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A note on nonresidually solvable hyperlinear one-relator groups

Jon P. Bannon and Nicholas Noblett

mathematical sciences publishers

2010 vol. 3, no. 3

INVOLVE 3:3(2010)

A note on nonresidually solvable hyperlinear one-relator groups

Jon P. Bannon and Nicholas Noblett

(Communicated by David R. Larson)

We prove that various nonresidually finite, nonresidually solvable groups of the form $\langle a, b \,|\, r^{r^w} = r^2 \rangle$ are sofic.

This paper concerns the sofic property discussed in the survey [Pestov 2008]. Particularly, we address Question 4.10 in that paper: the problem of Nate Brown asking whether or not every one-relator group is sofic. In [Bannon 2010], it is proved that the example in [Baumslag 1969] of a nonresidually finite nonresidually solvable one-relator group is a sofic group. The purpose of this paper is to exhibit more such examples in the following large class of nonresidually solvable one-relator groups introduced in [Baumslag et al. 2007]. Let $\mathbb{F}_2 = \langle a, b \,|\, \rangle$ denote the free group on two generators. Let $r, w \in \mathbb{F}_2$ be two elements that do not commute. In [Baumslag et al. 2007], the authors show that the group

$$\Gamma_{r,w} = \langle a, b \,|\, r^{r^w} = r^2 \rangle = \langle a, b \,|\, r = [r, (r^{-1})^w] \rangle$$

has the same finite quotients as the group

$$\langle a, b \,|\, r \rangle,$$

and is therefore not residually finite. We point out that none of the groups $\Gamma_{r,w}$ are residually solvable, since $r = [r, (r^{-1})^w]$ lies in every derived subgroup of $\Gamma_{r,w}$. In [Bannon 2010], it is shown that the group $\Gamma_{ab,a}$ is sofic. The proof in [Bannon 2010] uses [Dykema 2010, Corollary 3.4], that HNN extensions of sofic groups over amenable subgroups remain sofic. The proof in [Bannon 2010] uses the fact that $\Gamma_{ab,a}$ is an HNN extension of an amenable one-relator group. We shall extend this result to certain other of the groups $\Gamma_{r,w}$. If r and w generate $\mathbb{F}_2$, then $\Gamma_{r,w}$ embeds naturally as a subgroup of $\Gamma_{ab,a}$, and since the sofic property passes to subgroups, $\Gamma_{r,w}$ is sofic. The first result of this short note is that there exist r, w that do not generate $\mathbb{F}_2$, yet the group $\Gamma_{r,w}$ is sofic. More precisely, we prove:

MSC2000: primary 46L10; secondary 20F65.

Keywords: mathematics.

Noblett is an undergraduate at Siena College, Loudonville, NY.

Theorem 1. *The group $\Gamma_{a,b^{-1}ab}$ is sofic.*

Proof. Since $\Gamma_{a,b^{-1}ab} = \langle a, b \,|\, (bab^{-1})^{-2}a^{-1}(bab^{-1})^{-1}a(bab^{-1})a^{-1}(bab^{-1})a\rangle$, following [McCool and Schupp 1973], we let $a_0 = a$ and $a_{-1} = bab^{-1}$ and realize $\Gamma_{a,b^{-1}ab}$ as the HNN extension

$$\langle a_0, a_{-1}, t \,|\, (a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1}a_0, t^{-1}a_{-1}t = a_0\rangle$$

of the one-relator group $H_1 = \langle a_0, a_{-1} \,|\, a_0(a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1}\rangle$, where by the Freiheitssatz $\langle a_{-1}\rangle$ and $\langle a_0\rangle$ are copies of $\mathbb{Z}$ which in the HNN extension we identify by identifying a_{-1} with a_0. Letting $b_1 = a_0a_{-1}a_0^{-1}$ and $b_0 = a_{-1}$ we may identify H_1 as the HNN extension

$$\langle b_0, b_1, s \,|\, b_1^{-2}b_0^{-1}b_1b_0,\ s^{-1}b_1s = b_0\rangle$$

of the one-relator group $H_2 = \langle b_0, b_1, s \,|\, b_1^{-2}b_0^{-1}b_1b_0\rangle$, where we identify the two copies $\langle b_0\rangle$ and $\langle b_1\rangle$ of $\mathbb{Z}$ as above. By [Ceccherini-Silberstein and Grigorchuk 1997], the group H_2 is amenable, and hence by the argument in [Bannon 2010], the group H_1 is sofic. Since $\Gamma_{a,b^{-1}ab}$ is an HNN extension of a sofic group with respect to identified copies of the amenable group $\mathbb{Z}$, it follows that $\Gamma_{a,b^{-1}ab}$ is sofic. □

In this proof we used in an essential way that the identified subgroups are amenable and therefore invoke the full hypotheses of Corollary 3.4 of [Dykema 2010], whereas in [Bannon 2010], the group $\Gamma_{ab,a}$ is an HNN extension of an amenable group and so any pair of identified subgroups would work. We next illustrate that there are groups of the form $\Gamma_{r,w}$ that do not in an obvious way fall to the method of [Bannon 2010].

Theorem 2. *The group $\Gamma_{a,b^2} = \langle a, b \,|\, a = [a, (a^{-1})^{b^2}]\rangle$ is isomorphic to*

$$(G * \mathbb{Z})*_{\mathbb{F}_2},$$

where G is a one-relator amenable group.

Proof. Since $\Gamma_{a,b^2} = \langle a, b \,|\, a^{-2}(b^2ab^{-2})a(b^2ab^{-2})^{-1}\rangle$, then letting $a_0 = a$ and $a_{-2} = b^2ab^{-2}$ we have that Γ_{a,b^2} is isomorphic to the HNN extension

$$\langle a_0, a_{-1}, a_{-2}, t \,|\, a_0^{-2}a_{-2}a_0(a_{-2})^{-1}, t^{-1}a_{-2}t = a_{-1}, t^{-1}a_{-1}t = a_0\rangle$$

of the one-relator group $\langle a_0, a_{-1}, a_{-2} \,|\, a_0^{-2}a_{-2}a_0(a_{-2})^{-1}\rangle$, with the isomorphism from the free subgroup $\langle a_{-2}, a_{-1}\rangle$ with $\langle a_{-1}, a_0\rangle$ extending the set map that sends a_{-2} to a_{-1} and a_{-1} to a_0. But the relator $a_0^{-2}a_{-2}a_0(a_{-2})^{-1}$ does not involve a_{-1}, hence $\langle a_0, a_{-1}, a_{-2} \,|\, a_0^{-2}a_{-2}a_0(a_{-2})^{-1}\rangle = \langle a_{-1}\rangle * \langle a_0, a_{-2} \,|\, a_0^{-2}a_{-2}a_0(a_{-2})^{-1}\rangle$. □

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Received: 2010-07-26 Revised: Accepted: 2010-07-26

jbannon@siena.edu — *Siena College, Department of Mathematics, Loudonville, NY 12211, United States*

nb11nobl@siena.edu — *Siena College, Department of Mathematics, Loudonville, NY 12211, United States*

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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