}}$ . To proceed further, we need to have a description of the roots.

Writing  $g(x) = x^2 + bx + c$ , in case (a), the roots of  $g$  are  $r = -b/2 + p^{[(k+1)/2]}t$ , for  $t = 1, \dots, p^{[k/2]}$  by Lemma 2.3. If  $[k/2] \leq s < k$ , for each choice of  $\alpha = p^s u$ , there is exactly one factor  $\alpha(x - r) \pmod{p^k}$ . As there are  $p^{k-s-1}(p - 1)$  choices for  $u$ , and hence  $\alpha$ , we obtain the same number of distinct factors  $\alpha(x - r)$  for each  $s$ . If  $0 \leq s \leq [k/2]$ , then for each choice of  $\alpha = p^s u$ , there are  $p^{[k/2]-s}$  distinct factors  $\alpha(x - r) \pmod{p^k}$ . Hence there are  $p^{k+[k/2]-2s-1}(p - 1)$  distinct factors  $\alpha(x - r) \pmod{p^k}$  for each  $s$ . In total then, we have

$$\begin{aligned} |A_g| &= \sum_{s=0}^{[k/2]} (p - 1)p^{k+[k/2]-2s-1} + \left( \sum_{s=[k/2]+1}^{k-1} (p - 1)p^{k-s-1} + 1 \right) \\ &= \sum_{s=0}^{[k/2]} (p - 1)p^{k+[k/2]-2s-1} + p^{k-[k/2]-1} = p^{k-[k/2]} \left( \frac{p^{2[k/2]+1} + 1}{p + 1} \right), \end{aligned}$$

where the last equality is obtained by evaluating a geometric sum. We thus obtain the desired formula for case (a). In case (b), by Lemma 2.3, the roots of  $g$  are  $-\frac{1}{2}b \pm \frac{1}{2}ap^{d/2} + tp^{k-d/2} \pmod{p^k}$ , where  $a^2 \equiv v \pmod{p^k}$ ,  $t = 1, \dots, p^{d/2}$ . As in case (a), we let  $\alpha = p^s u$ , and consider the number of distinct factors  $h(x) = \alpha(x - r)$  for each choice of  $s$ . When  $s = k$ ,  $h(x) = \alpha = 0$  is the only factor. There are three additional cases:

- (1) Suppose  $k > s \geq k - d/2$ . Then  $k - s \leq d/2$  and all the roots of  $g$  are equivalent mod  $p^{k-s}$ . Since there are  $p^{k-s-1}(p - 1)$  distinct choices for  $\alpha$ , there are the same number of distinct factors  $\alpha(x - r)$ .
- (2) Suppose  $k - d/2 > s \geq d/2$ . Then  $d/2 < k - s \leq k - d/2$  and the roots of  $g$  determine two equivalence classes mod  $p^{k-s}$ . Thus for each  $s$ , there are a total of  $2p^{k-s-1}(p - 1)$  distinct factors  $\alpha(x - r)$ .
- (3) Suppose  $d/2 \geq s \geq 0$ . Then the roots of  $g$  determine  $2p^{d/2-s}$  equivalence classes mod  $p^{k-s}$  for each  $\alpha$ . Thus there are a total of  $2p^{k+d/2-2s-1}(p - 1)$  distinct factors  $\alpha(x - r)$ , for each  $s$ .

In total, when  $d < k - 1$ , we have for  $|A_g|$  the value

$$\begin{aligned} \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + \left( \sum_{s=d/2+1}^{k-d/2-1} 2(p-1)p^{k-s-1} + \sum_{s=k-d/2}^{k-1} (p-1)p^{k-s-1} + 1 \right) \\ = \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + 2p^{k-d/2-1} - p^{d/2}, \end{aligned}$$

which simplifies to the formula stated in (b). When  $d = k - 1$ , the second summation does not appear, and

$$\begin{aligned} |A_g| &= \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + \left( \sum_{s=k-d/2}^{k-1} (p-1)p^{k-s-1} + 1 \right) \\ &= \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + p^{d/2}, \end{aligned}$$

which again simplifies to the stated formula for (b). □

### 3. Proof of the main theorem

In this section, we let  $q = p^k$ . To prove Theorem 1.2, we will count the number of polynomial pairs  $(f, g)$ , where  $f, g \in \mathbb{Z}_q[x]$  are not relatively prime. Let  $f(x), g(x)$  be monic polynomials. Then by the division algorithm, there is a unique choice of polynomials  $q(x), r(x) \in \mathbb{Z}_q[x]$ , with  $q(x)$  monic, satisfying

$$f(x) = g(x)q(x) + r(x), \tag{1}$$

where  $r(x) = 0$  or  $\deg r(x) < \deg g(x)$ . Thus the pair  $(f, g)$  is uniquely determined by the triple  $(g, q(x), r(x))$ . From (1), any common divisor of  $f$  and  $g$  is a common divisor of  $g$  and  $r$  and vice-versa. We define

$$\begin{aligned} S_{m,d,q} &= \{(f, g) : f, g \in \mathbb{Z}_q[x] \text{ monic with } \deg f = m, \deg g = d, \\ &\quad f \text{ and } g \text{ not relatively prime}\}, \\ T_{m,q} &= \{(g, r) : g, r \in \mathbb{Z}_q[x] \text{ with } g \text{ monic of degree } m, \deg r < m, \\ &\quad g \text{ and } r \text{ not relatively prime}\}. \end{aligned}$$

**Lemma 3.1.** *If  $m \geq d$ , then  $|S_{m,d,q}| = q^{m-d}|T_{d,q}|$ .*

*Proof.* Let  $(g, r) \in T_{d,q}$ . Then each of the  $q^{m-d}$  monic polynomials  $q(x)$  with degree  $m - d$  gives rise via (1) to a unique pair  $(f, g) \in S_{m,d,q}$ . Conversely, the inverse map

$$(f, g) \mapsto (g, q, r) \mapsto (g, r)$$

is a  $q^{m-d}$ -to-1 map from  $S_{m,d,q}$  to  $T_{d,q}$ . □

Thus, proving Theorem 1.2 is reduced to calculating  $|T_{2,q}|$ . We begin with:

**Proposition 3.2.**  $|T_{1,q}| = q$ .

*Proof.* If  $(g, r) \in T_{1,q}$ , then  $g(x) = x - c$ . For  $g$  and  $r$  to have a common factor,  $r = 0$ . Hence  $T_{1,q}$  consists of the  $q$  pairs  $(x - c, 0)$ . □



We now determine  $|T_{2,q}|$ . By Lemma 2.3, we have  $|T_{2,q}| = B_1 + B_2 + B_3$ , where the  $B_i$  are defined by

$$B_1 = |\{(g, r) \in T_{2,q} : g \text{ is irreducible}\}|,$$

$$B_2 = |\{(g, r) \in T_{2,q} : \Delta_g \equiv 0 \pmod{p^k}\}|,$$

$$B_3 = |\{(g, r) \in T_{2,q} : \Delta_g \pmod{p^k} \text{ is a square, and, for each } d < k, \\ \Delta_g \equiv 0 \pmod{p^d} \text{ and } \Delta_g \not\equiv 0 \pmod{p^{d+1}}\}|.$$

**Lemma 3.3.** (a)  $B_1 = \frac{p^k}{2(p+1)}(p^{k+1} + 2p^k - p - p^{k-2\lfloor k/2\rfloor} - 1)$ .

(b)  $B_2 = p^{2k-\lfloor k/2\rfloor} \left( \frac{p^{2\lfloor k/2\rfloor+1} + 1}{p+1} \right)$ .

(c)  $B_3 = \frac{p^{2k-1-\lfloor (k-1)/2\rfloor} - p^{2k}}{2(p+1)(p^2+p+1)} \alpha$ , where

$$\alpha = (p+1)(p^2+p+1) - 2p^{k+1}(p+1)^2 - 2p^{k-\lfloor (k-1)/2\rfloor}(p+p^{-\lfloor (k-1)/2\rfloor}).$$

*Proof.* (a) Assume  $g \in \mathbb{Z}_{p^k}[x]$  is a monic, irreducible, quadratic polynomial. Since  $g$  has no factors,  $(g, r) \in T_{2,q}$  only when  $r = 0$ . Hence,  $B_1$  equals the number of monic, irreducible quadratic polynomials, which is given by Lemma 2.4.

(b) Assume  $g \in \mathbb{Z}_{p^k}[x]$  is a monic quadratic with  $\Delta_g \equiv 0 \pmod{p^k}$ . By Lemma 2.4, there are  $p^k$  such  $g$ . For each  $g$ ,  $|A_g|$  is given by Lemma 2.5(a). Thus

$$B_2 = p^k |A_g|.$$

(c) If  $(g, r) \in T_{2,q}$  is included in the pairs counted for  $B_3$ , then  $\Delta_g = p^d u$ , where  $0 \leq d < k$ ,  $d$  even, and  $u \in \mathbb{Z}_{p^k}^*$  is a square. For a fixed  $d$ ,  $u$ , satisfying these conditions, there are  $p^k$  polynomials  $g$  with  $\Delta_g = p^d u$  by Lemma 2.4(a). And for any such  $g$ ,  $|A_g|$  is given by Lemma 2.5(b). Now, for a fixed  $d$ , there are

$$\frac{1}{2}(p-1)p^{k-d-1}$$

choices for  $u$  that give distinct values for  $p^d u$ . Putting these results together, and replacing  $d$  by  $2d$ , we have

$$\begin{aligned} B_3 &= \sum_{d=0}^{\lfloor (k-1)/2\rfloor} \frac{1}{2}(p-1)p^{2k-d-1} \left( 2p^{k-2d} \left( \frac{p^{2d+1} + 1}{p+1} \right) - 1 \right) \\ &= \frac{p^{2k-1}(p-1)}{2(p+1)} \sum_{d=0}^{\lfloor (k-1)/2\rfloor} p^{-d} (2p^{k-2d}(p^{2d+1} + 1) - p - 1). \end{aligned} \quad (2)$$

Summing the geometric sequences, we have

$$\sum_{d=0}^{[(k-1)/2]} p^{-d}(-p-1) = -(p+1)p^{-[(k-1)/2]} \left( \frac{p^{[(k-1)/2]+1} - 1}{p-1} \right),$$

$$\sum_{d=0}^{[(k-1)/2]} p^{-d}(2p^{k-2d}(p^{2d+1} + 1)) = 2p^{k-[(k-1)/2]+1} \left( \frac{p^{[(k-1)/2]+1} - 1}{p-1} \right) + 2p^{k-3[(k-1)/2]} \left( \frac{p^{3[(k-1)/2]+3} - 1}{p^3 - 1} \right).$$

Substituting these equations in (2) and simplifying with the help of a computer algebra system, we obtain the desired expression.  $\square$

*Proof of Theorem 1.2.* There are  $q^m$  monic polynomials in  $\mathbb{Z}_q[x]$  with degree  $m$ . Hence there are  $q^{m+2}$  pairs of monic polynomials  $(f, g)$  with  $\deg f = m, \deg g = 2$ . By Lemma 3.1, the probability that a pair of these polynomials is relatively prime is

$$1 - \frac{|S_{m,2,q}|}{q^{m+2}} = 1 - \frac{|T_{2,q}|}{q^4}.$$

Now  $|T_{2,q}| = B_1 + B_2 + B_3$ , with the values of  $B_i$  given by Lemma 2.5. Manipulating this expression with the help of a computer algebra system, one obtains

$$|T_{2,q}| = \frac{p^k}{2(p+1)} D,$$

where  $D$  equals the expression

$$2p^{2k+1} + 2p^{2+k/2}(p-1) \left( \frac{p^{3k/2} - 1}{p^3 - 1} \right) + p^{1+k/2} + 3p^{k/2} + p^k - p - 2$$

when  $k$  is even, and  $D$  equals

$$2p^{2k+1} + 2(p-1) \left( \frac{p^{2(k+1)} - p^{(k+1)/2}}{p^3 - 1} \right) + 3p^{(k+1)/2} + p^{(k-1)/2} + p^k - 2p - 1,$$

when  $k$  is odd. When  $k$  is even, algebraic manipulation shows

$$2p^{2k+1} = 2(p+1)p^{2k} - 2p^{2k},$$

$$2p^{2+k/2}(p-1) \left( \frac{p^{3k/2} - 1}{p^3 - 1} \right) = 2p^{2k} - 2p^{2+k/2} + 2(1-p^2) \sum_{i=1}^{k/2-1} p^{2k-3i},$$

$$p^{1+k/2} + 3p^{k/2} = (p+1)(-2p^{1+k/2} + 3p^{k/2}) + 2p^{2+k/2},$$

$$p^k - p - 2 = -(p+1) \sum_{i=1}^{k-1} (-p)^i - 2(p+1).$$

Adding both sides, the left hand side sums to  $D$ . With  $f_k(x)$  defined as in the introduction, we then have

$$\frac{1}{2(p+1)}D = f_k(p).$$

Theorem 1.2 follows immediately for  $k$  even. Similar calculations establish it for  $k$  odd.  $\square$

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
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