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We explore a complex extension of finite calculus on the integer lattice of the complex plane. $f:\mathbb{Z}[i]\to\mathbb{C}$ satisfies the discretized Cauchy–Riemann equations at z if $\mathrm{Re}(f(z+1)-f(z)) = \mathrm{Im}(f(z+i)-f(z))$ and $\mathrm{Re}(f(z+i)-f(z)) = -\mathrm{Im}(f(z+1)-f(z))$. From this principle arise notions of the discrete path integral, Cauchy's theorem, the exponential function, discrete analyticity, and falling power series.

1. Introduction

The theory of finite (or discrete) calculus, that is, finite differences, has been well established. In addition, a unified theory of time scales has been formulated that encompasses both continuous and discrete calculus (for real variables) [Bohner and Peterson 2001]. The subject of complex analysis builds a continuous calculus on the complex plane. A remaining, natural question is what can we say about finite calculus on the complex plane? There are multiple approaches to addressing this question, and unbeknownst to the authors until after this work was completed, the question has been explored before under the monikers of *discrete analytic functions*, *preholomorphic functions*, and *monodiffric functions of the first kind* [Duffin 1956; Ferrand 1944; Isaacs 1941; 1952; Kiselman 2005; Mercat 2001]. Consequently, we do not claim mathematical originality for any of these results; we only hope to present these ideas in a fresh context. The reader is hereby warned that some familiar terms and theorem names will be used throughout this paper with a new meaning derived from the discretized context. To avoid confusion, invocations of these terms in their standard usage will be designated as *classical*.

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2. Definitions

Let $\mathbb{Z}[i] = \{x + iy : x \in \mathbb{Z}, y \in \mathbb{Z}\}$ denote the integer lattice in the complex plane. Let $f(z) = f(x, y) = u(x, y) + iv(x, y) : \Omega \to \mathbb{C}$, where Ω is a subset of $\mathbb{Z}[i]$. The partial derivative of u with respect to x, $\Delta_x u(x, y)$, can be calculated by a finite difference as

$$\Delta_x u(x, y) = u(x+1, y) - u(x, y)$$

or more simply as $\Delta_x u(z) = u(z+1) - u(z)$. Similarly,

$$\Delta_y u(x, y) = u(x, y+1) - u(x, y)$$

and again,

$$\Delta_y u(z) = u(z+i) - u(z).$$

This allows for the natural definition of

$$\Delta_x f = \Delta_x u + i\,\Delta_x v \quad \text{and} \quad \Delta_y f = \Delta_y u + i\,\Delta_y v.$$

Note that $\Delta_x u(z)$ is defined at $\{z \in \Omega : z+1 \in \Omega\}$ and $\Delta_y u(z)$ is defined at $\{z \in \Omega : z+i \in \Omega\}$. We have the following lemma for mixed partials:

Lemma 2.1. *If f is defined on a set Ω, then on $\{z \in \Omega : z+1, z+i, z+1+i \in \Omega\}$ we have*

$$\Delta_{xy} f(z) = \Delta_{yx} f(z).$$

Proof.
$$\begin{aligned}\Delta_{xy} f(z) &= \Delta_x(f(z+i) - f(z))\\ &= f(z+i+1) - f(z+1) - f(z+i) + f(z)\\ &= f(z+1+i) - f(z+i) - f(z+1) + f(z)\\ &= \Delta_y(f(z+1) - f(z)) = \Delta_{yx} f(z).\end{aligned}$$ □

Definition 2.2. The discrete function f is *holomorphic* at z if it satisfies the discrete Cauchy–Riemann equations at z:

$$\Delta_x u(z) = \Delta_y v(z) \text{ and } \Delta_y u(z) = -\Delta_x v(z).$$

Definition 2.3. The *partial derivative* of f with respect to z is

$$\Delta f = \frac{\Delta_x f - i\,\Delta_y f}{2} = \frac{f(z+1) - f(z) - i(f(z+i) - f(z))}{2}$$

and with respect to $\bar{z}$ is

$$\bar{\Delta} f = \frac{\Delta_x f + i\,\Delta_y f}{2} = \frac{f(z+1) - f(z) + i(f(z+i) - f(z))}{2}.$$

All partial derivative operators — Δ, $\bar{\Delta}$, Δ_x, Δ_y — are linear operators. There is no immediately apparent Leibniz product rule, or chain rule. In particular, the usual product of two holomorphic functions is not necessarily holomorphic. The Cauchy–Riemann equations imply that f is holomorphic if and only if $\bar{\Delta} f = 0$. If $\bar{\Delta} f = 0$ then $f(z+1) - f(z) = -i(f(z+i) - f(z))$ and, as in classical complex analysis,

$$\Delta f = \Delta_x f = -i\,\Delta_y f.$$

Definition 2.4. The *interior* of a set $\Omega \subset \mathbb{Z}[i]$ is the subset

$$\mathring{\Omega} = \{z \in \Omega : z+1 \in \Omega,\ z+i \in \Omega\}.$$

Note that for f to be holomorphic at z requires f is defined at z, $z+1$, and $z+i$. Hence, for f to be holomorphic on Ω necessitates that f is defined on G, where $\mathring{G} = \Omega$.

As in the classical case, holomorphic implies infinitely differentiable.

Theorem 2.5. *If f is holomorphic on Ω, then Δf is holomorphic on the interior of Ω.*

Proof. If $z \in \mathring{\Omega}$, then f is holomorphic at z, $z+1$, and $z+i$, so

$$\begin{aligned}\bar{\Delta}\Delta f(z) &= \bar{\Delta}\left(\frac{f(z+1) - f(z) - i(f(z+i) - f(z))}{2}\right)\\ &= \frac{\bar{\Delta} f(z+1) - \bar{\Delta} f(z) - i(\bar{\Delta} f(z+i) - \bar{\Delta} f(z))}{2}\\ &= 0.\end{aligned}$$ □

3. Formulas

Theorem 3.1. *Let $z_{n,j} := z + (n-j) + ji$ for $j = 0, \ldots, n$, that is, $\{z_{n,j}\}$ forms the hypotenuse of an isosceles triangle with right angle at z and base length n. If f is holomorphic on the interior of this triangle then*

$$f(z) = \left(\frac{1-i}{2}\right)^n \sum_{j=0}^{n} \binom{n}{j} i^j f(z_{n,j}).$$

Proof. We proceed by induction on n. When $n = 1$, we have

$$f(z) = \frac{1-i}{2}(f(z_{1,0}) + i f(z_{1,1})),$$

which is equivalent to $\bar{\Delta} f(z) = 0$. If f is holomorphic at $z_{n,j}$ then

$$f(z_{n,j}) = \frac{1-i}{2}(f(z_{n+1,j}) + i f(z_{n+1,j+1})).$$

Assuming the formula holds for n we write

$$\begin{aligned}
f(z) &= \left(\frac{1-i}{2}\right)^n \sum_{j=0}^{n} i^j \binom{n}{j} \left(\frac{1-i}{2}\right)(f(z_{n+1,j}) + i f(z_{n+1,j+1}))\\
&= \left(\frac{1-i}{2}\right)^{n+1} \left[\sum_{j=0}^{n} i^j \binom{n}{j} f(z_{n+1,j}) + \sum_{j=0}^{n} i^{j+1} \binom{n}{j} f(z_{n+1,j+1})\right]\\
&= \left(\frac{1-i}{2}\right)^{n+1} \left[\sum_{j=0}^{n} i^j \binom{n}{j} f(z_{n+1,j}) + \sum_{j=1}^{n+1} i^{j} \binom{n}{j-1} f(z_{n+1,j})\right]\\
&= \left(\frac{1-i}{2}\right)^{n+1} \left[\sum_{j=0}^{n+1} i^j \binom{n}{j} f(z_{n+1,j}) + \sum_{j=0}^{n+1} i^{j} \binom{n}{j-1} f(z_{n+1,j})\right]\\
&= \left(\frac{1-i}{2}\right)^{n+1} \sum_{j=0}^{n+1} i^j f(z_{n+1,j}) \left(\binom{n}{j} + \binom{n}{j-1}\right)\\
&= \left(\frac{1-i}{2}\right)^{n+1} \sum_{j=0}^{n+1} i^j \binom{n+1}{j} f(z_{n+1,j}). \qquad \square
\end{aligned}$$

Corollary 3.2. *If $M_n = \max |f(z+j+ik)|$ for $j+k=n$ and $j,k \geq 0$,*

$$|f(z)| \leq 2^{n/2} M_n.$$

Proof. $|f(z)| \leq (1/\sqrt{2})^n \sum_{j=0}^{n} \binom{n}{j} M_n \leq (1/\sqrt{2})^n 2^n M_n = 2^{n/2} M_n.$ $\square$

This formula, unlike the classical Cauchy estimate, grows as $n \to \infty$. So the veracity of Liouville's Theorem in this context remains in doubt. Theorem 3.4 presents a higher-order formula as a consequence of the following lemma.

Lemma 3.3.
$$\Delta^k f(z) = \left(\frac{1+i}{2}\right)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(z_{k,j}).$$

Proof. By definition,

$$\Delta^k f(z) = \Delta(\Delta^{k-1} f(z)) = \left(\frac{1+i}{2}\right)(\Delta^{k-1} f(z+1) - \Delta^{k-1} f(z+i)).$$

An induction argument similar to the proof of Theorem 3.1 holds. $\square$

Theorem 3.4. *Let $z_{n,j} := z + (n-j) + ji$ for $j = 0, \ldots, n$. Then $\{z_{n,j}\}$ forms the hypotenuse of an isosceles triangle with right angle at z and base length n. If f is holomorphic on the interior of this triangle then*

$$\Delta^k f(z) = i^k \left(\frac{1-i}{2}\right)^n \sum_{j=0}^{n} \left(i^j f(z_{n,j}) \sum_{l=0}^{k} i^l \binom{k}{l} \binom{n-k}{j-l} \right)$$

for all $n \geq k$.

Proof. Fix k and induct on n. The lemma establishes the case $n = k$. Assuming the formula holds for n we have

$$\Delta^k f(z) = i^k\left(\frac{1-i}{2}\right)^{n+1} \sum_{j=0}^{n}\left(i^j(f(z_{n+1,j}) + if(z_{n+1,j+1}))\sum_{l=0}^{k} i^l\binom{k}{l}\binom{n-k}{j-l}\right)$$

$$= i^k\left(\frac{1-i}{2}\right)^{n+1}\left(\sum_{j=0}^{n+1} i^j f(z_{n+1,j})\sum_{l=0}^{k} i^l\binom{k}{l}\binom{n-k}{j-l}\right.$$

$$\left. + \sum_{j=0}^{n+1} i^j f(z_{n+1,j})\sum_{l=0}^{k} i^l\binom{k}{l}\binom{n-k}{j-1-l}\right)$$

$$= i^k\left(\frac{1-i}{2}\right)^{n+1}\sum_{j=0}^{n+1}\left(i^j f(z_{n+1,j})\sum_{l=0}^{k} i^l\binom{k}{l}\binom{n+1-k}{j-l}\right). \qquad \square$$

Theorem 3.1 presents the value of $f(z)$ as a sum of function values along the hypotenuse of the triangle. The following formulas present the value of f at the other triangle vertices as a sum of function values on the opposing side. The proofs are similar to that of Theorem 3.1.

Proposition 3.5. *Let $z_j = z - ni + j$. Then*

$$f(z) = \sum_{j=0}^{n}(1-i)^{n-j} i^j\binom{n}{j} f(z_j).$$

Proposition 3.6. *Let $z_j = z - n + ij$. Then*

$$f(z) = \sum_{j=0}^{n}(1+i)^{n-j}(-i)^j\binom{n}{j} f(z_j).$$

In classical complex analysis, by using Green's theorem, we have a Cauchy formula for continuous, nonholomorphic functions. The discrete analogue of the Cauchy–Pompeiu–Green formula is:

Theorem 3.7. *For any function f defined on the isosceles, right triangle with base length $n \geq 1$,*

$$f(z) = \left(\frac{1-i}{2}\right)^n\left(\sum_{j=0}^{n} i^j\binom{n}{j} f(z_{n,j}) - (1-i)\sum_{l=0}^{n-1}\sum_{k=0}^{l} i^k(1+i)^{n-l}\binom{l}{k}\bar{\Delta} f(z_{l,k})\right),$$

where $z_{n,j} = z + (n-j) + ij$.

Proof. We proceed by induction on n. For $n = 1$, we need

$$f(z) = \frac{1-i}{2}\big(f(z_{1,0}) + if(z_{1,1}) - (1-i)(1+i)\bar{\Delta} f(z_{0,0})\big),$$

which holds by the definition of $\bar{\Delta}$. In general, from our base case,

$$f(z_{n,j}) = \frac{1-i}{2}\big(f(z_{n+1,j}) + if(z_{n+1,j+1}) - 2\bar{\Delta} f(z_{n,j})\big).$$

the induction hypothesis,

$$\begin{aligned} f(z) &= \left(\frac{1-i}{2}\right)^n \Bigg(\sum_{j=0}^{n} i^j \binom{n}{j} \left(\frac{1-i}{2}\right) \big(f(z_{n+1,j}) + i f(z_{n+1,j+1}) - 2\bar{\Delta} f(z_{n,j})\big) \\ &\qquad\qquad - (1-i) \sum_{l=0}^{n-1} \sum_{k=0}^{l} i^k (1+i)^{n-l} \binom{l}{k} \bar{\Delta} f(z_{l,k}) \Bigg) \\ &= \left(\frac{1-i}{2}\right)^{n+1} \Bigg(\sum_{j=0}^{n+1} i^j \binom{n+1}{j} f(z_{n+1,j}) - 2 \sum_{j=0}^{n} i^j \binom{n}{j} \bar{\Delta} f(z_{n,j}) \\ &\qquad\qquad - 2 \sum_{l=0}^{n-1} \sum_{k=0}^{l} i^k (1+i)^{n-l} \binom{l}{k} \bar{\Delta} f(z_{l,k}) \Bigg) \\ &= \left(\frac{1-i}{2}\right)^{n+1} \Bigg(\sum_{j=0}^{n+1} i^j \binom{n+1}{j} f(z_{n+1,j}) \\ &\qquad\qquad - (1-i) \sum_{l=0}^{n} \sum_{k=0}^{l} i^k (1+i)^{n+1-l} \binom{l}{k} \bar{\Delta} f(z_{l,k}) \Bigg). \qquad \square \end{aligned}$$

4. Discretization of polynomials

As in the study of discrete calculus of a real variable, we redefine powers so that the power rule holds. In the real variable case, if we consider falling powers $x^{\underline{0}} = 1$ and $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ for $n \geq 1$, the discrete derivative power rule follows:

Proposition 4.1. $\Delta_x x^{\underline{n}} = n x^{\underline{n-1}}$.

Proof.
$$\begin{aligned} \Delta_x x^{\underline{n}} &= (x+1)^{\underline{n}} - x^{\underline{n}} \\ &= (x+1)x\cdots(x-n+2) - x(x-1)\cdots(x-n+1) \\ &= (x+1-(x-n+1))x(x-1)\cdots(x-n+2) = n x^{\underline{n-1}}. \qquad \square \end{aligned}$$

To discretize z^n in the complex setting, first expand $z^n = (x+iy)^n$ in terms of x and y and replace each x^n with $x^{\underline{n}}$ and each y^n with $y^{\underline{n}}$. We will denote this polynomial as $z^{\underline{n}}$ or $\mathfrak{D}(z^n)$. Hence, our formal definition is

$$\mathfrak{D}(z^n) = z^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}} i^k.$$

Similarly, the discretization of a polynomial $p(z)$ will be denoted $\mathfrak{D}(p(z))$. These *complex falling powers* of z satisfy both the Cauchy–Riemann equations and the following power rule.

Theorem 4.2. $\Delta(z^{\underline{n}}) = n(z^{\underline{n-1}})$ *and* $\bar{\Delta}(z^{\underline{n}}) = 0$.

Proof. Considering the binomial expansion of $z^{\underline{n}}$, by Proposition 4.1,

$$\Delta_x z^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} (n-k) x^{\underline{n-k-1}} y^{\underline{k}} i^k \quad \text{and} \quad \Delta_y z^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} k y^{\underline{k-1}} i^k$$

and by a change of indices,

$$\Delta_y z^{\underline{n}} = \sum_{k=0}^{n-1} \binom{n}{k+1} x^{\underline{n-k-1}}(k+1) y^{\underline{k}} i^{k+1}.$$

We can simplify these expressions because

$$(n-k)\binom{n}{k} = \frac{(n-k)n!}{k!(n-k)!} = \frac{n(n-1)!}{k!(n-k-1)!} = n\binom{n-1}{k}$$

and

$$(k+1)\binom{n}{k+1} = \frac{(k+1)n(n-1)!}{(n-k-1)!(k+1)k!} = n\binom{n-1}{k}.$$

Using the definition for Δ and simplifying gives

$$\Delta z^{\underline{n}} = \frac{\sum_{k=0}^{n-1} 2n\binom{n-1}{k} x^{\underline{n-k-1}} y^{\underline{k}} i^k}{2} = n z^{\underline{n-1}}.$$

Similarly the definition of $\bar{\Delta}$ gives $\bar{\Delta} z^{\underline{n}} = 0$. □

Corollary 4.3. *If $p(z)$ is a polynomial then $\mathcal{D}(p'(z)) = \Delta(\mathcal{D}(p(z)))$.*

Proof. In both cases the derivative operators are linear. □

In the real case, $x^{\underline{n}} = \sum_{j=0}^{n} s(n, j) x^j$ where $s(n, j)$ are Stirling numbers of the first kind, so we also have the formula

$$z^{\underline{n}} = \sum_{k=0}^{n} i^k \binom{n}{k} \Big(\sum_{j=0}^{n-k} s(n-k, j) x^j \Big) \Big(\sum_{l=0}^{k} s(k, l) y^l \Big).$$

Note that if $n > 1$ then $z^{\underline{n}}$ is not holomorphic in the classical sense.

The definition of complex falling powers may seem unmotivated, so we furnish an example. Consider the difference equation $\Delta F(z) = 2z$. In accordance with the power rule, the solution should be an analogue of z^2. The function F must be of the form

$$z^2 + A\bar{z}^2 - \frac{(1+i)(1+A)}{2} z + B\bar{z} + C,$$

and so

$$\bar{\Delta} F(z) = 2A\bar{z} + \frac{(1+A)(1-i)}{2} + B.$$

Examples of solutions include:

$$\frac{z(z-1) + z(z-i)}{2} + C \quad \text{and} \quad z^2 - \frac{1+i}{2} z - \frac{1-i}{2} \bar{z} + C$$

with the latter being the general solution with $\bar{\Delta} F = 0$; the particular holomorphic solution with $C = 0$ is what we've denoted $z^{\underline{2}}$.

Proposition 4.4. $\{a + bi : a, b \geq 0$ *and* $a + b < n\}$ *are zeros of* $z^{\underline{n}}$.

Proof. If for each $k = 0, \dots, n$ we have either $x^{\underline{n-k}} = 0$ or $y^{\underline{k}} = 0$, then

$$z^{\underline{n}} = \sum_{k=0}^{n} i^k \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}} = 0.$$

The zeros of $x^{\underline{j}}$ are given precisely by $\{x \in \mathbb{Z} : 0 \le x < j\}$ since

$$x^{\underline{j}} = x(x-1)\cdots(x-j-1).$$

So $z^{\underline{n}} = 0$ if for each $k = 0, \dots, n$ we have either $0 \le x < n-k$ or $0 \le y < k$. This condition is met precisely if $x \ge 0,\ y \ge 0$, and $x + y < n$. □

5. Power series and continuation

Lemma 5.1 (Weak Identity Theorem). *If f and g are holomorphic functions which, for some z_0 agree on the line* $\operatorname{Im} z = \operatorname{Im} z_0$, $\operatorname{Re} z \ge \operatorname{Re} z_0$ *then f and g agree for all z such that* $\operatorname{Re} z \ge \operatorname{Re} z_0$ *and* $\operatorname{Im} z \ge \operatorname{Im} z_0$.

Proof. Without loss of generality, assume $z_0 = 0$. If f and g agree on the positive real line, then, since both are holomorphic, they have a unique holomorphic extension to the points above this line. □

The standard Schwarz reflection principle for holomorphic continuation does not hold. Falling power series can be represented as

$$\sum_{n=0}^{\infty} a_n (z - z_0)^{\underline{n}}.$$

Regions of convergence have vastly different shapes from those in the classical case.

Theorem 5.2. *The falling power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^{\underline{n}}$$

converges for at least all z such that $\operatorname{Re} z \ge \operatorname{Re} z_0$ *and* $\operatorname{Im} z \ge \operatorname{Im} z_0$.

Proof. We will prove this for $z_0 = 0$ and the proof can be carried out similarly for other finite z_0. By Proposition 4.4 the zeros of $z^{\underline{n}}$ include

$$\{a + bi : a, b \ge 0 \text{ and } a + b < n\}.$$

For any point $a + bi$ in the first quadrant, there exists n with $a + b < n$. Thus the terms of the series $z^{\underline{k}}$ with $k > n$ will be 0 for $z = a + bi$. A sum of a finite number of terms is trivially convergent. Since $a + bi$ was arbitrary, the falling power series converges for every point in the first quadrant. □

The first quadrant may not be the only place a falling power series centered at 0 converges. For instance, the series

$$\sum_{n=0}^{\infty} \frac{z^{\underline{n}}}{(n+1)!}$$

evaluated at $z = -1$ is the alternating harmonic series and thus converges to $\ln 2$.

Proposition 5.3. *If the falling power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{\underline{n}}$$

converges on a domain Ω, *then*

$$\Delta f(z) = \sum_{n=0}^{\infty} \Delta(a_n(z-z_0)^{\underline{n}}) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{\underline{n-1}}$$

for $z \in \mathring{\Omega}$.

Proof. For any point $z \in \mathring{\Omega}$, the series converges at $\{z, z+1, z+i\}$. So

$$\begin{aligned}
\Delta f(z) &= \frac{f(z+1) - f(z) - i(f(z+i) - f(z))}{2} \\
&= \frac{\sum a_n (z+1)^{\underline{n}} - \sum a_n (z)^{\underline{n}} - i\left(\sum a_n (z+i)^{\underline{n}} - \sum a_n (z)^{\underline{n}}\right)}{2} \\
&= \sum a_n \left(\frac{(z+1)^{\underline{n}} - z^{\underline{n}} - i((z+i)^{\underline{n}} - z^{\underline{n}})}{2}\right) \\
&= \sum a_n \Delta z^{\underline{n}}.
\end{aligned}$$

□

Definition 5.4. A function is *analytic* if it can be written locally as a convergent falling power series.

Proposition 5.5. *Analytic on* Ω *implies holomorphic on* $\mathring{\Omega}$.

Proof. As in the proof of Proposition 5.3, for $z \in \mathring{\Omega}$, the $\bar{\Delta}$ can be applied to the series term by term. For each n, $\bar{\Delta} z^{\underline{n}} = 0$ and so $\bar{\Delta} f(z) = 0$. Thus f is holomorphic on $\mathring{\Omega}$. □

This brings us one of the main results dealing with falling power series.

Theorem 5.6. *Holomorphic implies analytic.*

Proof. We may assume $z_0 = 0$ and the series converges everywhere in the first quadrant (Theorem 5.2). By interpolation, we can form a unique falling power series which agrees with the function f on the positive real line according to the recurrence relations

$$a_0 = f(0) \quad \text{and} \quad a_n = \frac{f(n) - \sum_{k=0}^{n-1} k^{\underline{n}} a_k}{n^{\underline{n}}}.$$

From Proposition 5.5, we know that the series is holomorphic, and by the weak identity theorem, since f agrees with this power series on the real line, then it agrees with the series in the whole first quadrant, and f is analytic there. □

From the proof of Theorem 5.6, follows the usual Taylor expression.

Corollary 5.7 (Taylor's theorem). *A holomorphic function f is locally given by the falling power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{\Delta^n f(z_0)}{n!} (z - z_0)^{\underline{n}}.$$

6. Elementary functions

First, a discrete analogue of the exponential function:

Proposition 6.1. *If* $\Delta f = f$ *and* $\bar{\Delta} f = 0$, *then*

$$f(x + iy) = 2^x (1+i)^y f(0).$$

Proof. Setting $f(z) = \Delta f(z)$ gives

$$f(z) = \frac{f(z+1) - f(z) - i(f(z+i) - f(z))}{2},$$

and $\bar{\Delta} f(z) = 0$ gives

$$\frac{f(z+1) - f(z) + i(f(z+i) - f(z))}{2} = 0.$$

After some simplification, we obtain

$$f(z+1) = 2f(z) \quad \text{and} \quad f(z+i) = (1+i) f(z).$$

With these two functional equations,

$$f(x+iy) = 2^x f(iy) = 2^x (1+i)^y f(0).$$ □

Definition 6.2. The *discrete complex exponential* is given by

$$\exp(z) = \exp(x+iy) = 2^x (1+i)^y.$$

Note that it satisfies a law of exponents, i.e., $\exp(z+w)=\exp(z)\exp(w)$. As a falling power series, for z in the first quadrant,

$$\exp(z)=\sum_{n=0}^{\infty}\frac{z^{\underline{n}}}{n!}.$$

Analogous to classical complex analysis where $e^{2\pi ik}=1$, we have:

Proposition 6.3. $\exp(z)=1$ *if and only if* $z=(4-8i)k$ *for some integer* k.

Proof. $\arg(2^x(1+i)^y)=y\cdot\arg(1+i)=y\pi/4$, which is a multiple of 2π if and only if y is a multiple of 8. Next, $|\exp(x+iy)|=|2^x|\cdot|1+i|^y=2^{x+y/2}$, which equals 1 if and only if $2x=-y$. We may conclude that $\exp(x+iy)=1$ if and only if $x+iy=(4-8i)k$ for some integer k. □

Next, we look for analogues of sine and cosine.

Proposition 6.4. *If* $-\Delta^2 f=f$ *and* $\bar{\Delta}f=0$ *then*

$$f(x+iy)=(1-i)^x2^y f(0).$$

Proof. If f is holomorphic at z, then $\Delta f(z)=\Delta_x f(z)=-i\Delta_y f(z)$. Hence,

$$-\Delta^2 f(z)=-\Delta_y\Delta_y f(z)=\Delta_y f(z+i)-\Delta_y f(z)=f(z+2i)-2f(z+i)+f(z).$$

Setting equal to $f(z)$ gives $f(z+2i)=2f(z+i)$, or by change of variables

$$f(z+i)=2f(z).$$

Also,

$$\begin{aligned}-\Delta^2 f(z)&=i\Delta_y\Delta_x f(z)=i\Delta_y f(z+1)-i\Delta_y f(z)\\&=if(z+1+i)-if(z+1)-if(z+i)+if(z).\end{aligned}$$

Setting equal to $f(z)$ and substituting $f(z+1+i)=2f(z+1)$ and $f(z+i)=2f(z)$ gives $(1+i)f(z)=if(z+1)$, or

$$f(z+1)=(1-i)f(z).$$

Combining results yields the solution $f(x+iy)=(1-i)^x2^y f(0)$. □

Motivated by the classical equation, $e^{x+iy}=e^x(\cos y+i\sin y)$, let us find an analogue for $\exp(x+iy)=2^x(1+i)^y$, by setting

$$c(t)=\operatorname{Re}(1+i)^t\quad\text{and}\quad s(t)=\operatorname{Im}(1+i)^t$$

for $t\in\mathbb{Z}$. With these definitions on the real line, define $c(x+iy)$ and $s(x+iy)$ for $y\geq 0$ by holomorphic extension to the upper half-plane.

Proposition 6.5. *For $y > 0$,*

$$c(x+iy) = \frac{(1-i)^x 2^y}{2} \quad \textit{and} \quad s(x+iy) = \frac{(1-i)^x 2^y}{-2i}.$$

Proof. By Proposition 6.4, the functions

$$(x+iy) \mapsto \frac{(1-i)^x 2^y}{2} \quad \text{and} \quad (x+iy) \mapsto \frac{(1-i)^x 2^y}{-2i}$$

are holomorphic everywhere. Hence by Lemma 5.1, it is sufficient to show that equality holds on the line $\operatorname{Im} z = 1$. Let $x \in \mathbb{Z}$. Since c is holomorphic at x, by Proposition 3.5,

$$\begin{aligned} c(x+i) &= (1-i)c(x) + ic(x+1) \\ &= (1-i)\operatorname{Re}(1+i)^x + i\operatorname{Re}(1+i)^{x+1} \\ &= (1-i)\operatorname{Re}(1+i)^x + i(\operatorname{Re}(1+i)^x - \operatorname{Im}(1+i)^x) \\ &= \operatorname{Re}(1+i)^x - i\operatorname{Im}(1+i)^x = \overline{(1+i)^x} = (1-i)^x, \end{aligned}$$

which equals $\frac{(1-i)^x 2^y}{2}$ for $y = 1$. Similarly,

$$\begin{aligned} s(x+i) &= (1-i)s(x) + is(x+1) \\ &= (1-i)\operatorname{Im}(1+i)^x + i\operatorname{Im}(1+i)^{x+1} \\ &= (1-i)\operatorname{Im}(1+i)^x + i(\operatorname{Re}(1+i)^x + \operatorname{Im}(1+i)^x) \\ &= i\operatorname{Re}(1+i)^x + \operatorname{Im}(1+i)^x = i\overline{(1+i)^x} = i(1-i)^x, \end{aligned}$$

which equals $\frac{(1-i)^x 2^y}{-2i}$ for $y = 1$. □

7. Path integration

Definition 7.1. A *path* γ of length n is a sequence $\{\gamma_j\}_{j=0}^n \subset \mathbb{Z}[i]$ such that

$$|\gamma_j - \gamma_{j-1}| = 1,$$

for every integer j such that $1 \le j \le n$. A closed path satisfies $\gamma_0 = \gamma_n$.

Definition 7.2. A *simply connected domain* Ω is a path-connected set of points $\{z \in \mathbb{Z}[i]\}$ with no holes, i.e., Ω is such that the interior of every closed path set lies inside Ω.

Definition 7.3. A *corner* of a path γ is a point γ_j with $0 < j < n$ such that

$$|\gamma_{j+1} - \gamma_{j-1}| \neq 2.$$

Definition 7.4. The *path integral* of f along γ is

$$\int_\gamma f(z) = \sum_{j=1}^{n} f\big(\min\{x_j, x_{j-1}\} + i\min\{y_j, y_{j-1}\}\big)(\gamma_j - \gamma_{j-1}),$$

where $x_j = \operatorname{Re}\gamma_j$ and $y_j = \operatorname{Im}\gamma_j$ for $0 \le j \le n$.

Lemma 7.5. *If γ is a path from γ_0 to γ_n with no corners and f is holomorphic everywhere along the path, then*

$$\int_\gamma \Delta f(z) = f(\gamma_n) - f(\gamma_0).$$

Proof. For a horizontal path oriented from left to right having no corners, $\gamma_j - \gamma_{j-1}$ is constant and equal to 1, so

$$\int_\gamma \Delta f(z) = \int_\gamma \Delta_x f(z) = \int_\gamma f(z+1) - f(z) = \sum_{j=1}^{n} f(\gamma_j) - f(\gamma_{j-1}),$$

which telescopes leaving $\int_\gamma \Delta f(z) = f(\gamma_n) - f(\gamma_0)$. For a horizontal path oriented from right to left, $\gamma_j - \gamma_{j-1} = -1$, so

$$\int_\gamma \Delta f(z) = -\sum_{j=1}^{n} f(\gamma_{j-1}) - f(\gamma_j) = f(\gamma_n) - f(\gamma_0).$$

For a vertical path oriented from bottom to top, $\gamma_j - \gamma_{j-1} = i$, so

$$\int_\gamma \Delta f(z) = \int_\gamma -i\,\Delta_y f(z) = -i\sum_{j=1}^{n} \big(f(\gamma_j) - f(\gamma_{j-1})\big)\, i = f(\gamma_n) - f(\gamma_0).$$

For a vertical path oriented from top to bottom, $\gamma_j - \gamma_{j-1} = -i$, so

$$\int_\gamma \Delta f(z) = \int_\gamma -i\,\Delta_y f(z) = -i\sum_{j=1}^{n} \big(f(\gamma_{j-1}) - f(\gamma_j)\big)(-i) = f(\gamma_n) - f(\gamma_0). \qquad \square$$

Theorem 7.6 (Fundamental Theorem). *If γ is a path from γ_0 to γ_n and f is holomorphic everywhere along the path, then*

$$\int_\gamma \Delta f(z) = f(z_n) - f(z_0).$$

Proof. Decompose γ into a union of paths having no corners:

$$\gamma = \gamma^1 + \gamma^2 + \cdots + \gamma^m.$$

Then

$$\begin{aligned}\int_\gamma \Delta f &= \int_{\gamma^1} \Delta f + \int_{\gamma^2} \Delta f + \cdots + \int_{\gamma^m} \Delta f \\ &= (f(\gamma^m_{n_m}) - f(\gamma^m_0)) + \cdots + (f(\gamma^1_{n_1}) - f(\gamma^1_0)) = f(\gamma_n) - f(\gamma_0). \qquad \square\end{aligned}$$

Corollary 7.7. *If $\Delta f(z) = 0$ on a path-connected set Ω, then $f(z)$ is constant on Ω.*

Proof. If z and w are in Ω, there exists a path in Ω from z to w. Since $\int_\gamma \Delta f(z) = 0$, it follows that $f(w) = f(z)$. $\square$

Lemma 7.8 (Goursat's lemma). *Let T be a unit square given by the path*

$$\{z,\ z+1,\ z+1+i,\ z+i,\ z\},$$

and suppose f is holomorphic at z. Then $\int_T f(z) = 0$.

Proof. $\int_T f(z) = f(z) + if(z+1) - f(z+i) - if(z) = 2i\,\bar{\Delta} f(z) = 0$. $\square$

The following corollary immediately follows.

Corollary 7.9 (Morera's theorem). *Let f be a function defined on a set G. If $\int_T f(z) = 0$ for all unit squares T whose interior point is contained in the interior of G, then f is holomorphic on the interior of G.*

Theorem 7.10 (Cauchy's theorem). *Let Ω be a simply connected domain and let γ be a closed path in Ω. Then*

$$\int_\gamma f(z) = 0,$$

for each function f that is holomorphic on Ω.

Proof. γ can be written as a canceling sum of unit squares, $T_1 + T_2 + \cdots + T_m$. Since Ω is simply connected, all of these squares lie in the interior of Ω. By Goursat's lemma, $\int_{T_1} f(z) = \int_{T_2} f(z) = \cdots = \int_{T_n} f(z) = 0$, and so

$$\int_\gamma f(z) = \int_{T_1} f(z) + \int_{T_2} f(z) + \cdots + \int_{T_n} = 0. \qquad \square$$

Theorem 7.11. *If f is holomorphic on a simply connected domain, Ω, then f has a primitive in Ω.*

Proof. Fix $z_0 \in \Omega$. Let $F(z) = \int_\gamma f(w)$ where $\gamma_0 = z_0$ and $\gamma_n = z$. By Cauchy's Theorem, this function is path-independent and well-defined.

$$\begin{aligned}
\Delta F(z) &= \Delta \int_\gamma f(w) \\
&= \frac{\int_{z_0}^{z+1} f(w) - \int_{z_0}^{z} f(w) - i\left(\int_{z_0}^{z+i} f(w) - \int_{z_0}^{z} f(w)\right)}{2} \\
&= \frac{\int_{z_0}^{z} f(w) + f(z) - \int_{z_0}^{z} f(w) - i\left(\int_{z_0}^{z} f(w) + if(z) - \int_{z_0}^{z} f(w)\right)}{2} \\
&= f(z).
\end{aligned}$$

$\square$

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