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The probability of relatively prime polynomials in $\mathbb{Z}_{p^k}[x]$

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Let $\mathrm{P}_R(m, n)$ denote the probability that two randomly chosen monic polynomials $f, g \in R[x]$ of degrees m and n, respectively, are relatively prime. Let $q = p^k$ be a prime power. We establish an explicit formula for $\mathrm{P}_R(m, 2)$ when $R = \mathbb{Z}_q$, the ring of integers mod q.

1. Introduction

Given two polynomials $f(x)$, $g(x)$ chosen at random, what is the probability that they are relatively prime? For a ring R, we say that two polynomials $f, g \in R[x]$ are relatively prime if there is no monic polynomial of positive degree that divides both f and g. Let $\mathrm{P}_R(m, n)$ denote the probability that two randomly chosen monic polynomials f, $g \in R[x]$ of degrees m and n, respectively, are relatively prime. If R has an infinite number of elements, then $\mathrm{P}_R(m, n) = 1$, so we restrict our attention to finite rings R. Let $R = \mathbb{F}_q$, the finite field with q elements. The formula, $\mathrm{P}_{\mathbb{F}_q}(m, m) = 1 - 1/q$ was proved in [Corteel et al. 1998]. When $q = p = 2$, Reifegerste [2000] gave a combinatorial proof that $\mathrm{P}_{\mathbb{F}_2}(m, m) = 1/2$. Benjamin and Bennett subsequently found a beautifully simple proof generalizing these results:

Theorem 1.1 [Benjamin and Bennett 2007]. *If* $m, n \geq 1$, *then* $\mathrm{P}_{\mathbb{F}_q}(m, n) = 1 - \dfrac{1}{q}$.

This can be generalized in at least two ways. Hou and Mullen [2009] have generalized Theorem 1.1 by considering the problem of relatively prime polynomials in several variables over a finite field. In earlier work, Gao and Panario [2006] considered the probability distribution of the greatest common divisor of l randomly chosen monic single-variable polynomials in $\mathbb{F}_q[x]$ with degrees $n_1, \ldots, n_l$ as the $n_i \to \infty$. In this paper, we restrict ourselves to single-variable polynomials and explore a different perspective.

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As the formula in Theorem 1.1 only depends on the number of elements in the field $\mathbb{F}_q$, one can ask whether the same formula holds when R is another ring with q elements. For example, if $R = \mathbb{Z}_q$, the integers mod q, does the same formula hold? It does not, but the formula for $\mathrm{P}_{\mathbb{F}_q}(m,n)$ can be viewed as a first approximation to the formula for $\mathrm{P}_{\mathbb{Z}_q}(m,n)$. In this paper, we prove an explicit formula for $\mathrm{P}_{\mathbb{Z}_{p^k}}(m,2)$ for p odd.

For each positive integer k, we define a monic polynomial $f_k(x) \in \frac{1}{2}\mathbb{Z}[x]$ by

$$f_k(x) = x^{2k} + (1-x)\sum_{i=0}^{(k-3)/2} x^{(k+3)/2+3i} + \frac{1}{2}\sum_{i=0}^{k-1}(-x)^i + \tfrac{1}{2}x^{(k-1)/2} - 1,$$

for k odd, and

$$f_k(x) = x^{2k} + (1-x)\sum_{i=1}^{k/2-1} x^{2k-3i} - \frac{1}{2}\sum_{i=1}^{k-1}(-x)^i - x^{k/2+1} + \tfrac{3}{2}x^{k/2} - 1,$$

for k even. The polynomial $f_k(x)$ has degree $2k$ and its coefficients have absolute value at most 2.

Theorem 1.2. *Let p be an odd prime and let $m, k \geq 1$ be integers. The probability that two randomly chosen monic polynomials in $\mathbb{Z}_{p^k}[x]$ of degrees m and 2, respectively, are relatively prime is*

$$\mathrm{P}_{\mathbb{Z}_{p^k}}(m,2) = 1 - \frac{1}{p^{3k}} f_k(p).$$

When $k = 1$, we rediscover $\mathrm{P}_{\mathbb{F}_p}(m,2) = 1 - 1/p$. For small values of k, we have

$$\mathrm{P}_{\mathbb{Z}_{p^2}}(m,2) = 1 - \frac{1}{p^2} + \frac{1}{p^4} - \frac{2}{p^5} + \frac{1}{p^6},$$
$$\mathrm{P}_{\mathbb{Z}_{p^3}}(m,2) = 1 - \frac{1}{p^3} + \frac{1}{p^5} - \frac{1}{p^6} - \frac{1}{2p^7} + \frac{1}{2p^9},$$
$$\mathrm{P}_{\mathbb{Z}_{p^4}}(m,2) = 1 - \frac{1}{p^4} + \frac{1}{p^6} - \frac{1}{p^7} + \frac{1}{2p^9} - \frac{1}{p^{10}} - \frac{1}{2p^{11}} + \frac{1}{p^{12}}.$$

As an immediate corollary to Theorem 1.2, we obtain:

Corollary 1.3. *Given $k \geq 1$, there exists a monic polynomial*

$$g_k(x) = \sum a_i x^i \in \tfrac{1}{2}\mathbb{Z}[x]$$

with degree $2k-2$ and $|a_i| \leq 2$, such that

$$\mathrm{P}_{\mathbb{Z}_{p^k}}(m,2) = 1 - \frac{1}{p^k} + \frac{1}{p^{3k}} g_k(p) \quad \textit{for all odd primes } p \textit{ and all } m \geq 1.$$

We obtain Theorem 1.2 and its corollary by adapting the arguments of Benjamin and Bennett [2007], who proved Theorem 1.1 by a clever use of the Euclidean algorithm in $\mathbb{F}_q[x]$. While $\mathbb{Z}_{p^k}[x]$ does not have the Euclidean algorithm, due to the existence of noninvertible elements in $\mathbb{Z}_{p^k}$, it does have a division algorithm for monic polynomials. This division algorithm, together with some facts about polynomial factorization of quadratics in $\mathbb{Z}_{p^k}[x]$, suffices to prove Theorem 1.2 for odd primes p. It appears that our arguments can also be used to prove the formula for $\mathrm{P}_{\mathbb{Z}_{p^k}}(m, 2)$ when $p = 2$, and also a formula for $\mathrm{P}_{\mathbb{Z}_{p^k}}(m, 3)$, but the details are much more involved and have not yet been fully worked through. However, the present approach does not seem able to establish a formula for $\mathrm{P}_{\mathbb{Z}_{p^k}}(m, n)$ for general $m, n \geq 4$ as the number of cases to consider in the proof grows as a function of $\min(m, n)$.

2. Arithmetic in $\mathbb{Z}_{p^k}[x]$

In this section, we establish some basic results on the rings $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{p^k}[x]$. Recall that $\mathbb{Z}_n$ denotes the ring of integers mod n. We will make use of Hensel's lemma [Gouvêa 1997, page 70] in the following form:

Lemma 2.1 (Hensel's lemma). *Let $f(x) \in \mathbb{Z}_{p^k}[x]$ be a polynomial and denote its reduction mod p by $\bar{f}(x) \in \mathbb{Z}_p[x]$. Suppose there exists $u_0 \in \mathbb{Z}_p$ with $\bar{f}(u_0) = 0$ in $\mathbb{Z}_p$ and $\bar{f}'(u_0) \neq 0$ in $\mathbb{Z}_p$. Then there exists a unique $u \in \mathbb{Z}_{p^k}$, with $f(u) = 0$ in $\mathbb{Z}_{p^k}$ and $u \equiv u_0 \bmod p$.*

We start by counting the squares in $\mathbb{Z}_{p^k}$ and its unit subgroup $\mathbb{Z}^*_{p^k}$.

Lemma 2.2. *Let p be an odd prime and $k \geq 1$.*

(a) *$\mathbb{Z}^*_{p^k}$ has $\frac{1}{2}p^{k-1}(p-1)$ squares.*

(b) *Let d be even, with $0 \leq d < k$. There are $\frac{1}{2}(p-1)p^{k-1-d}$ nonzero squares $x \in \mathbb{Z}_{p^k}$ with $x \in p^d\mathbb{Z}_{p^k} \setminus p^{d+1}\mathbb{Z}_{p^k}$.*

(c) *There are $1 + \dfrac{1}{2(p+1)}(p^{k+1} - p^{1-k+2[k/2]})$ squares in $\mathbb{Z}_{p^k}$.*

Proof. (a) We first note that the $(p-1)/2$ squares $x = 1^2, \ldots, (\frac{p-1}{2})^2$ are distinct nonzero squares in both $\mathbb{Z}_p$ and $\mathbb{Z}_{p^k}$. Now consider a unit $u \in \mathbb{Z}_{p^k}$ satisfying $u \equiv 1 \bmod p$. Letting $f(x) = x^2 - u \in \mathbb{Z}_{p^k}[x]$, and $u_0 = 1$, by Lemma 2.1, u is a square in $\mathbb{Z}_{p^k}$. Thus the p^{k-1} units $u \in \mathbb{Z}_{p^k}$ with $u \equiv 1 \bmod p$ are squares. Hence, the $\frac{1}{2}p^{k-1}(p-1)$ distinct units xu are all squares and every unit square can be seen to be of this form.

(b) Let $x \in \mathbb{Z}_{p^k}$ satisfy $x \in p^d\mathbb{Z}_{p^k} \setminus p^{d+1}\mathbb{Z}_{p^k}$. Let $x = (p^t u)^2 = p^{2t}u^2$, where u is a unit. To satisfy the given conditions, $t = d/2$, u^2 is a unit square in $\mathbb{Z}^*_{p^k}$, and $u^2 \equiv u_1^2 \bmod p^{k-d}$. Hence, the number of distinct x equals the number of unit squares in $\mathbb{Z}_{p^{k-d}}$, which is given by (a).

(c) Every nonzero square can be written as $p^{2d}u$, where u is a unit square and $0 \le 2d < n$. Counting the square 0, the total sum is, thanks to (b),

$$1+\tfrac{1}{2}(p-1)\sum_{d=0}^{[(k-1)/2]} p^{k-1-2d}.$$

This expression simplifies to the claimed formula. □

For $g(x) = x^2 + bx + c \in \mathbb{Z}_{p^k}[x]$, define the discriminant $\Delta_g = b^2 - 4c$. As when $k = 1$, we can describe the number of roots of $g(x) \in \mathbb{Z}_{p^k}[x]$ using Δ_g.

Lemma 2.3. *Let p be an odd prime, $k \ge 1$, and $g(x) = x^2 + bx + c \in \mathbb{Z}_{p^k}[x]$.*

(a) *Δ is a square mod p^k if and only if g is reducible.*

(b) *If $\Delta \equiv 0 \bmod p^k$, then g has the $p^{[k/2]}$ roots given by $\dfrac{-b}{2} + p^{[(k+1)/2]}t \bmod p^k$, where $t = 1, \dots, p^{[k/2]}$.*

(c) *Suppose $\Delta \equiv p^d u \bmod p^k$ is a nonzero square with $0 \le d < k$, d even, $u \in \mathbb{Z}^*_{p^k}$ a square. Choose a such that $u \equiv a^2 \bmod p^k$. Then g has the $2p^{d/2}$ roots*

$$-\tfrac{1}{2}b \pm \tfrac{1}{2}ap^{d/2} + tp^{k-d/2} \mod p^k, \qquad \text{where } t = 1, \dots, p^{d/2}.$$

Proof. Since p is odd, we have $g(x) = (x+b/2)^2 - \Delta/4$. Hence $r = -(b+z)/2$ is a root of $g(x)$ if and only if z is a solution of the equation $z^2 \equiv \Delta \bmod p^k$. Condition (a) is thus proved. Condition (b) follows as well as the roots of the equation $z^2 \equiv 0 \bmod p^k$ are $z \equiv p^{[(k+1)/2]}t \bmod p^k$, for $t = 1, \dots, p^{[k/2]}$, or equivalently, $z \equiv 2p^{[(k+1)/2]}t \bmod p^k$, for $t = 1, \dots, p^{[k/2]}$. (c) By the hypothesis, d is even and $a \not\equiv 0 \bmod p$. The solutions to the equation $z^2 \equiv p^d a^2 \bmod p^k$ have the form $z \equiv p^{d/2}w \bmod p^k$, where $w \in \mathbb{Z}_{p^k}$ is a solution of $x^2 \equiv a^2 \bmod p^{k-d}$. Hensel's lemma (using the polynomial $f(x) = x^2 - a^2$), shows that the solutions to this latter equation are the $w \in \mathbb{Z}_{p^k}$ satisfying $w \equiv \pm a \bmod p^{k-d}$. Thus $w = \pm a + tp^{k-d}$, for $t = 1, \dots, p^d$, or equivalently, as 2 is a unit mod p^d, $w = \pm a + 2tp^{k-d}$ for $t = 1, \dots, p^d$. Now two roots $z = p^{d/2}w$ and $z_1 = p^{d/2}w_1$ are equal precisely when the signs in the expressions for w and w_1 agree and the respective parameters t and t_1 satisfy $t \equiv t_1 \bmod p^{d/2}$. Hence we have shown that the original equation $z^2 \equiv p^d a^2 \bmod p^k$ has the $2p^{d/2}$ distinct roots given by $z = \pm ap^{d/2} + 2tp^{k-d/2}$, for $t = 1, \dots, p^{d/2}$. □

Lemma 2.4. *Let p be an odd prime and $k \ge 1$.*

(a) *Given $\Delta \in \mathbb{Z}_{p^k}$, there are p^k monic, quadratic polynomials $g \in \mathbb{Z}_{p^k}[x]$ with $\Delta_g \equiv \Delta \bmod p^k$.*

(b) *There are*
$$\frac{p^k}{2(p+1)}(p^{k+1}+2p^k-p-p^{k-2[k/2]}-1)$$
monic, irreducible, quadratic polynomials $g \in \mathbb{Z}_{p^k}[x]$.

Proof. If $g = x^2+bx+c$, then $\Delta_g = b^2-4c$. Since 4 is invertible mod p^k, for every Δ, $b \in \mathbb{Z}_{p^k}$, there is a unique choice of c such that $\Delta_g \equiv \Delta \bmod p^k$. Since there are p^k choices for b, (a) is proved. Now g is irreducible precisely when Δ_g is not a square. Let S be the number of squares in $\mathbb{Z}_{p^k}$. Then for each $b \in \mathbb{Z}_{p^k}$, there are $p^k - S$ choices for c such that b^2-4c is not a square. Thus, using the formula for S given by Lemma 2.2(c), there are
$$p^k(p^k-S) = \frac{p^k}{2(p+1)}\left(p^{k+1}+2p^k-2p+p^{1-k+2[k/2]}-2\right)$$
irreducible polynomials g. Simplification gives (b). □

Given a monic, quadratic polynomial $g \in \mathbb{Z}_{p^k}[x]$, we define the set
$$A_g = \{h \in \mathbb{Z}_{p^k}[x] : \deg h \le 1 \text{ and } g, h \text{ are not relatively prime}\},$$
and let $|A_g|$ denote its cardinality. We note that in the definition of A_g, we allow nonmonic polynomials h.

Lemma 2.5. *Let p be an odd prime and $g(x)$ be a monic quadratic polynomial in* $\mathbb{Z}_{p^k}[x]$.

(a) *If* $\Delta_g \equiv 0 \bmod p^k$, *then*
$$|A_g| = p^{k-[k/2]}\left(\frac{p^{2[k/2]+1}+1}{p+1}\right).$$

(b) *Assume* $\Delta_g \in \mathbb{Z}_{p^k}$ *is a nonzero square. Let* $\Delta_g \equiv p^d v \bmod p^k$, *where d is even,* $0 \le d < k$, *and* $v \in (\mathbb{Z}^*_{p^k})^2$. *Then*
$$|A_g| = 2p^{k-d/2}\left(\frac{p^{d+1}+1}{p+1}\right) - p^{d/2}.$$

Proof. We first note that a linear factor of $g(x)$ must have the form $u(x-r)$, where $u, r \in \mathbb{Z}_{p^k}$, u is a unit, and r is a root of g. Therefore, the elements $h(x) \in A_g$ are exactly the polynomials $h(x) = \alpha(x-r)$, for some $\alpha \in \mathbb{Z}_{p^k}$ and some root $r \in \mathbb{Z}_{p^k}$ of g. Hence, to calculate $|A_g|$, we need to count the number of distinct $h(x)$ of this form.

Suppose r_1 and r_2 are two roots of g and $\alpha(x-r_1) \equiv \beta(x-r_2) \bmod p^k$. Then $\beta \equiv \alpha \bmod p^k$ and $\alpha(r_1-r_2) \equiv 0 \bmod p^k$. Let $\alpha = p^s u$, with $u \in \mathbb{Z}^*_{p^k}$. If $s = k$, then $\alpha = 0$ is the only choice. Now suppose $s < k$. Then there are $p^{k-s-1}(p-1)$ distinct choices for u giving rise to distinct α. For each such α, we need to calculate the number of roots of g in $\mathbb{Z}_{p^{k-s}}$. To proceed further, we need to have a description of the roots.

Writing $g(x)=x^2+bx+c$, in case (a), the roots of g are $r=-b/2+p^{[(k+1)/2]}t$, for $t=1,\dots,p^{[k/2]}$ by Lemma 2.3. If $[k/2]\le s<k$, for each choice of $\alpha=p^s u$, there is exactly one factor $\alpha(x-r) \bmod p^k$. As there are $p^{k-s-1}(p-1)$ choices for u, and hence α, we obtain the same number of distinct factors $\alpha(x-r)$ for each s. If $0\le s\le [k/2]$, then for each choice of $\alpha=p^s u$, there are $p^{[k/2]-s}$ distinct factors $\alpha(x-r) \bmod p^k$. Hence there are $p^{k+[k/2]-2s-1}(p-1)$ distinct factors $\alpha(x-r) \bmod p^k$ for each s. In total then, we have

$$\begin{aligned}|A_g| &= \sum_{s=0}^{[k/2]}(p-1)p^{k+[k/2]-2s-1}+\Bigg(\sum_{s=[k/2]+1}^{k-1}(p-1)p^{k-s-1}+1\Bigg)\\ &= \sum_{s=0}^{[k/2]}(p-1)p^{k+[k/2]-2s-1}+p^{k-[k/2]-1}=p^{k-[k/2]}\left(\frac{p^{2[k/2]+1}+1}{p+1}\right),\end{aligned}$$

where the last equality is obtained by evaluating a geometric sum. We thus obtain the desired formula for case (a). In case (b), by Lemma 2.3, the roots of g are $-\frac{1}{2}b\pm\frac{1}{2}ap^{d/2}+tp^{k-d/2} \bmod p^k$, where $a^2\equiv v \bmod p^k$, $t=1,\dots,p^{d/2}$. As in case (a), we let $\alpha=p^s u$, and consider the number of distinct factors $h(x)=\alpha(x-r)$ for each choice of s. When $s=k$, $h(x)=\alpha=0$ is the only factor. There are three additional cases:

(1) Suppose $k>s\ge k-d/2$. Then $k-s\le d/2$ and all the roots of g are equivalent mod p^{k-s}. Since there are $p^{k-s-1}(p-1)$ distinct choices for α, there are the same number of distinct factors $\alpha(x-r)$.

(2) Suppose $k-d/2>s\ge d/2$. Then $d/2<k-s\le k-d/2$ and the roots of g determine two equivalence classes mod p^{k-s}. Thus for each s, there are a total of $2p^{k-s-1}(p-1)$ distinct factors $\alpha(x-r)$.

(3) Suppose $d/2\ge s\ge 0$. Then the roots of g determine $2p^{d/2-s}$ equivalence classes mod p^{k-s} for each α. Thus there are a total of $2p^{k+d/2-2s-1}(p-1)$ distinct factors $\alpha(x-r)$, for each s.

In total, when $d<k-1$, we have for $|A_g|$ the value

$$\begin{aligned}&\sum_{s=0}^{d/2}2(p-1)p^{k+d/2-2s-1}+\Bigg(\sum_{s=d/2+1}^{k-d/2-1}2(p-1)p^{k-s-1}+\sum_{s=k-d/2}^{k-1}(p-1)p^{k-s-1}+1\Bigg)\\ &\qquad\qquad=\sum_{s=0}^{d/2}2(p-1)p^{k+d/2-2s-1}+2p^{k-d/2-1}-p^{d/2},\end{aligned}$$

which simplifies to the formula stated in (b). When $d = k-1$, the second summation does not appear, and

$$|A_g| = \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + \left(\sum_{s=k-d/2}^{k-1} (p-1)p^{k-s-1} + 1 \right)$$
$$= \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + p^{d/2},$$

which again simplifies to the stated formula for (b). □

3. Proof of the main theorem

In this section, we let $q = p^k$. To prove Theorem 1.2, we will count the number of polynomial pairs (f, g), where $f, g \in \mathbb{Z}_q[x]$ are not relatively prime. Let $f(x), g(x)$ be monic polynomials. Then by the division algorithm, there is a unique choice of polynomials $q(x), r(x) \in \mathbb{Z}_q[x]$, with $q(x)$ monic, satisfying

$$f(x) = g(x)q(x) + r(x), \tag{1}$$

where $r(x) = 0$ or $\deg r(x) < \deg g(x)$. Thus the pair (f, g) is uniquely determined by the triple $(g, q(x), r(x))$. From (1), any common divisor of f and g is a common divisor of g and r and vice-versa. We define

$$S_{m,d,q} = \{(f, g) : f, g \in \mathbb{Z}_q[x] \text{ monic with } \deg f = m,\ \deg g = d,\ f \text{ and } g \text{ not relatively prime}\},$$

$$T_{m,q} = \{(g, r) : g, r \in \mathbb{Z}_q[x] \text{ with } g \text{ monic of degree } m,\ \deg r < m,\ g \text{ and } r \text{ not relatively prime}\}.$$

Lemma 3.1. *If $m \geq d$, then $|S_{m,d,q}| = q^{m-d}|T_{d,q}|$.*

Proof. Let $(g, r) \in T_{d,q}$. Then each of the q^{m-d} monic polynomials $q(x)$ with degree $m - d$ gives rise via (1) to a unique pair $(f, g) \in S_{m,d,q}$. Conversely, the inverse map

$$(f, g) \mapsto (g, q, r) \mapsto (g, r)$$

is a q^{m-d}-to-1 map from $S_{m,d,q}$ to $T_{d,q}$. □

Thus, proving Theorem 1.2 is reduced to calculating $|T_{2,q}|$. We begin with:

Proposition 3.2. $|T_{1,q}| = q$.

Proof. If $(g, r) \in T_{1,q}$, then $g(x) = x - c$. For g and r to have a common factor, $r = 0$. Hence $T_{1,q}$ consists of the q pairs $(x - c, 0)$. □

We now determine $|T_{2,q}|$. By Lemma 2.3, we have $|T_{2,q}| = B_1 + B_2 + B_3$, where the B_i are defined by

$$B_1 = \big|\{(g,r) \in T_{2,q} : g \text{ is irreducible}\}\big|,$$

$$B_2 = \big|\{(g,r) \in T_{2,q} : \Delta_g \equiv 0 \bmod p^k\}\big|,$$

$$B_3 = \big|\{(g,r) \in T_{2,q} : \Delta_g \bmod p^k \text{ is a square, and, for each } d < k,\ \Delta_g \equiv 0 \bmod p^d \text{ and } \Delta_g \not\equiv 0 \bmod p^{d+1}\}\big|.$$

Lemma 3.3. (a) $B_1 = \dfrac{p^k}{2(p+1)}(p^{k+1} + 2p^k - p - p^{k-2[k/2]} - 1)$.

(b) $B_2 = p^{2k-[k/2]}\left(\dfrac{p^{2[k/2]+1}+1}{p+1}\right)$.

(c) $B_3 = \dfrac{p^{2k-1-[(k-1)/2]} - p^{2k}}{2(p+1)(p^2+p+1)}\alpha$, *where*

$$\alpha = (p+1)(p^2+p+1) - 2p^{k+1}(p+1)^2 - 2p^{k-[(k-1)/2]}(p + p^{-[(k-1)/2]}).$$

Proof. (a) Assume $g \in \mathbb{Z}_{p^k}[x]$ is a monic, irreducible, quadratic polynomial. Since g has no factors, $(g,r) \in T_{2,q}$ only when $r = 0$. Hence, B_1 equals the number of monic, irreducible quadratic polynomials, which is given by Lemma 2.4.

(b) Assume $g \in \mathbb{Z}_{p^k}[x]$ is a monic quadratic with $\Delta_g \equiv 0 \bmod p^k$. By Lemma 2.4, there are p^k such g. For each g, $|A_g|$ is given by Lemma 2.5(a). Thus

$$B_2 = p^k|A_g|.$$

(c) If $(g,r) \in T_{2,q}$ is included in the pairs counted for B_3, then $\Delta_g = p^d u$, where $0 \le d < k$, d even, and $u \in \mathbb{Z}^*_{p^k}$ is a square. For a fixed d, u, satisfying these conditions, there are p^k polynomials g with $\Delta_g = p^d u$ by Lemma 2.4(a). And for any such g, $|A_g|$ is given by Lemma 2.5(b). Now, for a fixed d, there are

$$\tfrac{1}{2}(p-1)p^{k-d-1}$$

choices for u that give distinct values for $p^d u$. Putting these results together, and replacing d by $2d$, we have

$$\begin{aligned} B_3 &= \sum_{d=0}^{[(k-1)/2]} \frac{1}{2}(p-1)p^{2k-d-1}\left(2p^{k-2d}\left(\frac{p^{2d+1}+1}{p+1}\right) - 1\right) \\ &= \frac{p^{2k-1}(p-1)}{2(p+1)} \sum_{d=0}^{[(k-1)/2]} p^{-d}\big(2p^{k-2d}(p^{2d+1}+1) - p - 1\big). \end{aligned} \tag{2}$$

Summing the geometric sequences, we have

$$\sum_{d=0}^{[(k-1)/2]} p^{-d}(-p-1) = -(p+1)p^{-[(k-1)/2]}\Big(\frac{p^{[(k-1)/2]+1}-1}{p-1}\Big),$$

$$\sum_{d=0}^{[(k-1)/2]} p^{-d}\big(2p^{k-2d}(p^{2d+1}+1)\big) = 2p^{k-[(k-1)/2]+1}\Big(\frac{p^{[(k-1)/2]+1}-1}{p-1}\Big) \\ +2p^{k-3[(k-1)/2]}\Big(\frac{p^{3[(k-1)/2]+3}-1}{p^3-1}\Big).$$

Substituting these equations in (2) and simplifying with the help of a computer algebra system, we obtain the desired expression. □

Proof of Theorem 1.2. There are q^m monic polynomials in $\mathbb{Z}_q[x]$ with degree m. Hence there are q^{m+2} pairs of monic polynomials (f, g) with $\deg f = m$, $\deg g = 2$. By Lemma 3.1, the probability that a pair of these polynomials is relatively prime is

$$1-\frac{|S_{m,2,q}|}{q^{m+2}} = 1-\frac{|T_{2,q}|}{q^4}.$$

Now $|T_{2,q}| = B_1+B_2+B_3$, with the values of B_i given by Lemma 2.5. Manipulating this expression with the help of a computer algebra system, one obtains

$$|T_{2,q}| = \frac{p^k}{2(p+1)}D,$$

where D equals the expression

$$2p^{2k+1}+2p^{2+k/2}(p-1)\Big(\frac{p^{3k/2}-1}{p^3-1}\Big)+p^{1+k/2}+3p^{k/2}+p^k-p-2$$

when k is even, and D equals

$$2p^{2k+1}+2(p-1)\Big(\frac{p^{2(k+1)}-p^{(k+1)/2}}{p^3-1}\Big)+3p^{(k+1)/2}+p^{(k-1)/2}+p^k-2p-1,$$

when k is odd. When k is even, algebraic manipulation shows

$$2p^{2k+1} = 2(p+1)p^{2k}-2p^{2k},$$

$$2p^{2+k/2}(p-1)\Big(\frac{p^{3k/2}-1}{p^3-1}\Big) = 2p^{2k}-2p^{2+k/2}+2(1-p^2)\sum_{i=1}^{k/2-1} p^{2k-3i},$$

$$p^{1+k/2}+3p^{k/2} = (p+1)(-2p^{1+k/2}+3p^{k/2})+2p^{2+k/2},$$

$$p^k-p-2 = -(p+1)\sum_{i=1}^{k-1}(-p)^i-2(p+1).$$

Adding both sides, the left hand side sums to D. With $f_k(x)$ defined as in the introduction, we then have

$$\frac{1}{2(p+1)}D = f_k(p).$$

Theorem 1.2 follows immediately for k even. Similar calculations establish it for k odd. □

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