involve
a journal of mathematics

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msp

2010 vol. 3, no. 4

INVOLVE 3:4 (2010)
https://doi.org/10.2140/involve.2010.3.363

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(Communicated by Zuhair Nashed)

Chaos theory examines the iterates of continuous functions to draw conclusions about long-term behavior. As this relatively new theory has evolved, one difficulty still present is the absence of universally agreed upon definitions. On the other hand, function spaces and equicontinuity are well established concepts with mathematical definitions that are universally accepted. We will present some theorems that display the natural connections between chaos and equicontinuity.

1. Introduction

Modern dynamical systems theory began in 1890, when Henri Poincaré was studying the three-body problem. He discovered the existence of aperiodic points that approach neither infinity nor a fixed point. Although this chaotic behavior was observed in 1890, it was not until around 1960 that chaos was formally studied. The invention of the electronic computer made studying chaos possible, by allowing one to iterate a simple function many times.

It can be said that chaos occurs when a deterministic system appears to be random. Edward Lorenz observed this phenomenon in 1961, when small round off errors led to unexpected results. The sensitive dependence on initial conditions caused his deterministic system to appear random. This can be examined using chaos theory.

There is no universal agreement on what the precise definition of a chaotic system should be. Many authors use different definitions to describe similar concepts [Devaney 1986; Li and Yorke 1975; Martelli 1999; Robinson 1999]. It is easy to see how this could be a problem and it would be useful to unify the terminology. A good starting point would be relating chaos to the classic idea of equicontinuity.

Well over one hundred years ago, equicontinuous families of functions were introduced in [Arzelà 1895; Ascoli 1884]. Equicontinuity allowed Giulio Ascoli and Cesare Arzelà to understand the behavior of families of continuous functions.

MSC2000: 37D45, 54C05, 54C35.
Keywords: equicontinuity, chaos.

Since chaos describes the behavior of a family of iterates of a continuous function, it seems natural to examine chaotic systems in terms of equicontinuity.

2. Equicontinuity

Ascoli and Arzelá used equicontinuity to explain which families of continuous functions will be "well behaved". Their results now have many important applications throughout mathematics. We will first introduce their classic definition.

A function space is defined to be the set of functions from a space X into a space Y, denoted by Y^X. For our purposes we restrict our function space to the set of continuous functions from X to Y, denoted by $\mathscr{C}(X, Y)$.

Definition 2.1. Let (Y, d) be a metric space. Let $\mathscr{F}$ be a subset of the function space $\mathscr{C}(X, Y)$. If $x_0 \in X$, the set $\mathscr{F}$ of functions is said to be *equicontinuous at* x_0 if given $\epsilon > 0$, there is a neighborhood U of x_0 such that for all $x \in U$ and all $f \in \mathscr{F}$,

$$d\,(f(x),\, f(x_0)) < \epsilon.$$

If the set $\mathscr{F}$ is equicontinuous at x_0 for each $x_0 \in X$, it is said simply to be *equicontinuous*.

Many times in topology, it is important to know when a set will be compact. Informally speaking, if a family of continuous functions is compact, then it will be well behaved. More precisely, if a sequence from a compact family of continuous functions converges in the supremum metric, then it must converge to a continuous function. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. As Ascoli and Arzelá determined, equicontinuity is the additional property needed to assure that a closed and bounded subset of a family of continuous functions will be compact.

Theorem 2.2 (Arzelà–Ascoli theorem). *Let X be a compact space, and consider $\mathscr{C}(X, \mathbb{R}^n)$ in the* sup *metric. A subset $\mathscr{F}$ of $\mathscr{C}(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded, and equicontinuous.*

3. Chaos

Chaos theory describes behavior that is the diametric opposite of well behaved. The first time the word *chaos* was used to describe this mathematical phenomenon was in [Li and Yorke 1975], where the authors described when the iterates of a continuous function on an interval of the real line exhibit chaotic behavior. The definition of a chaotic function in that paper does not conveniently generalize, so most subsequent authors have chosen to use a different definition of chaos.

One commonly cited definition of a chaotic function, given in Robert Devaney's book [1986], involves two important characteristics: topological transitivity and sensitive dependence on initial conditions. But as for chaos itself, multiple definitions exist for these two ideas (see [Devaney 1986; Martelli 1999; Robinson 1999; 2004]). We will adopt those used in [Robinson 2004].

Definition 3.1. $f : X \to X$ is said to be *topologically transitive* if there exists $x \in X$ such that $\{f^n(x) \mid n \in \mathbb{Z}^+\}$ is dense in X.

Definition 3.2. Let f be a map on a metric space X. The map has *sensitive dependence on initial conditions at* x_0, provided that there exists $\epsilon > 0$ such that, for any $\delta > 0$, there exists a y_0 such that $d(x_0, y_0) < \delta$ and a $n > 0$ such that

$$d(f^n(x_0), f^n(y_0)) \geq \epsilon.$$

The map has *sensitive dependence on initial conditions on a set* A, provided that it has sensitive dependence on initial conditions at every points $x_0 \in A$.

These are the two characteristic properties of a chaotic function, according to [Robinson 1999].

Definition 3.3. A subset $S \subseteq X$ is said to be *invariant* under f provided $f(S) = S$.

Definition 3.4. A map f on a metric space X is said to be *chaotic on an invariant set* Y provided (i) f restricted to Y is topologically transitive and (ii) f restricted to Y has sensitive dependence on initial conditions.

This idea of sensitive dependence on initial conditions appears to be related to an equicontinuous family of functions. The following section will explore the connections between chaos and equicontinuity.

4. Connections

If a family of iterates of a continuous function is equicontinuous at a point, then all iterates of nearby points will remain close together. This idea appears contrary to sensitive dependence on initial conditions, so the following theorem is natural.

Theorem 4.1 [Henry and Trapp 1998]. *Let X be a metric space, $f : X \to X$ be a continuous function, $x \in X$, and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. Then f has sensitive dependence on initial conditions at x, if and only if $\mathcal{F}$ is not equicontinuous at x.*

Proof. Suppose that $\mathcal{F}$ is not equicontinuous at $x \in X$. Then there exists an $\epsilon > 0$ such that, for any $\delta > 0$, there exists a y such that $d(x, y) < \delta$ and an $n > 0$ such that $d(f^n(x), f^n(y)) \geq \epsilon$. Therefore, $\mathcal{F}$ being equicontinuous at x is the negation of f having sensitive dependence on initial conditions at x. $\square$

It becomes clear that equicontinuity is closely related to chaos. There are additional natural connections, so we continue to show the various forms in which equicontinuity appears. One definition that appears throughout texts on chaos is stability.

Definition 4.2. A point p is *Lyapunov stable* provided given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - p| < \delta$ then $\left|f^j(x) - f^j(p)\right| < \epsilon$, for all $j \geq 0$.

Theorem 4.3. *Let X be a metric space, $f : X \to X$ be a continuous function, and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. Then $\mathcal{F}$ is equicontinuous at $x \in X$ if and only if x is Lyapunov stable.*

Proof. First suppose that $\mathcal{F}$ is equicontinuous at $x \in X$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that

$$d(f^n(x), f^n(y)) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Without loss of generality, we may assume $\delta \leq \epsilon$. Thus, $d(x, y) < \epsilon$. Therefore,

$$d(f^n(x), f^n(y)) < \epsilon \quad \text{for all } n \geq 0.$$

So f is Lyapunov stable at x. Now if f is Lyapunov stable at x, then $\mathcal{F}$ is clearly equicontinuous at x. □

Definition 4.4 (Stable fixed point [Henry and Trapp 1998]). Let p be a fixed point of f. We call p a stable fixed point provided there is a neighborhood U of p such that

$$\lim_{n\to\infty} \operatorname{diam}(f^n(U)) = 0.$$

Definition 4.5. A point p is called *periodic* if $f^n(p) = p$ for some $n \in \mathbb{N}$. The smallest such n is the *period* of p.

Two theorems relating this definition to equicontinuity are proved in [Henry and Trapp 1998]. We say that p is a stable periodic point if and only if p is a stable fixed point of f^n for some $n \in \mathbb{N}$.

Theorem 4.6 [Henry and Trapp 1998]. *Let p be a fixed point of f. If p is stable then the iterates of f are equicontinuous at p.*

Theorem 4.7 [Henry and Trapp 1998]. *If p is a stable periodic point, then f has equicontinuous iterates at p.*

Another definition that appears extensively in chaos theory is that of an ω-limit set.

Definition 4.8. A point y is an *ω-limit point of x for f* provided there exists a sequence of n_k going to infinity as k goes to infinity such that

$$\lim_{k\to\infty} d(f^{n_k}(x), y) = 0.$$

The set of all ω-limit points of x for f is called the *ω-limit set of x* and is denoted by $\omega(x)$ or $\omega(x, f)$.

A useful characterization of an ω-limit set is given by the following theorem.

Theorem 4.9 [Robinson 1999]. *Let $f : X \to X$ be a continuous function on a metric space X. Then for any $x \in X$, $\omega(x) = \bigcap_{N \geq 0} \mathrm{cl}(\bigcup_{n \geq N} \{f^n(x)\})$.*

Notice that if $\omega(x) = X$, then $\{f^n(x) \mid n \in \mathbb{Z}^+\}$ is dense in X. This is useful for showing a system is topologically transitive. We will now prove a theorem that allows us to connect ω-limit sets to equicontinuity.

Theorem 4.10. *Let X be a metric space, $f : X \to X$ be a continuous function, and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. Now suppose that $x \in X$ is a point at which $\mathcal{F}$ is equicontinuous. If $y \in X$ and $x \in \mathrm{cl}(\bigcup_{n \geq 0} \{f^n(y)\})$, then $\mathcal{F}$ is equicontinuous at y.*

Proof. Suppose $x \in X$ is a point at which $\mathcal{F}$ is equicontinuous and

$$x \in \mathrm{cl}\big(\textstyle\bigcup_{n \geq 0} \{f^n(y)\}\big).$$

Let $\epsilon > 0$ be given. By equicontinuity of $\mathcal{F}$, there exists a $\delta_1 > 0$ such that $d(x, \alpha) < \delta_1$ implies that $d(f^n(x), f^n(\alpha)) < \epsilon/2$, for all $n \in \mathbb{N}$. Without loss of generality, we may take $\delta_1 < \epsilon$. Now let $x \in \mathrm{cl}(\bigcup_{n \geq 0} \{f^n(y)\})$. So there exists an $m \in \mathbb{N}$ such that $d(x, f^m(y)) < \delta_1/2$. Since any finite set of continuous functions is equicontinuous, there exists $\delta_2 > 0$ such that $d(y, \beta) < \delta_2$ implies that $d(f^i(y), f^i(\beta)) < \delta_1/2$ for $1 \leq i \leq m$. Thus

$$d(x, f^m(\beta)) \leq d(x, f^m(y)) + d(f^m(y), f^m(\beta)) < \delta_1/2 + \delta_1/2 = \delta_1.$$

Hence for $j > m$,

$$d(f^j(y), f^j(\beta)) \leq d(f^j(y), f^{j-m}(x)) + d(f^{j-m}(x), f^j(\beta)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

So $d(y, \beta) < \delta_2$ implies that for all $n \in \mathbb{N}$,

$$d(f^n(y), f^n(\beta)) \leq \max\{d(f^i(y), f^i(\beta)), d(f^j(y), f^j(\beta)) \mid 1 \leq i \leq m,\ j > m\} < \epsilon. \quad \square$$

Corollary 4.11. *Let X be a metric space, $f : X \to X$ be a continuous function, and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. Now suppose that $x \in X$ is a point at which $\mathcal{F}$ is equicontinuous. If $x \in \omega(y)$ then $\mathcal{F}$ is equicontinuous at $y \in X$.*

Proof. Let $\mathcal{F}$ be equicontinuous at $x \in X$ and suppose $x \in \omega(y)$. But

$$x \in \omega(y) \subseteq \mathrm{cl}\Big(\bigcup_{n \geq 0} \{f^n(y)\}\Big),$$

so $\mathcal{F}$ is equicontinuous at x. $\square$

The previous theorems have shown how chaos can be recast in terms of equicontinuity. This is possible because of the intimate connection between chaos and equicontinuity as the following theorem and its corollaries show.

Theorem 4.12. *Let X be a metric space, $f : X \to X$ be a continuous function, and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. Suppose there exists a point $x \in X$ such that $\mathcal{F}$ is equicontinuous at x. If $\omega(y) = X$, then $\mathcal{F}$ is equicontinuous on $\{f^n(y) \mid n \in \mathbb{Z}^+\}$.*

Proof. If $\omega(y) = X$, then $x \in \omega(y)$. Since x is a point at which $\mathcal{F}$ is equicontinuous, y must also be a point at which $\mathcal{F}$ is equicontinuous. But notice that for any $n \in \mathbb{N}$, $\omega(f^n(y)) = \omega(y) = X$. So for all $n \in \mathbb{N}$, $f^n(y)$ is also a point at which $\mathcal{F}$ is equicontinuous. Therefore, $\mathcal{F}$ is equicontinuous on $\{f^n(y) \mid n \in \mathbb{Z}^+\}$. □

Corollary 4.13 [Kolyada 2004]. *Let X be a metric space, $f : X \to X$ be a continuous onto function, and $\mathcal{F} = \{f^n \mid n \in \mathbb{Z}^+\}$. Suppose that there exists a point $x \in X$ such that $\omega(x) = X$. Then $\mathcal{F}$ is equicontinuous on a dense subset of X, if and only if f is not chaotic on X.*

Proof. First suppose that f is not chaotic on X. Since $\omega(x) = X$, f is topologically transitive on X. Thus f must not have sensitive dependence on initial conditions on X. Hence there exists a point at which $\mathcal{F}$ is equicontinuous. Since $\omega(x) = X$, $\mathcal{F}$ is equicontinuous on $\{f^n(x) \mid n \in \mathbb{Z}^+\}$. But $\{f^n(x) \mid n \in \mathbb{Z}^+\}$ is dense in X.

Now suppose that $\mathcal{F}$ is not equicontinuous on a dense subset of X. Since $\omega(x) = X$, there are no points in X at which $\mathcal{F}$ is equicontinuous. Thus f has sensitive dependence on initial conditions on X. Therefore f is chaotic on X. □

Definition 4.14 (Minimal Set [Robinson 1999]). A set S is a minimal set for f provided (i) S is a closed, nonempty, invariant set and (ii) if B is a closed, nonempty, invariant subset of S, then $B = S$.

Lemma 4.15 [Robinson 1999]. *Let X be a metric space, $f : X \to X$ a continuous map, and $S \subseteq X$ a nonempty compact subset. Then, S is a minimal set if and only if $\omega(x) = S$ for all $x \in S$.*

Corollary 4.16 [Kolyada 2004]. *Let X be a metric space, $f : X \to X$ be a continuous function, S be a nonempty compact minimal subset of X, and $\mathcal{F} = \{f^n \mid n \in \mathbb{N}\}$. Then either f is chaotic on S or $\mathcal{F}$ is equicontinuous on S.*

Proof. First suppose that there is a point $x \in S$ such that $\mathcal{F}$ is equicontinuous at x. Then since S is minimal, $x \in \omega(y)$ for all $y \in S$. Thus $\mathcal{F}$ is equicontinuous on S. But if there are no points in S at which $\mathcal{F}$ is equicontinuous, then f is chaotic on S. □

5. Conclusion

Many definitions have been used to describe when a continuous function will be chaotic. Since we are relating chaos to equicontinuity, we propose to define chaos in terms of a family of continuous functions. Abstracting the connections from the previous section, we offer the following definition of a chaotic family of continuous functions.

Definition 5.1. Let (Y, d) be a metric space. Let $\mathscr{F}$ be a subset of the function space $\mathscr{C}(X, Y)$. Then $\mathscr{F}$ is *chaotic* if there exists $x \in X$ such that $\{f(x) \mid f \in \mathscr{F}\}$ is dense in Y and $\mathscr{F}$ is not equicontinuous at x.

Now suppose that we let X be a metric space, $f : X \to X$ be a continuous function, and $\mathscr{F} = \{f^n \mid n \in \mathbb{N}\}$. Then the definition of f being chaotic with our first definition of chaos is equivalent to $\mathscr{F}$ being chaotic with this definition.

It is our belief that chaos terminology should be unified. We have displayed the intrinsic relations of the classical concept of equicontinuity and the modern idea of chaos, hoping to help unify the definitions within chaos theory.

Acknowledgments

I thank my mentor, Dr. Don Reynolds, for his guidance, wisdom, and support. I also thank the Office of Research and Sponsored Programs from the University of Wisconsin – Eau Claire for providing financial support.

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Received: 2009-12-07 Accepted: 2010-11-12

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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Typeset in LaTeX

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2010 vol. 3 no. 4