involve
a journal of mathematics

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2010 vol. 3, no. 4

INVOLVE 3:4 (2010)
https://doi.org/10.2140/involve.2010.3.459

Computing corresponding values of the Neumann and Dirichlet boundary values for incompressible Stokes flow

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(Communicated by Kenneth S. Berenhaut)

We consider the Stokes equation for a flow through a partially obstructed channel and determine the relationship between Dirichlet boundary values (velocities) and Neumann boundary values (forces) for the FEM discrete form. For the steady state case we find a linear relationship. For the transient case the relationship depends on the time stepping procedure and includes the relationship at prior states. We resolve the issue for trapezoid and Adams–Bashford-2 time stepping. Since Stokes flow may be considered as the startup phase of Navier–Stokes flow, we give particular attention to a flow with a startup function.

1. Introduction

Our interest in boundary value questions for incompressible Stokes flow arises from the following setting. Commonly, finite element methods (FEM) are used to derive approximate solutions for the vector field of an incompressible fluid flow. These techniques involve first rendering a discrete form of the Navier–Stokes equation for the spatial variables via FEM and then employing finite difference techniques (FDM) to realize the flow in time. In this context the nonlinearity of the Navier–Stokes equations requires knowledge of the prior flow state at each time step. In practice the flow is assumed to begin at rest and then pass through a Stokes phase when the Reynolds number is small. The end step of this phase then provides the initial step data for the time step FDM applied to the Navier–Stokes equations. Authors often emphasize the importance of the Stokes phase to success in the resulting calculations with the Navier–Stokes phase [Gresho and Sani 2000].

When setting up the linear system of equations for a flow problem, the boundary values are initially applied in the Stokes phase then carried forward to the Navier–Stokes phase. For the case of a channel flow past an obstruction, authors commonly

MSC2000: 47N40.
Keywords: incompressible Stokes flow, finite element method, boundary values, computational fluid dynamics.

set values for the velocity field at the inflow edge, that is, they set Dirichlet boundary values. From a mathematical point of view, the flow problem could just as well be set up by assuming values for the force at that edge, that is, Neumann boundary values. This leads us to inquire how these two approaches differ if at all. Indeed, in [Gresho et al. 1981] the authors demonstrate the calculated flow vector field of an obstructed channel flow based on Neumann boundary values at the inflow edge. Interestingly, the authors state that the Neumann values are derived Dirichlet values. In particular they have postulated values for the velocity at the inflow, converted these velocities to forces and then proceeded with the Neumann boundary values.

In our investigation we determine a simple relationship between Neumann and Dirichlet boundary values for the steady state case. Carrying this forward we consider two common FDM techniques used for nonsteady or transient flows, *trapezoid* and *Adams-Bashford-2*. There are correspondences in the nonsteady case, but they are more complicated. In this case it is clear that an initial setting of forces or velocities result in very different outcomes. Indeed, by setting a startup function for force and then calculating the corresponding startup function for velocity results in a different startup velocity function at each applied node.

Although it is always mathematically possible to set Neumann boundary values for a node at the flow, this is not the case for Dirichlet boundary values. Indeed, the admissibility of Dirichlet boundary values lies in the physics not the mathematics. As our investigation is mathematical or linear algebraic, we decided to define a term to identify linear systems which admit Dirichlet boundary values.

In Section 2 we state the notation for the linear system arising from the Galerkin FEM applied to an incompressible Stokes flow. We also use this section to introduce an example. Later we use this example to demonstrate the results of Sections 3 and 4. In Section 3 we consider the steady state problem. Here we state results in a manner which is applicable to the nonsteady case. Finally the nonsteady case is handled in Section 4. Here we derive formulae relating Neumann boundary values to Dirichlet values and vice-versa. In both sections we provide point plots which demonstrate the formula for the example case.

We have included a note at the end to delineate the details of the example case.

2. Preliminaries

We begin by stating the governing equations for an incompressible Stokes flow. As we are primarily concerned with laminar flow we state the equations is two spatial dimensions.

Let $\vec{u} = \vec{u}(t, x, y) = (u(t, x, y), v(t, x, y))$ be a time dependent vector field in $\mathbb{R}^2$, and $P = P(t, x, y)$ be a real valued function. In addition $\vec{G} = (g_1, g_2)$ is a time dependent vector field. In these equations $\vec{u} = (u, v)$ denotes the velocity field, P is the pressure and $\vec{G}$ represents external body forces such as gravity. Further suppose that $\vec{u}$ and P are sufficiently differentiable to support the following.

$$\frac{\partial \vec{u}}{\partial t} + \nabla P - \nu \nabla \cdot (\nabla \vec{u} + (\nabla \vec{u})^T) - \vec{G} = 0, \tag{2-1}$$

$$\nabla \cdot \vec{u} = 0, \tag{2-2}$$

Equation (2-1) is the *Stokes equation*. It is the Navier–Stokes equation with the inertial or convection term removed. It applies to viscous fluid flows with small Reynolds number. Equation (2-2) is referred to as the *continuity equation*. It arises from the incompressibility assumption. Alternatively (2-1) and (2-2) may be referred to as the *Stokes equations* governing an incompressible flow at low Reynolds number. There are equivalent formulations for these equations [see 1] that are derived from the given pair. Additionally, a fourth-order equation may be derived from these. This equation states a relationship for velocity without reference to pressure. The formulation given here is convenient for our purposes.

If Ω is the domain of the flow, then Ω is a connected compact set in $\mathbb{R}^2$. Take $[0, T]$ as the time interval. When t is fixed, then P lies in $L^2[\Omega] = L^2$ and both u and v are elements of $H^1 = \{u : \Omega \to \mathbb{R} : \int_\Omega \nabla u \cdot \nabla u < \infty\}$. Finally $\vec{G}$ represents external body forces such as gravity. Equation (2-1) restated in terms of the coordinate functions yields

$$\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} - \nu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x}\Big) - g_1 = 0, \tag{2-3}$$

$$\frac{\partial v}{\partial t} + \frac{\partial P}{\partial x} - \nu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\Big) - g_2 = 0. \tag{2-4}$$

Below we suppose that the external body forces do not play a significant role in the flow and will ignore this term.

A discrete form of the Stokes equation is derived from FEM techniques applied to the spatial variables. If the flow is transient ($\partial \vec{u}/\partial t \neq 0$) then the resulting discrete equations yield approximate solutions via finite difference techniques.

For the purposes of the theory and examples developed below, we base the FEM on the (Q_1^4, Q_0^1) model. This model supposes the decomposition of the flow domain into the union of rectangles. The vertices of the rectangles are velocity nodes and centroids of the rectangles are the pressure nodes. For the succeeding examples we use a channel flow obstructed by a square obstruction (see Figure 1).

Denoting the partition by $\Omega = \bigcup_{e=1}^{s} \Omega^e$, we define finite dimensional subspaces V of H^1 and W of L^2. V is defined as the linear space of first-order polynomials $\{\phi_i^e : i = 1, 2, 3, 4;\ e = 1, \ldots, s\}$, where each ϕ_i^e is supported by Ω^e and equal to the i-th Lagrange polynomial on Ω^e. In turn W is the span of constant functions,

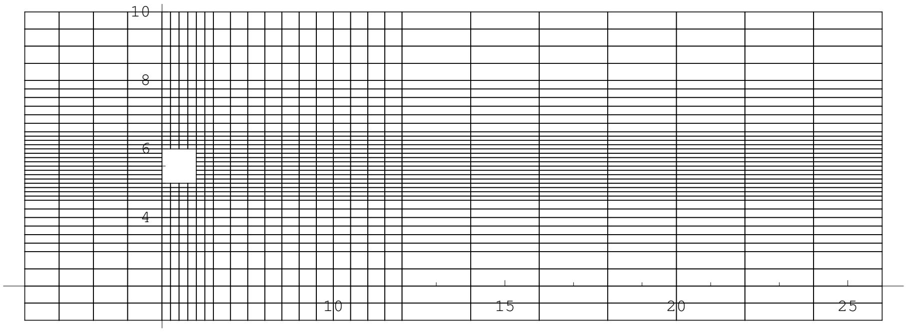

Figure 1. Decomposition of an obstructed channel into rectangular elements.

$\{P^e : e = 1, \ldots, m\}$ supported by the elements. The Galerkin FEM proceeds by seeking elements $\tilde{u}$ and $\tilde{v}$ in V and $\tilde{P}$ in W so that the residual (Equations (2-1) and (2-2) and evaluated at these functions) is L^2 orthogonal to V. In particular,

$$(R_1(\tilde{u}, \tilde{v}, \tilde{P}), \phi_i^e) = \int_\Omega R_1(\tilde{u}, \tilde{v}, \tilde{P})\phi_i^e = 0, \tag{2-5}$$

$$(R_2(\tilde{u}, \tilde{v}), \phi_i^e) = \Omega \int_\Omega R_2(\tilde{u}, \tilde{v}) P^e = 0. \tag{2-6}$$

Expanding these equations and using the divergence theorem to linearize the second-order term, we arrive at the following linear system for each element, e:

$$\begin{pmatrix} M_1^e & 0 & 0 \\ 0 & M_2^e & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \\ \dot{P} \end{pmatrix} + \begin{pmatrix} K_1^e & K_{12}^2 & L_1^e \\ K_{21}^e & K_2^e & L_2^e \\ (L_1^e)^T & (L_2^e)^T & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ P \end{pmatrix} = \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{24} \\ g \end{pmatrix},$$

where the dot represents differentiation with respect to t. The matrix entries are

$$M_1^e(i, j) = M_2^e(i, j) = \int_{\Omega^e} \phi_i^e \phi_j^e, \qquad K_{12}^e(i, j) = K_{21}^e(j, i) = \nu \int_{\Omega^2} \frac{\partial \phi_i^e}{\partial y} \frac{\partial \phi_j^e}{\partial x},$$

$$K_1^e(i, j) = \nu \int_{\Omega^e} 2 \frac{\partial \phi_i^e}{\partial x} \frac{\partial \phi_j^e}{\partial x} + \frac{\partial \phi_i^e}{\partial y} \frac{\partial \phi_j^e}{\partial y}, \qquad L_1^e(i, 1) = -\int_{\Omega^e} \frac{\partial \phi_i^e}{\partial x},$$

$$K_2^e(i, j) = \nu \int_{\Omega^e} \frac{\partial \phi_i^e}{\partial x} \frac{\partial \phi_j^e}{\partial x} + 2 \frac{\partial \phi_i^e}{\partial y} \frac{\partial \phi_j^e}{\partial y}, \qquad L_2^e(i, 1) = -\int_{\Omega^e} \frac{\partial \phi_i^e}{\partial y}.$$

On the right hand side we have

$$f_{1i}^e = \int_{\Gamma^e} \Big(2\frac{\partial \tilde{u}}{\partial x}, \frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\Big)\phi_i^e \cdot \vec{n}, \quad f_{2i}^e = \int_{\Gamma^e} \Big(\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y}, 2\frac{\partial \tilde{v}}{\partial y}\Big)\phi_i^e \cdot \vec{n},$$

from the application of the divergence theorem to (2-5). Whereas (2-6) yields

$$g = \int_{\Gamma^e} (\tilde{u}, \tilde{v}) \cdot \vec{n}.$$

In both cases Γ^e denotes the boundary of Ω^e. Using standard processes [Huebner et al. 2001] we assemble these s linear systems in a single $(2m+s) \times (2m+s)$ system (where m is the number of nodes and s. is the number of elements). This is done by first identifying the corresponding node for each vertex of a single element. Then adding the linear equations which refer to a common node. The resulting system can be expressed in compact form as

$$\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{P} \end{pmatrix} + \begin{pmatrix} K & L \\ L^T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ P \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Here M is $m \times m$ symmetric, K is $2m \times 2m$ symmetric and positive definite (from the underlying physics) and L is $2m \times s$.

3. Boundary values for the steady state problem

We begin our study of boundary values by considering the steady state problem. In this case we need only consider the equation

$$\begin{pmatrix} K & L \\ L^T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ P \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{3-1}$$

as the discrete form of the steady state Stokes equation. We assert that the coefficient matrix of (3.1) is nonsingular. This assertion is equivalent to the statement that the flow has a unique solution in the discrete form stated in (3.1). In general this is not the case, but it may be achieved by imposition of boundary conditions at the channel edges and at the obstruction, as well as by the choice of model. The underlying physics assures us that the matrix K is symmetric and positive definite, as well as sparse and diagonally dominant. L is sparse.

Our primary concern is with the entries of f for the nodes along the inflow boundary. On the one hand we may designate a value for f_i. In this case the designated value implements driving forces applied along the inflow edge as is evident from the expression for f_i given in the previous section.These boundary values are then called *Neumann* or *natural*. Alternatively we may designate the velocity components on this boundary. This alternative is implemented at the i-th node by replacing the i-th row of the coefficient matrix to the i-th row of the identity matrix, denoted $\vec{e}_i$ and then setting f_i to the desired velocity. These boundary values are referred to as *Dirichlet* or *essential*.

For simplicity of notation we write Equation (3-1) as $A\vec{u}=\vec{f}$. Now since A is nonsingular, then for any choice of $\vec{f}$ the system has a unique solution. Consider the process, just described, used to set Dirichlet boundary values. For this to be meaningful, the resulting coefficient matrix must be row-equivalent to A. Otherwise the resulting linear system, $B\vec{u}=\hat{f}$ would no longer represent the discrete form of the same differential equation. With this in mind we begin our analysis with the following definition, where for a matrix A, $A_{(i)}$ denotes the i-th row of A.

Definition 3.1. Let $A\vec{u} = \vec{f}$ be an n-by-n linear system of equations and take i, $1 \le i \le n$. Then we say that a Dirichlet boundary condition at f_i is *algebraically admissible* provided A is row-equivalent to B where the $A_{(j)} = B_{(j)}$ for each $j \ne i$ and $B_{(i)} = \vec{e}_i^{\,T}$.

From the definition it is apparent that a Dirichlet condition at f_i is algebraically admissible if there are elementary row matrices $E_0, E_1, \dots, E_m$ with

$$B = \Big(\prod_{j\ne i} E_j\Big) E_0 A,$$

where E_0 is type 2, representing the multiplication of row i of A by a nonzero scalar, and for $j \ne 0$, E_j is type-3, representing the operation of adding to the i-th row a scalar multiple of some other row.

For the case at hand, a linear system arising from the FEM discrete form of the steady state Stokes equation, the matrix K is positive definite symmetric, sparse and diagonally dominant. Therefore for each $i \le 2m$,

$$\vec{e}_i^{\,T} = \sum_j \alpha_j K_{(j)}.$$

Since K is diagonally dominant, α_i is not zero. If $E_{\beta s+t}$ denotes the row operation of adding β times the s-th row to the t-th row and $E_{\beta s}$ denotes the elementary operation of multiplying the s-th row by nonzero β, then

$$\vec{e}_i^{\,T} = \Big(\prod_{j\ne i} E_{\alpha_j j+i}\Big) E_{\alpha_i i} A.$$

Therefore, in this case a Dirichlet condition at f_i is algebraically admissible for each $i \le 2m$.

Theorem 3.1. *Let $A\vec{u} = \vec{f}$ be an n-by-n linear system of equations and take i, with $1 \le i \le n$. Suppose that a Dirichlet boundary condition at f_i is algebraically admissible. Then there exists a nonsingular linear transformation N such that* *the Dirichlet assumption for u_i is the i-th coordinate of $N\vec{f}$. Further all other coordinates of $N\vec{f}$ are unchanged.*

Proof. From the comment following Definition 3.1, it suffices to set

$$N = \Big(\prod_{j \neq i} E_j\Big) E_0.$$

Now the remaining assertions are immediate. □

Next, supposing that A is nonsingular, we can get a specific representation for the elementary row operations. First we set up the notation. Set $E_0 = E_{\alpha_i i}$ and $E_j = E_{\alpha_k k+i}$. Now we may suppose that N has n factors by setting $E_j = E_{\alpha_j j+i}$ for each $j \neq i$ where $\alpha_j = 0$ if row j is not involved in reducing the i-th row of A. Finally define the column n-tuple $\vec{\alpha} = \alpha_{(j)}$.

Corollary 3.2. *If A is nonsingular, then $\vec{\alpha} = (A^T)^{-1}\vec{e}_i$, is the i-th column of $(A^T)^{-1}$.*

Proof. With the notation just introduced,

$$(E_{\alpha_i i} A)_{(i)} = \alpha_i A_{(i)} \quad \text{and} \quad (E_{\alpha_j j+i} A)_{(i)} = \alpha_j A_{(j)} + A_{(i)}.$$

Therefore,

$$(NA)_{(i)} = \sum_{j \neq i} \alpha_j A_{(j)} + \alpha_i A_{(i)} = \sum_j \alpha_j A_{(j)}.$$

Restating this as an expression for $(\vec{e}_i)^T$ we get

$$(\vec{e}_i)^T = \sum_j \alpha_j A_{(j)} = \Big(\sum_j (A^T)^{(j)} \alpha_j\Big)^T = (A^T \vec{\alpha})^T. \qquad \square$$

This yields the desired expression for $\vec{\alpha} = (A^T)^{-1}\vec{e}_i$, which is indeed the i-th column of $(A^T)^{-1}$.

In the case of (3-1), A is symmetric and we have:

Corollary 3.3. *If A is the coefficient matrix for the FEM discrete form of the Stokes equation, then $\vec{\alpha}$ is the i-th column of A^{-1}.*

Next we turn to the relationship between the Neumann boundary value $\vec{f}$ and the Dirichlet boundary value u_i at the i-th node.

Corollary 3.4. *Suppose that A is nonsingular and algebraically admits a Dirichlet condition at f_i then the Dirichlet value, u_i is related to $\vec{f}$ via $u_i = \vec{\alpha} \cdot \vec{f}$ where $\vec{\alpha} = (A^T)^{-1}\vec{e}_i$. (Here $\vec{\alpha} \cdot \vec{f}$ denotes the ordinary inner product in $\mathbb{R}^n$.)*

Proof. The relation

$$A\vec{u} = \vec{f}$$

yields,

$$u_i = \vec{e}_i \cdot \vec{u} = \vec{e}_i^{\,T} \vec{u} = (A^T \vec{\alpha})^T \vec{u} = (\vec{\alpha})^T A\vec{u} = (\vec{\alpha})^T \vec{f} = \vec{\alpha} \cdot \vec{f}. \qquad \square$$

We end this section by considering the following problem. Given a linear system with Dirichlet conditions applied, what is the corresponding linear system without Dirichlet boundary conditions, but rather Neumann boundary conditions.

Theorem 3.5. *Let A be an n-by-n matrix, which algebraically admits a Dirichlet boundary condition on the i-th row. Suppose that B is row-equivalent to A via the nonsingular matrix N as in Theorem 3.1 Consider a linear system $B\vec{u} = \hat{f}$, then the equivalent linear system $A\vec{u} = \vec{f}$ satisfies $f_j = \hat{f}_j$ for each $j \neq i$ and $f_i = \vec{\beta} \cdot \hat{f}$, where $\vec{\beta} = (B^T)^{-1}(A_{(i)})^T$.*

Proof. With the notation of Equation (3-1), Take N nonsingular so that $B = NA$ and $\hat{f} = N\vec{f}$. Now N is a product of elementary row matrices. Hence the same is true of N^{-1}. In particular,

$$N^{-1} = \Big(\Big(\prod_{j\neq i} E_{\alpha_j j+i}\Big) E_{\alpha_i i}\Big)^{-1} = (E_{\alpha_i i})^{-1}\Big(\prod_{j\neq i} E_{\alpha_j j+i}\Big)^{-1} = \Big(\prod_{j\neq i} E_{\beta_j j+i}\Big) E_{\beta_i i},$$

where $\beta_i = \alpha_i^{-1}$ and $\beta_j = -\alpha_j/\alpha_i$ otherwise. Setting $\vec{\beta} = (\beta_i)$, it now follows that $\vec{\beta} \cdot \hat{f} = f_i$. In turn

$$A_{(i)} = (N^{-1}B)_{(i)} = \sum_j \beta_j B_{(j)} = \sum_j (B^T)^{(j)} \beta_j = (B^T\vec{\beta})^T. \qquad \square$$

So $(A_{(i)})^T = B^T\vec{\beta}$ or $(B^T)^{-1}(A_{(i)})^T = \vec{\beta}$. The following point plots show first a set of given forces at points along the inflow edge of the example flow (Figure 2, left). We used B-splines to fit a continuous function to the given data. With this function we were able to infer forces at the inflow nodes and compute f via one point quadrature. Then we used Corollary 3.4 to compute the velocities shown in the second plot (Figure 2, right). As indicated by the mathematics, the calculated flow using either the Neumann or the Dirichlet boundary values produces identical velocity fields.

4. Boundary values for the transient flow

In this section we modify our results of the section to the case of a nonsteady Stokes flow. Our particular concern with Stokes flows is their application as the initial phase of a Navier–Stokes flow. In this setting it is natural to suppose that there is a velocity or force startup function, $v(t)$ with $t \epsilon (0, T]$, implemented at the inflow edge. In this section we will consider the discrete case of the nonsteady Stokes equation and determine the relationship between a velocity startup and a force startup.

For the nonsteady Stokes flow the spatial problem is realized via finite element techniques while the time dependent problem is developed via finite difference techniques. There are several competing finite difference techniques. For each, the

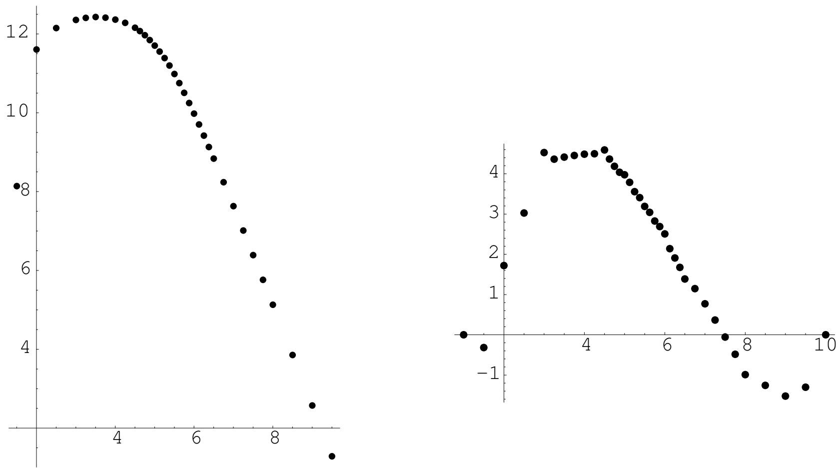

Figure 2. Left: derived forces on inflow edge. Right: computed velocities at the inflow edge.

function relating forces to velocities and pressures is distinct. We will develop two cases, the *trapezoid* (TR) method and the *Adam-Bashford-2* (AB-2) method. We begin with TR. Here we use superscripts to designate time steps.

$$\begin{pmatrix} M+\frac{1}{2}\Delta t\,K & \Delta t\,L \\ \frac{1}{2}\Delta t\,L^T & 0 \end{pmatrix}\begin{pmatrix} u^n \\ v^n \\ P^n \end{pmatrix} = \begin{pmatrix} M-\frac{1}{2}\Delta t\,K & -\Delta t\,L \\ -\frac{1}{2}\Delta t\,L^T & 0 \end{pmatrix}\begin{pmatrix} u^{n-1} \\ v^{n-1} \\ P^{n-1} \end{pmatrix} + \begin{pmatrix} \Delta t f^n \\ t g^{n-1} - \frac{1}{2}\Delta t\,L^T \begin{pmatrix} u^{n-1} \\ v^{n-1} \end{pmatrix} \end{pmatrix}, \tag{4-1}$$

where

$$g^n = L^T \begin{pmatrix} u^n \\ v^n \end{pmatrix}.$$

The term on the right, f, which is related to force is superscripted as we may suppose it varies with time. Further we suppose that the fluid starts at rest, so for $t = t_1$, $u^0 = v^0 = P^0 = 0$. Hence (4-1) becomes

$$\begin{pmatrix} M+\frac{1}{2}\Delta t\,K & \Delta t\,L \\ \frac{1}{2}\Delta t\,L^T & 0 \end{pmatrix}\begin{pmatrix} u^1 \\ v^1 \\ P^1 \end{pmatrix} = \begin{pmatrix} \Delta t f^n \\ 0 \end{pmatrix}. \tag{4-2}$$

As in Section 3, we use a notationally simplified version of these equations:

$$A\vec{u}^n = C\vec{u}^{n-1} + \Delta t \begin{pmatrix} f^n \\ g^{n-1} \end{pmatrix}. \tag{4-3}$$

For N nonsingular, we have

$$NA\vec{u}^n = NC\vec{u}^{n-1} + N\Delta t \begin{pmatrix} f^n \\ g^{n-1} \end{pmatrix}. \tag{4-4}$$

As in the steady state case, restriction of these two equations to the example flow assures us that A is nonsingular and that the Dirichlet boundary condition is algebraically admissible at each inflow edge node. Assuming that the fluid starts at rest implies that Equation (4-3) reduces to

$$\frac{1}{\Delta t} A\vec{u}^1 = \begin{pmatrix} f^1 \\ 0 \end{pmatrix} \quad \text{at } n = 1.$$

This equation is essentially the same as the one considered in Section 3 except that the coefficient matrix is not symmetric. Nevertheless, Corollary 3.4 and Theorem 3.5 apply to the present setting.

Theorem 4.1. *Consider the TR time step development of the nonsteady Stokes flow represented by (4-3). Suppose that A is nonsingular and that Neumann boundary values are set at $t = t_n$ via the coordinates of f^n. Then a boundary value f_i^n may be replaced by a Dirichlet boundary value*

$$u_i^n = (NC)_{(i)}\vec{u}^{n-1} + \vec{\alpha} \cdot \begin{pmatrix} f^n \\ 0 \end{pmatrix},$$

where $\vec{\alpha} = \Delta t (A^T)^{-1}\vec{e}_i$. Hence, $\vec{\alpha}$ is Δt times the i-th column of $(A^T)^{-1}$. In addition

$$N = \Big(\prod_{j \neq i} E_{\alpha_j j+i}\Big) E_{\alpha_i i}.$$

Proof. The assertion for $t = t_1$ follows immediately from Corollary 3.4. For $n > 1$, we need to first let

$$C\vec{u}^{n-1} + \begin{pmatrix} f^n \\ g^{n-1} \end{pmatrix}$$

take the role of $\vec{f}$ in Corollary 3.4 to get

$$\begin{aligned} u_i^n &= (NC)_{(i)}\vec{u}^{n-1} + N \begin{pmatrix} f^n \\ g^{n-1} \end{pmatrix} = NC_{(i)}\vec{u}^{n-1} + \vec{\alpha} \cdot \begin{pmatrix} f^n \\ g^{n-1} \end{pmatrix} \\ &= (NC)_{(i)}\vec{u}^{n-1} + \vec{\alpha} \cdot \begin{pmatrix} f^n \\ 0 \end{pmatrix}. \end{aligned} \tag{4-5}$$

The final equality holds since if the upper left hand block of A is k-by-k then $\alpha_j = 0$ for $j > k$. Indeed, the upper left block is itself nonsingular, so by Dirichlet admissibility, the i-th row is row-equivalent to the i-th row of the k-by-k identity matrix. Finally, since the lower right block of A is zero, it now follows that

$$\vec{\alpha} \cdot \begin{pmatrix} f^n \\ g^{n-1} \end{pmatrix} = \vec{\alpha} \cdot \begin{pmatrix} f^n \\ 0 \end{pmatrix}.$$

The remaining assertions follow as in Section 3. □

Notice that for $n > 1$, the calculation of the Dirichlet boundary value requires the prior state. Therefore the results for the steady state problem do not carry over directly to the nonsteady flow. In particular even if f_i^n is fixed for each $n > 1$, u_i^n will vary with n.

We now particularize Theorem 4.1 to the case of a startup function for the force along the inflow edge. For this purpose we need to develop some notation. First (4-3) becomes

$$A\vec{u}^n = C\vec{u}^{n-1} + \Delta t \begin{pmatrix} \varphi(t_n) f \\ g^{n-1} \end{pmatrix}, \tag{4-6}$$

where $\varphi : (0, T] \to (0, 1]$ designates the startup function and $f = (f_i)$ with $f_i = 0$ for each node which is not on the inflow edge and $f_i = 1$ at the inflow edge. In turn (4-5) becomes

$$\begin{aligned} u_i^n &= (N_i C)_{(i)}\, \vec{u}^{n-1} + \Delta t \varphi(t_n) \overrightarrow{\alpha} \cdot f \\ &= (N_i C)_{(i)}\, \vec{u}^{n-1} + \Delta t \varphi\,(t_n) \left((A^T)^{-1}\right)_{(i)} f, \end{aligned} \tag{4-7}$$

where N is now subscripted to identify the row operations applied to the i-th row of A. Next we define $u = u_i$ and $u_i = \left((A^T)^{-1}\right)_{(i)} f$ as in Section 3. We can consider a corresponding startup function for Dirichlet boundary values at the inflow edge.

Corollary 4.2. *Suppose that $\varphi : (0, T] \to (0, 1]$ denotes a startup function for the force along the inflow edge of the transient Stokes flow. Let ν be a second function defined on the time steps and taking values in $\mathbb{R}^\ell$, where ℓ designates the number of nodes on the inflow edge. If ν is defined by*

$$\nu(t_n)_i = \frac{1}{u_i}(N_i C)_{(i)}\vec{u}^{n-1} + \Delta t \varphi(t_n),$$

then $\nu(t_n)_i u_i = u_i^n$.

Proof. The result follows immediately from (4-7). □

The startup force function results in separate velocity startup functions, one defined at each of designated nodes. We illustrate this result in the following plots. Figure 3 shows a burst startup function $\varphi(t) = 1 - e^{-t/0.1}$. The subsequent three point plots (Figure 4) show the corresponding velocity plots at selected nodes: 2, 6 and 12 along the inflow edge (see Figure 1). Note that the velocity startup functions though distinct are very similar. Indeed they appear linear. Lastly, Figure 5 shows the final value ($t = 0.5$) for the velocity startup function at each node. This plot is

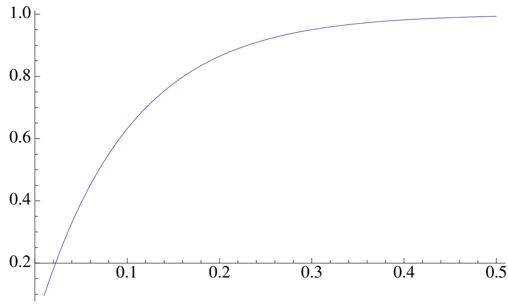

Figure 3. Burst startup function.

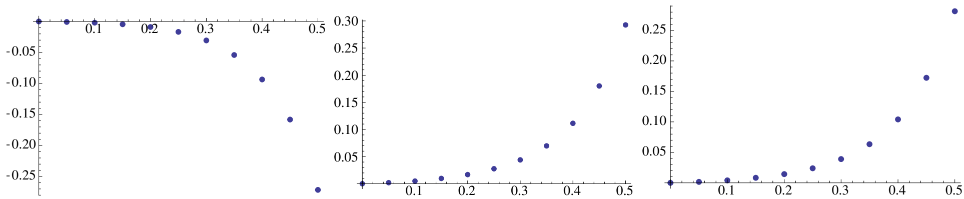

Figure 4. TR: Dirichlet values at nodes 2, 6, 12 (left to right).

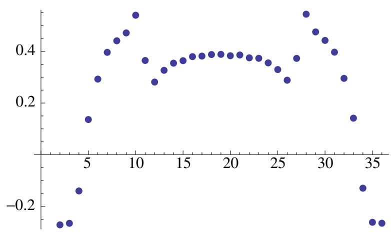

Figure 5. TR: inflow velocities at $t = 0.5$.

symmetric. The differences from node to node appear consistent with incompressibility as the flow reacts to the obstruction.

The next result considers the reverse setting where we begin with a Dirichlet boundary value and derive the corresponding Neumann value.

Theorem 4.3. *Consider the TR time step development of the nonsteady Stokes flow represented by (4-3). Suppose that A is nonsingular and algebraically admits Dirichlet boundary values along the inflow edge. Suppose for $t = t_n$ that Dirichlet boundary values are set on this edge via the coordinates of u^n to yield*

$$DA\vec{u}^n = \begin{pmatrix} u^n \\ \Delta t L P^{n-1} + g^{n-1} \end{pmatrix},$$

where D is a product of matrices N as described in Section 2. Fix a node i, then the corresponding Neumann boundary value at the i-th node is

$$f_i = \vec{\beta} \cdot \hat{f}, \quad \text{where } \vec{\beta} = (B^T)^{-1}(A_{(i)})^T \text{ and } B = NA.$$

Proof. We proceed as in Theorem 3.5. First we set

$$N = \Big(\prod_{j \neq i} E_{\alpha_j\, j+i}\Big) E_{\alpha_i\, i},$$

as in Theorem 3.1 and

$$N^{-1} = \Big(\Big(\prod_{j \neq i} E_{\alpha_j\, j+i}\Big) E_{\alpha_i\, i}\Big)^{-1} = (E_{\alpha_i\, i})^{-1}\Big(\prod_{j \neq i} E_{\alpha_j\, j+i}\Big)^{-1} = \Big(\prod_{j \neq i} E_{\beta_j\, j+i}\Big) E_{\beta_i\, i},$$

where $\beta_i = \alpha_i^{-1}$ and $\beta_j = -\alpha_j/\alpha_i$ otherwise. Setting $\vec{\beta} = (\beta_i)$, it now follows that $\vec{\beta} \cdot u^n = f_i$ and

$$A_{(i)} = (N^{-1}B)_{(i)} = \sum_j \beta_j B_{(j)} = \sum_j (B^T)^{(j)} \beta_j = (B^T \vec{\beta})^T. \qquad \square$$

We turn next to the case of AB-2. This FDM procedure computes the current velocity field in terms of the weighted average of the prior two time steps via

$$\begin{pmatrix} M & \frac{3}{2}\Delta t\, L \\ L^T & 0 \end{pmatrix} \begin{pmatrix} u^n \\ v^n \\ P^{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{3}{2}\Delta t\, K + M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^{n-1} \\ v^{n-1} \\ P^{n-1} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\Delta t\, K & \frac{1}{2}\Delta t\, L \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^{n-2} \\ v^{n-2} \\ P^{n-2} \end{pmatrix} \begin{pmatrix} \Delta t f^n \\ g^{n-1} \end{pmatrix}, \tag{4-8}$$

where

$$g^n = L^T \begin{pmatrix} u^n \\ v^n \end{pmatrix} = P^n.$$

For a fluid starting at rest we have for $n = 1$

$$\begin{pmatrix} M & \frac{3}{2}\Delta t\, L \\ L^T & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ v^1 \\ P^1 \end{pmatrix} = \begin{pmatrix} \Delta t f^1 \\ 0 \end{pmatrix}. \tag{4-9}$$

As before it is convenient to restate (4-8) in a simplified form:

$$B\vec{u}^n = D\vec{u}^{n-1} + E\vec{u}^{n-2} + \begin{pmatrix} \Delta t f^n \\ g^{n-1} \end{pmatrix}. \tag{4-10}$$

For N nonsingular,

$$NB\vec{u}^n = ND\vec{u}^{n-1} + NE\vec{u}^{n-2} + N\begin{pmatrix} \Delta t f^n \\ g^{n-1} \end{pmatrix}. \tag{4-11}$$

The next results are analogous to Theorem 4.1, Corollary 4.2 and Theorem 4.3.

Theorem 4.4. *Consider the AB-2 time step development of the nonsteady Stokes flow given by (4-8), (4-10). Suppose that Neumann boundary values are set at time step $t = t_n$ via the coordinates of f^n. Then the boundary value f_i^n may be replaced by a Dirichlet boundary value*

$$u_i^n = (ND)_{(i)}\vec{u}^{n-1} + (NE)_{(i)}\vec{u}^{n-2} + \vec{\alpha} \cdot \begin{pmatrix} f^n \\ 0 \end{pmatrix},$$

where

$$\vec{\alpha} = \Delta t (B^T)^{-1}\vec{e}_i \quad \textit{and} \quad N = \Big(\prod_{j \neq i} E_{\alpha_j j+i}\Big) E_{\alpha_i i}.$$

Hence, $\vec{\alpha}$ is Δt times the i-th column of $(B^T)^{-1}$.

Proof. The expression for u_i^n follows from (4-11) and the given decomposition of N as a product of elementary matrices. The given expression results from

$$\sum_i \alpha_i B(i) = (NB)_{(i)} = \vec{e}_i^{\,T}.$$

The final statement is immediate. □

Turning to a startup function for force we have:

Corollary 4.5. *Suppose the setting of Theorem 4.4 and suppose that*

$$\gamma(0, T] :\to (0, 1]$$

denotes a startup function for the force along the inflow edge of the transient Stokes flow. Determine a second function, δ, defined at the time steps and taking values in $\mathbb{R}^\ell$, where ℓ designates the number of nodes in the inflow edge by

$$\delta(t_n)_i = \frac{1}{u_i}(N_i D)_{(i)}\vec{u}^{n-1} + \frac{1}{u_i}(N_i E)_{(i)}\vec{u}^{n-2} + \Delta t \gamma(t_n).$$

Then $\delta(t_n)_i u_i = u_i^n$, where $u_i = ((B^T)^{-1})_{(i)} f$.

Proof. As in the TR case we now particularize (4-11) for the startup function, γ, the product of elementary operations associated to the i-th inflow edge node, N_i, and then consider the i-th entry of the result to get

$$\vec{u}_i^n = (N_i D)_{(i)}\vec{u}^{n-1} + (N_i E)_{(i)}\vec{u}^{n-2} + \Delta t \gamma(t_n)((B^T)^{-1})_{(i)} f.$$

The result is now immediate. □

Finally we consider the reverse case.

Theorem 4.6. *Consider the AB-2 time step development of the nonsteady Stokes flow represented by (4-10). Suppose that for $t = t_n$ Dirichlet boundary values are set along the inflow edge via the corresponding coordinates of u_n to yield*

$$FB\vec{u}_n = \begin{pmatrix} u^n \\ g^{n-1} \end{pmatrix},$$

where F is a product of matrices N as described in Theorem 4.4. Fix a node i, then the corresponding Neumann boundary value at the i-th node is $f_i = \vec{\beta} \cdot \hat{f}$, where $\vec{\beta} = (G^T)^{-1}(B_{(i)})^T$ and $G = NB$.

Proof. We proceed as with Theorem 4.3. First we set $N = \Big(\prod_{j \neq i} E_{\alpha_j\, j+i}\Big) E_{\alpha_i i}$, as in Theorem 3.1, then resolve

$$N^{-1} = \Big(\prod_{j \neq i} E_{\beta_j\, j+i}\Big) E_{\beta_i i},$$

where $\beta_i = \alpha_i^{-1}$ and $\beta_j = -\alpha_j/\alpha_i$ otherwise. Setting $\vec{\beta} = (\beta_i)$, we have $f_i = \vec{\beta} \cdot \hat{f}$. Finally,

$$B_{(i)} = (N^{-1}G)_{(i)} = \sum_j \beta_j G_{(j)} = (G^T \vec{\beta})^T,$$

which yields the desired expression for $\vec{\beta}$. □

The plots in Figures 6 and 7 show output for AB-2. They are analogous to Figures 4 and 5.

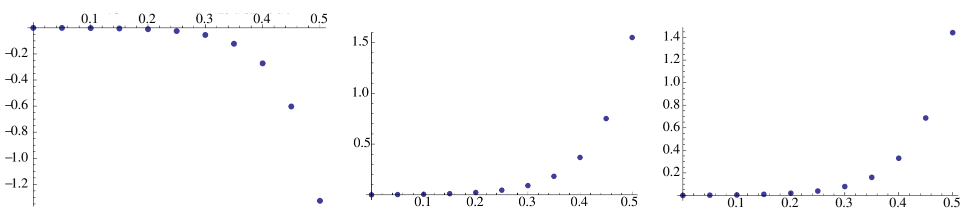

Figure 6. AB-2: Dirichlet values at nodes 2, 6 and 12 (left to right).

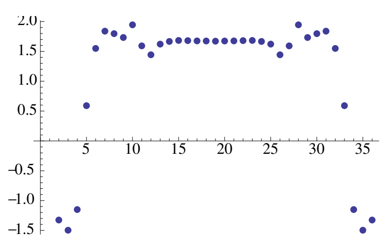

Figure 7. AB-2: inflow velocities at $t = 0.5$.

A note on the illustrations

All programming was done in Mathematica.

Geometry. Channel: lower left vertex at (1, 1); upper right vertex at (26, 10).
Obstruction: lower left vertex at (5, 5); upper right vertex at (6, 6).

FEM. 976 elements: 1052 velocity nodes; 976 pressure nodes; 2104+976 degrees of freedom; max x increment = 1.0; min x increment = 0.25; max y increment = 0.5; min y increment = 0.125.

FDM. $\Delta t = 0.05$.

Fluid. water, $\nu = 0.89$.

Boundary values. All surfaces are nonslip and nonpenetrating. Dirichlet boundary values are set to zero. The outflow edge is included in the Neumann boundary with values set to zero. Transient flows are started at rest.

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Received: 2007-08-10 Accepted: 2010-10-23

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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2010 vol. 3 no. 4

Identification of localized structure in a nonlinear damped harmonic oscillator using Hamilton's principle 349
THOMAS VOGEL AND RYAN ROGERS

Chaos and equicontinuity 363
SCOTT LARSON

Minimum rank, maximum nullity and zero forcing number for selected graph families 371
EDGARD ALMODOVAR, LAURA DELOSS, LESLIE HOGBEN, KIRSTEN HOGENSON, KAITLYN MURPHY, TRAVIS PETERS AND CAMILA A. RAMÍREZ

A numerical investigation on the asymptotic behavior of discrete Volterra equations with two delays 393
IMMACOLATA GARZILLI, ELEONORA MESSINA AND ANTONIA VECCHIO

Visual representation of the Riemann and Ahlfors maps via the Kerzman–Stein equation 405
MICHAEL BOLT, SARAH SNOEYINK AND ETHAN VAN ANDEL

A topological generalization of partition regularity 421
LIAM SOLUS

Energy-minimizing unit vector fields 435
YAN DIGILOV, WILLIAM EGGERT, ROBERT HARDT, JAMES HART, MICHAEL JAUCH, ROB LEWIS, CONOR LOFTIS, ANEESH MEHTA, ESTHER PEREZ, LEOBARDO ROSALES, ANAND SHAH AND MICHAEL WOLF

Some conjectures on the maximal height of divisors of $x^n - 1$ 451
NATHAN C. RYAN, BRYAN C. WARD AND RYAN WARD

Computing corresponding values of the Neumann and Dirichlet boundary values for incompressible Stokes flow 459
JOHN LOUSTAU AND BOLANLE BOB-EGBE