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A note on nonresidually solvable hyperlinear one-relator groups

Jon P. Bannon and Nicholas Noble

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A note on nonresidually solvable hyperlinear one-relator groups

Jon P. Bannon and Nicholas Noblett

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We prove that various nonresidually finite, nonresidually solvable groups of the form $\langle a, b \mid r^{r^w} = r^2 \rangle$ are sofic.

This paper concerns the sofic property discussed in the survey [Pestov 2008]. Particularly, we address Question 4.10 in that paper: the problem of Nate Brown asking whether or not every one-relator group is sofic. In [Bannon 2010], it is proved that the example in [Baumslag 1969] of a nonresidually finite nonresidually solvable one-relator group is a sofic group. The purpose of this paper is to exhibit more such examples in the following large class of nonresidually solvable one-relator groups introduced in [Baumslag et al. 2007]. Let $\mathbb{F}_2 = \langle a, b \mid \rangle$ denote the free group on two generators. Let $r, w \in \mathbb{F}_2$ be two elements that do not commute. In [Baumslag et al. 2007], the authors show that the group

$$\Gamma_{r,w} = \langle a, b \mid r^{r^w} = r^2 \rangle = \langle a, b \mid r = [r, (r^{-1})^w] \rangle$$

has the same finite quotients as the group

$$\langle a, b \mid r \rangle,$$

and is therefore not residually finite. We point out that none of the groups $\Gamma_{r,w}$ are residually solvable, since $r = [r, (r^{-1})^w]$ lies in every derived subgroup of $\Gamma_{r,w}$. In [Bannon 2010], it is shown that the group $\Gamma_{ab,a}$ is sofic. The proof in [Bannon 2010] uses [Dykema 2010, Corollary 3.4], that HNN extensions of sofic groups over amenable subgroups remain sofic. The proof in [Bannon 2010] uses the fact that $\Gamma_{ab,a}$ is an HNN extension of an amenable one-relator group. We shall extend this result to certain other of the groups $\Gamma_{r,w}$. If r and w generate \mathbb{F}_2 , then $\Gamma_{r,w}$ embeds naturally as a subgroup of $\Gamma_{ab,a}$, and since the sofic property passes to subgroups, $\Gamma_{r,w}$ is sofic. The first result of this short note is that there exist r, w that do not generate \mathbb{F}_2 , yet the group $\Gamma_{r,w}$ is sofic. More precisely, we prove:

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Theorem 1. *The group $\Gamma_{a,b^{-1}ab}$ is sofic.*

Proof. Since $\Gamma_{a,b^{-1}ab} = \langle a, b \mid (bab^{-1})^{-2}a^{-1}(bab^{-1})^{-1}a(bab^{-1})a^{-1}(bab^{-1})a \rangle$, following [McCool and Schupp 1973], we let $a_0 = a$ and $a_{-1} = bab^{-1}$ and realize $\Gamma_{a,b^{-1}ab}$ as the HNN extension

$$\langle a_0, a_{-1}, t \mid (a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1}a_0, t^{-1}a_{-1}t = a_0 \rangle$$

of the one-relator group $H_1 = \langle a_0, a_{-1} \mid a_0(a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1} \rangle$, where by the Freiheitssatz $\langle a_{-1} \rangle$ and $\langle a_0 \rangle$ are copies of \mathbb{Z} which in the HNN extension we identify by identifying a_{-1} with a_0 . Letting $b_1 = a_0a_{-1}a_0^{-1}$ and $b_0 = a_{-1}$ we may identify H_1 as the HNN extension

$$\langle b_0, b_1, s \mid b_1^{-2}b_0^{-1}b_1b_0, s^{-1}b_1s = b_0 \rangle$$

of the one-relator group $H_2 = \langle b_0, b_1, s \mid b_1^{-2}b_0^{-1}b_1b_0 \rangle$, where we identify the two copies $\langle b_0 \rangle$ and $\langle b_1 \rangle$ of \mathbb{Z} as above. By [Ceccherini-Silberstein and Grigorchuk 1997], the group H_2 is amenable, and hence by the argument in [Bannon 2010], the group H_1 is sofic. Since $\Gamma_{a,b^{-1}ab}$ is an HNN extension of a sofic group with respect to identified copies of the amenable group \mathbb{Z} , it follows that $\Gamma_{a,b^{-1}ab}$ is sofic. \square

In this proof we used in an essential way that the identified subgroups are amenable and therefore invoke the full hypotheses of Corollary 3.4 of [Dykema 2010], whereas in [Bannon 2010], the group $\Gamma_{ab,a}$ is an HNN extension of an amenable group and so any pair of identified subgroups would work. We next illustrate that there are groups of the form $\Gamma_{r,w}$ that do not in an obvious way fall to the method of [Bannon 2010].

Theorem 2. *The group $\Gamma_{a,b^2} = \langle a, b \mid a = [a, (a^{-1})^{b^2}] \rangle$ is isomorphic to*

$$(G * \mathbb{Z}) *_{\mathbb{F}_2},$$

where G is a one-relator amenable group.

Proof. Since $\Gamma_{a,b^2} = \langle a, b \mid a^{-2}(b^2ab^{-2})a(b^2ab^{-2})^{-1} \rangle$, then letting $a_0 = a$ and $a_{-2} = b^2ab^{-2}$ we have that Γ_{a,b^2} is isomorphic to the HNN extension

$$\langle a_0, a_{-1}, a_{-2}, t \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1}, t^{-1}a_{-2}t = a_{-1}, t^{-1}a_{-1}t = a_0 \rangle$$

of the one-relator group $\langle a_0, a_{-1}, a_{-2} \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle$, with the isomorphism from the free subgroup $\langle a_{-2}, a_{-1} \rangle$ with $\langle a_{-1}, a_0 \rangle$ extending the set map that sends a_{-2} to a_{-1} and a_{-1} to a_0 . But the relator $a_0^{-2}a_{-2}a_0(a_{-2})^{-1}$ does not involve a_{-1} , hence $\langle a_0, a_{-1}, a_{-2} \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle = \langle a_{-1} \rangle * \langle a_0, a_{-2} \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle$. \square

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no. 3

Gracefulness of families of spiders	241
PATRICK BAHLS, SARA LAKE AND ANDREW WERTHEIM	
Rational residuacity of primes	249
MARK BUDDEN, ALEX COLLINS, KRISTIN ELLIS LEA AND STEPHEN SAVIOLI	
Coexistence of stable ECM solutions in the Lang–Kobayashi system	259
ERICKA MOCHAN, C. DAVIS BUENGER AND TAMAS WIANDT	
A complex finite calculus	273
JOSEPH SEABORN AND PHILIP MUMMERT	
$\zeta(n)$ via hyperbolic functions	289
JOSEPH D’AVANZO AND NIKOLAI A. KRYLOV	
Infinite family of elliptic curves of rank at least 4	297
BARTOSZ NASKRĘCKI	
Curvature measures for nonlinear regression models using continuous designs with applications to optimal experimental design	317
TIMOTHY O’BRIEN, SOMSRI JAMROENPINYO AND CHINNAPHONG BUMRUNGSUP	
Numerical semigroups from open intervals	333
VADIM PONOMARENKO AND RYAN ROSENBAUM	
Distinct solution to a linear congruence	341
DONALD ADAMS AND VADIM PONOMARENKO	
A note on nonresidually solvable hyperlinear one-relator groups	345
JON P. BANNON AND NICOLAS NOBLETT	