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A note on moments in finite von Neumann algebras

Jon Bannon, Donald Hadwin and Maureen Jeffery

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By a result of the second author, the Connes embedding conjecture (CEC) is false if and only if there exists a self-adjoint noncommutative polynomial $p(t_1, t_2)$ in the universal unital C^*-algebra $\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*,\ 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2\rangle$ and positive, invertible contractions x_1, x_2 in a finite von Neumann algebra $\mathcal{M}$ with trace τ such that $\tau(p(x_1, x_2)) < 0$ and $\mathrm{Tr}_k(p(A_1, A_2)) \geq 0$ for every positive integer k and all positive definite contractions A_1, A_2 in $M_k(\mathbb{C})$. We prove that if the real parts of all coefficients but the constant coefficient of a self-adjoint polynomial $p \in \mathcal{A}$ have the same sign, then such a p cannot disprove CEC if the degree of p is less than 6, and that if at least two of these signs differ, the degree of p is 2, the coefficient of one of the t_i^2 is nonnegative and the real part of the coefficient of t_1t_2 is zero then such a p disproves CEC only if either the coefficient of the corresponding linear term t_i is nonnegative or both of the coefficients of t_1 and t_2 are negative.

1. Introduction

The Connes embedding conjecture (CEC) is true if every separable type II_1 factor $\mathcal{M}$ embeds in a tracial ultrapower $\mathcal{R}^\omega$ of the amenable type II_1 factor $\mathcal{R}$. This question concerns the matricial approximation of the elements of a type II_1 factor $\mathcal{M}$ with faithful normal trace state τ in the sense we now recall. For an N-tuple $(x_1, \ldots, x_N)$ of self-adjoint elements in $\mathcal{M}$, $R > 0$, $n, k \in \mathbb{N}$ and $\varepsilon > 0$, we let

$$\Gamma_R(x_1, \ldots, x_N : n, k, \varepsilon)$$

denote the set of tuples $(A_1, \ldots, A_N)$ of those $k \times k$ self-adjoint matrices over $\mathbb{C}$ of operator norm at most R satisfying

$$\left| \tau(x_{i_1} x_{i_2} \ldots x_{i_p}) - \frac{1}{k} \mathrm{Tr}(A_{i_1} A_{i_2,} \ldots A_{i_p}) \right| < \varepsilon,$$

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whenever $1 \le p \le n$ and $(i_1, i_2, \dots, i_p) \in \{1, 2, \dots, N\}^p$. We call the elements of $\Gamma_R(x_1, \dots, x_N : n, k, \varepsilon)$ *approximating microstates* for $(x_1, \dots, x_N)$ of precision (n, ε) using $k \times k$ matrices of norm at most R. A separable type II_1 factor $\mathcal{M}$ embeds in an ultrapower $\mathcal{R}^\omega$ if and only if for all tuples $(x_1, \dots, x_N)$ of self-adjoint elements in $\mathcal{M}$, all $n \in \mathbb{N}$ and all $\varepsilon > 0$, it is possible to find $k \in \mathbb{N}$ and $R > 0$ such that $\Gamma_R(x_1, \dots, x_N : n, k, \varepsilon) \neq \varnothing$. In [Rădulescu 1999] it is proved that this statement is true under the restriction that $n \in \{2, 3\}$, and that if the statement were true for $n = 4$, the CEC would follow.

Our paper concerns the following reformulation of the CEC:

Theorem 1.1 [Hadwin 2001, Corollary 2.3]. *Let $\mathcal{H}$ be a separable Hilbert space. The Connes embedding conjecture is false if and only if there is a positive integer n, a noncommutative polynomial $p(t_1, t_2, \dots, t_n)$ in the universal unital C^*-algebra $\mathcal{A}_n = \langle t_1, t_2, \dots, t_n : t_j = t_j^*, -1 < t_j \le 1$ for $1 \le j \le n\rangle$ and an n-tuple $(x_1, \dots, x_n)$ of self-adjoint contractions in $B(H)$ such that*

(i) $\mathrm{Tr}_k(p(A_1, A_2, \dots, A_n)) \ge 0$ *for every positive integer k and every n-tuple $(A_1, \dots, A_n)$ of self-adjoint contractions $A_1, A_2, \dots, A_n$ in $M_k(\mathbb{C})$, and*

(ii) $W^*(x_1, x_2, \dots, x_n)$ *has a faithful tracial state τ and $\tau(p(x_1, x_2, \dots, x_n)) < 0$.*

It is well known that a separable type II_1 factor $\mathcal{M}$ embeds in an $\mathcal{R}^\omega$ if and only if $\mathcal{M} \otimes M_k(\mathbb{C})$ does for all $k \in \mathbb{N}$. If $\mathcal{M}$ is generated by k self-adjoint elements then $\mathcal{M} \otimes M_k(\mathbb{C})$ is generated by two self-adjoint elements [Sinclair and Smith 2008, Proposition 16.1.1]. Whenever $x \in B(H)$ is a self-adjoint contraction and $\varepsilon > 0$, it follows (e.g., by the continuous functional calculus for x) that

$$\frac{(1+\varepsilon)+x}{2+\varepsilon}$$

is a positive invertible contraction. Therefore, if we replace $\mathcal{A}_n$ by

$$\mathcal{A} = \big\langle t_1, t_2 : t_j = t_j^*,\ 0 < t_j \le 1 \text{ for } 1 \le j \le 2\big\rangle,$$

and repeat the argument in [Hadwin 2001, Section 2], we obtain the following.

Theorem 1.2. *Let $\mathcal{H}$ be a separable Hilbert space. The Connes embedding conjecture is false if and only if there is a noncommutative polynomial $p(t_1, t_2)$ in the universal unital C^*-algebra $\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*$, with $0 < t_j \le 1$ for $1 \le j \le 2\rangle$, and positive, invertible contractions x_1 and x_2 in $B(H)$ such that*

(i) $\mathrm{Tr}_k(p(A_1, A_2)) \ge 0$ *for every positive integer k and all positive definite contractions A_1 and A_2 in $M_k(\mathbb{C})$, and*

(ii) $W^*(x_1, x_2)$ *has a faithful tracial state τ and $\tau(p(x_1, x_2)) < 0$.*

Also note that if a polynomial $p \in \mathscr{A}$ satisfies (i) and (ii) in the theorem, then so does the polynomial $p + p^*$. We may therefore assume that the polynomial appearing in the theorem is self-adjoint.

Note that, even if we restrict our attention in Theorem 1.1 (or Theorem 1.2) to the case where the degree of p is less than or equal to 3, we cannot use [Rădulescu 1999] to rule out the possibility of finding such a p that will disprove the CEC, because existing methods only allow us to use, when $R' < R$, the existence of a microstate in $\Gamma_R(x_1, \ldots, x_N : n, k, \varepsilon)$ to guarantee the existence of a microstate in $\Gamma_{R'}(x_1, \ldots, x_N : n', k, \varepsilon')$, where $\varepsilon' < \varepsilon$ and $n' > n$ — that is, decreasing R comes at the expense of increasing n. See, for example, Proposition 2.4 of [Voiculescu 1994] or Lemma 4 of [Dostál and Hadwin 2003]. Even if this difficulty were overcome, there is no guarantee that the matrices in any approximating microstates found would be positive definite. It behooves us, therefore, to either look for a noncommutative polynomial that may be used to disprove the CEC as prescribed in Theorem 1.1, or to proceed inductively, by degree, to show that such a polynomial cannot exist.

In Section 2 of this paper we prove, in Corollary 2.5 and Theorem 2.6 that if the real parts of all coefficients but the constant coefficient of a self-adjoint noncommutative polynomial $p \in \mathscr{A}$ share the same sign, then such a p cannot disprove the CEC if the degree of p is less than 6. We prove in Section 3 that if the degree of a self-adjoint noncommutative polynomial $p \in \mathscr{A}$ is 2, the real part of the coefficient of t_1t_2 is zero and the coefficient of one of the t_i^2 is nonnegative, then such a p disproves the CEC only if either the coefficient of the corresponding linear term t_i is nonnegative or if both of the coefficients of t_1 and t_2 are negative.

From here on in this paper, the symbols t_1 and t_2 will denote the standard generators of the universal C^*-algebra

$$\mathscr{A} = \langle t_1, t_2 : t_j = t_j^*,\ 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2\rangle.$$

We refer the reader to [Kadison and Ringrose 1983; Sinclair and Smith 2008] for the basic theory of finite von Neumann algebras.

2. τ-symmetrizable monomials

We prove that if the real parts of all coefficients but the constant coefficient of self-adjoint $p \in \mathscr{A}$ share the same sign, and the constant coefficient is positive, then p cannot disprove the CEC if its degree is less than six. Let $\mathscr{M}$ be a finite von Neumann algebra with faithful trace state τ, and $0 < x_1, x_2 \leq 1$ self-adjoint contractions in $\mathscr{M}$.

Definition 2.1. A *symmetric expression* in x_1, x_2 is a finite sequence

$$(w_0, w_1, \ldots, w_{N-1}, w_N)$$

of elements in $\mathcal{M}$, where $N \in \mathbb{N}$, $w_k = x_i^s$ with $i \in \{1, 2\}$, $s \in \{1, 1/2\}$ and $w_k = w_{N-k}$ for all $k \in \{0, 1, \ldots, N\}$. A monic monomial $m(x_1, x_2) = x_{i_1} x_{i_2} \ldots x_{i_l} \in \mathcal{M}$ with $i_j \in \{1, 2\}$ for $j \in \{1, 2, \ldots, l\}$ is τ-symmetrizable if there exists a symmetric expression $(w_0, w_1, \ldots, w_{N-1}, w_N)$ in x_1, x_2 such that

$$\tau(x_{i_1} x_{i_2} \ldots x_{i_l}) = \tau(w_0 w_1 \ldots w_{N-1} w_N).$$

The element $w_0 w_1 \ldots w_{N-1} w_N \in \mathcal{M}$ is called the element associated to the symmetric expression $(w_0, w_1, \ldots, w_{N-1}, w_N)$.

Lemma 2.2. *If* $(w_0, w_1, \ldots, w_{N-1}, w_N)$ *is a symmetric expression in* x_1, x_2, *then the associated element* $w_0 w_1 \ldots w_{N-1} w_N$ *in* $\mathcal{M}$ *is a nonnegative contraction.*

Proof. We prove this by induction on $N + 1$. If $N + 1 = 1$, then $N = 0$ and the result is clear from the assumptions on the x_i.

Assume now that the result holds for $N + 1 \leq l$, that is, for all symmetric expressions $(w_0, w_1, \ldots, w_{j-1}, w_j)$ in x_1, x_2 with $j < l$. Let $(w_0, w_1, \ldots, w_{l-1}, w_l)$ be a symmetric expression in x_1, x_2. Then so is $(w_1, \ldots, w_{l-1})$. By the induction hypothesis, $w_1 \ldots w_{l-1} \in \mathcal{M}$ is a nonnegative contraction. Since $w_0 = w_l = x_i^s$ for some $i \in \{1, 2\}$ and $s \in \{1, \frac{1}{2}\}$, we have

$$0 \leq w_0 w_1 \ldots w_{l-1} w_l = x_i^s w_1 \ldots w_{l-1} x_i^s \leq x_i^{2s} \leq x_i \leq 1. \qquad \square$$

Remark 2.3. It is a straightforward exercise to verify that every monic noncommutative monomial $m(x_1, x_2)$ of degree less than six is τ-symmetrizable in any finite von Neumann algebra $\mathcal{M}$ with faithful trace state τ. (Here, of course, it is essential that $0 < x_1, x_2 \leq 1$!)

Corollary 2.4. *If* $m(x_1, x_2) = x_{i_1} x_{i_2} \ldots x_{i_l} \in \mathcal{M}$ *is a* τ*-symmetrizable monic monomial*, *then* $1 - \tau(m(x_1, x_2)) \geq 0$.

Proof. Since m is τ-symmetrizable, there exists a symmetric expression

$$(w_0, w_1, \ldots, w_{N-1}, w_N)$$

in x_1, x_2 such that

$$\tau(x_{i_1} x_{i_2} \ldots x_{i_l}) = \tau(w_0 w_1 \ldots w_{N-1} w_N).$$

By Lemma 2.2 and the fact that τ is a state, $\tau(w_0 w_1 \ldots w_{N-1} w_N) \leq 1$. $\square$

In the following two results, $J = J \setminus \{0\}$ denotes a finite index set, and for all $j \in J$, $c_j \in \mathbb{C}$, and $m_j(t_1, t_2) \neq 1$ denotes a monic monomial in $\mathcal{A}$.

Corollary 2.5. *If* $0 < x_1, x_2 \leq 1$ *in* $\mathcal{M}$ *and* $p(t_1, t_2) = c_0 1 + \sum_{j \in J} c_j m_j(t_1, t_2)$ *is a self-adjoint noncommutative polynomial in* $\mathcal{A}$ *such that*, *such that* $c_0 > 0$, $\mathrm{Re}(c_j) \geq 0$ *for all* $j \in J$, $p(1, 1) \geq 0$ *and* $m_j(x_1, x_2)$ *is* τ*-symmetrizable for every* $j \in J$, *then* $\tau(p(x_1, x_2)) \geq 0$.

Proof. This is trivial application of Corollary 2.4. $\square$

Theorem 2.6. *If* $0 < x_1, x_2 \leq 1$ *in* $\mathcal{M}$ *and* $p(t_1, t_2) = c_0 1 + \sum_{j \in J} c_j m_j(t_1, t_2)$ *is a self-adjoint noncommutative polynomial in* $\mathcal{A}$ *such that* $c_0 > 0$, $\operatorname{Re}(c_j) < 0$ *for all* $j \in J$, $p(1, 1) \geq 0$ *and* $m_j(x_1, x_2)$ *is* τ*-symmetrizable for every* $j \in J$, *then* $\tau(p(x_1, x_2)) \geq 0$.

Proof. Suppose $p(t_1, t_2)$ satisfies the hypotheses. We have

$$p(1, 1) = c_0 1 + \sum_{j \in J} c_j \geq 0,$$

and therefore

$$\tau(p(x_1, x_2)) \geq \sum_{j \in J} c_j (m_j(x_1, x_2) - 1) \geq 0. \qquad \square$$

3. Degrees 1 and 2

In degree 1 it is convenient to consider the statement of Theorem 1.1 above. The next result rules out the possibility of finding a polynomial p of degree 1 that will disprove the CEC via Theorem 1.1. Observe that if $p(s, t) = c_0 + c_1 s + c_2 t = \bar{c}_0 + \bar{c}_1 s + \bar{c}_2 t$ for any real numbers $-1 \leq s, t \leq 1$ and that $p(s, t) \geq 0$ for any such s and t, then $c_0 \geq |c_1 + c_2|$.

Theorem 3.1. *Let* $\mathcal{H}$ *be a separable Hilbert space. Let* x_1 *and* x_2 *be self-adjoint contraction operators in* $B(H)$ *such that* $W^*(x_1, x_2)$ *has a faithful trace state* τ. *If* $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 = \bar{c}_0 + \bar{c}_1 t_1 + \bar{c}_2 t_2$ *is a self-adjoint polynomial in* $\mathcal{A}$ *with* $c_0 \geq |c_1 + c_2|$ *then* $\tau(p(x_1, x_2)) \geq 0$.

Proof. Observe that $\tau(c_0 + c_1 x_1 + c_2 x_2) = c_0 + c_1 \tau(x_1) + c_2 \tau(x_2) \geq c_0 - |c_1 + c_2|$, since $-1 \leq \tau(x_i) \leq 1$ for $i \in \{1, 2\}$. $\square$

We now turn to degree 2. We first prove in Theorem 3.4 that if

$$p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_4 t_1 t_2 + \bar{c}_4 t_2 t_1 + c_5 t_2^2$$

is a quadratic, self-adjoint noncommutative polynomial such that either c_4 is the only nonzero degree 2 term with $2\operatorname{Re}(c_4) \neq 0$ or one of c_3 or c_5 is positive, then whenever $p(s, t)$ is nonnegative for all real numbers $0 < s, t \leq 1$, it follows that $\operatorname{Tr}_k(p(A, B)) \geq 0$ for all positive definite contractions A and B in $M_k(\mathbb{C})$, for any $k \in \mathbb{N}$.

To prove the result above, we shall need the fact that any positive definite square matrix has strictly positive entries on its main diagonal. This is a direct consequence of Sylvester's minorant criterion for positive definiteness.

Lemma 3.2. *Let* $A = (A_{ij})_{i=1}^{k} \in M_k(\mathbb{C})$ *be positive definite. Then* $A_{ii} > 0$ *for all* $i \in \{1, 2, \ldots, k\}$.

Proof. We prove this by induction on k. Recognize that the case $k = 1$ is clear. Assume the claim holds for $k=l$, and that $A=(A_{ij})_{i=1}^{l+1}$ is a positive definite matrix. By Sylvester's criterion, $A=(A_{ij})_{i=1}^{l}$ is also positive definite, and therefore, by the induction hypothesis, $A_{ii}>0$ if $i\in\{1, 2, \dots l\}$. We need only show $A_{(l+1)(l+1)}>0$. Let $v \in \mathbb{C}^{l+1}$ be the vector with 1 in its $(l+1)$-st row and zero elsewhere. Then $\langle Av, v\rangle = A_{(l+1)(l+1)} > 0$ by the positive definiteness of A. □

We now observe that if a polynomial is nonnegative on $(0, 1] \times (0, 1]$, then its constant term must be nonnegative.

Lemma 3.3. *If* $p(s,t)=c_0+c_1s+c_2t+c_3s^2+2\operatorname{Re}(c_4)st+c_5t^2\geq 0$ *for all real numbers* $0 < s, t \leq 1$, *then* $c_0 \geq 0$.

Proof. For any $\varepsilon > 0$ we have

$$0 < p(\varepsilon, \varepsilon) = c_0 + \big(c_1 + c_2 + (c_3 + 2\operatorname{Re}(c_4) + c_5)\varepsilon\big)\varepsilon;$$

hence $c_0 \geq 0$. □

Theorem 3.4. *Let* $p(t_1,t_2)=c_0+c_1t_1+c_2t_2+c_3t_1^2+c_4t_1t_2+\bar{c}_4t_2t_1+c_5t_2^2$ *be a self-adjoint noncommutative polynomial in* $\mathscr{A}$. *Suppose*

$$p(s,t)=c_0+c_1s+c_2t+c_3s^2+2\operatorname{Re}(c_4)st+c_5t^2\geq 0$$

for all real numbers $0 < s, t \leq 1$, *and either* $c_3 = 0$, $c_5 = 0$ *and* $2\operatorname{Re}(c_4) \neq 0$ *or* $c_3 > 0$ *or* $c_5 > 0$. *Then* $\operatorname{Tr}_k(p(A, B)) \geq 0$ *for any positive definite contractions* A, B *in* $M_k(\mathbb{C})$.

Proof. For simplicity, let us assume $c_5 \geq 0$. Let A, B be positive definite contractions in $M_k(\mathbb{C})$. By the spectral theorem, we may assume $A = \operatorname{diag}(A_i)_{i=1}^{k}$ is diagonal. A simple computation establishes that, for all $i \in \{1, 2, \dots, k\}$,

$$(p(A, B))_{ii} = p(A_i, B_{ii}) + \sum_{j\in\{1,2,\dots,k\}\setminus\{i\}} c_5|B_{ij}|^2.$$

Since A is a positive definite contraction, each A_i satisfies $0 < A_i \leq 1$. If we could establish that the matrix $B_0 := \operatorname{diag}(B_{ii})_{i=1}^{k}$ is a positive definite contraction, then each $p(A_i, B_{ii})$ would follow nonnegative by assumption and therefore $\operatorname{Tr}_k(p(A, B)) = \sum_{i=1}^{k}(p(A, B))_{ii} \geq 0$. Positivity of B_0 is a simple consequence of the positive definiteness of B, since every diagonal entry of a positive definite matrix is strictly positive by Lemma 3.2. It remains to show that B_0 is a contraction, which is equivalent to proving that $I - B_0$ is positive semidefinite. We know, however, that $I-B$ is positive semidefinite, and hence that for all $\varepsilon>0$ that $(I + \varepsilon) - B$ is positive definite. Again as a consequence of Sylvester's criterion, $((I + \varepsilon) - B)_{ii} > 0$ for all $i \in \{1, 2, \dots, n\}$, therefore for all such i it follows that $1 + \varepsilon > B_{ii}$, and hence $1 \geq B_{ii}$. It follows that $I - B_0$ is positive semidefinite, hence B_0 is a contraction. □

Let $\mathcal{M}$ be a von Neumann algebra with faithful trace state τ. Below,

$$\langle x, y\rangle_2 = \tau(y^*x) \quad \text{and} \quad \|x\|_2^2 = \tau(x^*x)^{1/2}, \quad \text{for } x, y \in \mathcal{M}.$$

Let $n \in \mathbb{N}$ and x_1, x_2 be positive invertible contractions in $\mathcal{M}$. For every $k \in \mathbb{N}$, there are spectral projections $\{P_i^{(k)}\}_{i=1}^k$ in $\{1, x_1\}''$ such that $\tau(P_i^{(k)}) = 1/k$ for each i and

$$\Big\| x_1 - \sum_{i=1}^{k} \frac{i-1}{k} P_i^{(k)} \Big\| < \frac{1}{k}.$$

If $i = j$, let $V_{ij}^{(k)} = P_i^{(k)}$, and if $i \neq j$, let $V_{ij}^{(k)}$ be a partial isometry in $\mathcal{M}$ with initial projection $P_j^{(k)}$ (meaning that $V_{ij}^{(k)}(V_{ij}^{(k)})^* = P_j$) and final projection $P_i^{(k)}$ (meaning that $(V_{ij}^{(k)})^* V_{ij}^{(k)} = P_i^{(k)}$). We now prove that if x_2 is sufficiently close (in $\|\cdot\|_2$) to a positive definite element in the type I subfactor of $\mathcal{M}$ generated by $\{V_{ij}^{(k)}\}_{i,j=1}^k$, then $\tau(p(x_1, x_2)) \geq 0$ when p satisfies the hypotheses of Theorem 3.4. In the statement of the theorem, we regard x_2 as an operator matrix and compare it entry-wise to the element $(b_{ij} V_{ij}^{(k)})_{i,j=1}^k$.

Theorem 3.5. *Let $\mathcal{M}$ be a finite von Neumann algebra with faithful trace state τ, let x_1, x_2 be positive, invertible elements in $\mathcal{M}$, and adopt the notation in the previous paragraph. Let $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_4 t_1 t_2 + \bar{c}_4 t_2 t_1 + c_5 t_2^2$ be a self-adjoint noncommutative polynomial in $\mathcal{A}$. Suppose that*

$$p(s, t) = c_0 + c_1 s + c_2 t + c_3 s^2 + 2\operatorname{Re}(c_4) st + c_5 t^2 \geq 0$$

for all real numbers $0 < s, t \leq 1$, that either $c_3 = 0$, $c_5 = 0$ and $2\operatorname{Re}(c_4) \neq 0$ or $c_3 > 0$ or $c_5 > 0$, and that for all $k \in \mathbb{N}$ there exists a type I subfactor of $\mathcal{M}$ generated by $\{V_{ij}^{(k)}\}_{i,j=1}^k$ as in the previous paragraph, and a positive definite contraction $(b_{ij})_{i,j=1}^k \in M_k(\mathbb{C})$ such that

$$\big\| P_i^{(k)} x_2 P_j^{(k)} - b_{ij} V_{ij}^{(k)} \big\|_2 < \frac{1}{k^{100}}, \quad \textit{for all } i, j \in \{1, 2, \dots, k\}.$$

Then $\tau(p(x_1, x_2)) \geq 0$.

Proof. Let $D_k = \sum_{i=1}^{k} \frac{i-1}{k} P_i^{(k)}$ and $B_k = \sum_{i,j=1}^{k} b_{ij} V_{ij}^{(k)}$. Writing $x_1 = D_k + (x_1 - D_k)$ and $x_2 = B_k + (x_2 - B_k)$, we have

$$\begin{aligned}
&\tau(p(x_1), p(x_2)) \\
&\quad = c_0 + c_1 \tau\big(D_k + (x_1 - D_k)\big) + c_2 \tau\big(B_k + (x_2 - B_k)\big) + c_3 \tau\big((D_k + (x_1 - D_k))^2\big) \\
&\qquad\quad + 2\operatorname{Re}(c_4) \tau\big((D_k + (x_1 - D_k))(B_k + (x_2 - B_k))\big) + c_5 \tau\big((B_k + (x_2 - B_k))^2\big) \\
&\quad = p(\tau(D_k), \tau(B_k)) + c_1 \tau(x_1 - D_k) + c_2 \tau(x_2 - B_k) \\
&\qquad\quad + 2c_3 \tau(D_k(x_1 - D_k)) + c_3 \tau(x_1 - D_k)^2 + 2\operatorname{Re}(c_4) \tau(D_k(x_2 - B_k)) \\
&\qquad\quad + 2\operatorname{Re}(c_4) \tau(B_k(x_1 - D_k)) + 2\operatorname{Re}(c_4) \tau\big((x_1 - D_k)(x_2 - B_k)\big) \\
&\qquad\quad + 2c_5 \tau\big(B_k(x_2 - B_k)\big) + c_5 \tau\big((x_2 - B_k)^2\big).
\end{aligned}$$

Therefore, by the triangle and Cauchy–Schwartz inequalities and the fact that the operator norm dominates the $\|\cdot\|_2$-norm,

$$\left|\tau(p(x_1), p(x_2)) - p(\tau(D_k), \tau(B_k))\right| \leq \left(|c_1| + |c_2| + 3|c_3| + 6\,\mathrm{Re}(c_4) + 3c_5\right)\frac{1}{k}.$$

Since $W^*(D_k, B_k) \cong W^*(\mathrm{diag}((i-1)/k, i \in \{1, \ldots, k\}), (b_{ij})_{i,j=1}^k) \subseteq M_k(\mathbb{C})$ via the obvious trace-preserving $*$-isomorphism, it follows that

$$\tau(p(x_1), p(x_2)) \geq 0. \qquad \square$$

Proposition 3.6. *Let $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_4 t_1 t_2 + \bar{c}_4 t_2 t_1 + c_5 t_2^2$ be a self-adjoint noncommutative polynomial in $\mathscr{A}$ satisfying the hypotheses of Theorem 3.4, and let $\mathscr{M}$ be a finite von Neumann algebra with faithful trace state τ. If $0 < x_1, x_2 \leq 1$ in $\mathscr{M}$ then $\tau(p(x_1, x_2)) < 0$ if and only if*

$$c_5\|x_2 - \tau(x_2)\|_2^2 + c_3\|x_1 - \tau(x_1)\|_2^2 + 2\,\mathrm{Re}(c_4)\langle x_1 - \tau(x_1), x_2 - \tau(x_2)\rangle_2 < -p(\tau(x_1), \tau(y_1)).$$

Proof. Writing each $\tau(x_i x_j)$ as $\tau\big((x_i - \tau(x_i)1)(x_j - \tau(x_j)1)\big) + \tau(x_i)\tau(x_j)$, we see that

$$\tau(p(x_1, x_2)) = p(\tau(x_1), \tau(y_1)) + c_5\|x_2 - \tau(x_2)\|_2^2 + c_3\|x_1 - \tau(x_1)\|_2^2 + 2\,\mathrm{Re}(c_4)\langle x_1 - \tau(x_1), x_2 - \tau(x_2)\rangle_2.$$

The result follows. $\square$

In the rest of this section, we narrow down the possibilities for disproving the CEC using polynomials satisfying the hypotheses of Theorem 3.4 in the nonrotated case, where $\mathrm{Re}(c_4) = 0$. We point out that if $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$ is a self-adjoint noncommutative polynomial in $\mathscr{A}$ satisfying the hypotheses of Theorem 3.4 with both $c_5 \geq 0$ and $c_3 \geq 0$, then $\tau(p(x_1, x_2)) \geq 0$ by the proof of Proposition 3.6.

Theorem 3.7. *Let $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$ be a self-adjoint noncommutative polynomial in $\mathscr{A}$ satisfying the hypotheses of Theorem 3.4 with $c_3 > 0$, $c_5 < 0$ and such that $c_1 \geq 0$ and $c_2 \leq 0$. Then, for any finite von Neumann algebra $\mathscr{M}$ with faithful trace state τ, we have*

$$\tau(p(x_1, x_2)) \geq 0,$$

for any positive definite contractions x_1 and x_2 in $\mathscr{M}$.

Proof. Assume that $p(t_1, t_2)$ satisfies the hypotheses. Suppose that there exists a finite von Neumann algebra $\mathscr{M}$ with faithful trace state τ and positive definite contractions x_1 and x_2 such that $\tau(p(x_1, x_2)) < 0$. If $c_1 \geq 0$ and $c_2 \leq 0$, then

$$p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2,$$

so $c_0 + (c_1 + c_3\varepsilon)\varepsilon + c_2 + c_5 \geq 0$ for every $\varepsilon > 0$, and hence $c_0 \geq -c_5 - c_2$. Thus

$$\begin{aligned} 0 &> c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2 \\ &\geq -c_5 - c_2 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2 \\ &= -c_5(1 - t_2^2) + c_3 t_1^2 + c_1 t_1 - c_2(1 - t_2), \end{aligned}$$

and

$$0 > -c_5 \tau(1 - x_2^2) + c_3 \tau(x_1^2) + c_1 \tau(x_1) - c_2 \tau(1 - x_2) \geq 0.$$

This is a contradiction. □

Theorem 3.8. *Let $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$ be a self-adjoint noncommutative polynomial in $\mathscr{A}$ satisfying the hypotheses of Theorem 3.4 with $c_3 > 0$, $c_5 < 0$ and such that $c_1 < 0$ and $c_2 = 0$. Then for any finite von Neumann algebra $\mathscr{M}$ with faithful trace state τ,*

$$\tau(p(x_1, x_2)) \geq 0,$$

for any positive definite contractions x_1 and x_2 in $\mathscr{M}$.

Proof. Assume that $p(t_1, t_2)$ satisfies the hypotheses. Let $\mathscr{M}$ be a finite von Neumann algebra with faithful trace state τ and let x_1 and x_2 be positive definite contractions. If $c_1 < 0$ and $c_2 = 0$, then for every $\varepsilon > 0$ letting $t_1 = \varepsilon - c_1/(2c_3)$,

$$c_0 + c_3\varepsilon^2 - \frac{c_1^2}{4c_3} + c_5 \geq 0,$$

and therefore $c_0 \geq \dfrac{c_1^2}{4c_3} - c_5$. Then

$$\begin{aligned} p(t_1, t_2) &= c_0 + c_3\Big(t_1 + \frac{c_1}{2c_3}\Big)^2 - \frac{c_1^2}{4c_3} + c_5 t_2^2 \\ &\geq \frac{c_1^2}{4c_3} - c_5 + c_3\Big(t_1 + \frac{c_1}{2c_3}\Big)^2 - \frac{c_1^2}{4c_3} + c_5 t_2^2 = -c_5(1 - t_2^2) + c_3\Big(t_1 + \frac{c_1}{2c_3}\Big)^2. \end{aligned}$$

Therefore

$$\tau(p(x_1, x_2)) = -c_5 \tau(1 - x_2^2) + c_3 \tau\bigg(\Big(x_1 + \frac{c_1}{2c_3}\Big)^2\bigg) \geq 0.$$ □

The previous two theorems establish that any polynomial $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$ in $\mathscr{A}$ that has a chance to disprove the CEC must satisfy either $c_2 > 0$ or both $c_1 < 0$ and $c_2 < 0$.

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