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A generalization of even and odd functions

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We generalize the concepts of even and odd functions in the setting of complex-valued functions of a complex variable. If $n > 1$ is a fixed integer and r is an integer with $0 \le r < n$, we define what it means for a function to have type $r \bmod n$. When $n = 2$, this reduces to the notions of even ($r = 0$) and odd ($r = 1$) functions respectively. We show that every function can be decomposed in a unique way as the sum of functions of types-0 through $n-1$. When the given function is differentiable, this decomposition is compatible with the differentiation operator in a natural way. We also show that under certain conditions, the type r component of a given function may be regarded as a real-valued function of a real variable. Although this decomposition satisfies several analytic properties, the decomposition itself is largely algebraic, and we show that it can be explained in terms of representation theory.

1. Introduction

1.1. ***Background.*** The notions of *even* and *odd* functions are well-known to most students of high school and college algebra. A function $f : \mathbb{R} \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and is odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. These concepts are important in many areas of analysis, and there are numerous useful examples of even or odd functions. For example, the function $f(x) = \cos x$ is even, as is any polynomial in x whose nonzero coefficients all correspond to even powers of x. Although there are numerous functions that are neither even nor odd, every function $f : \mathbb{R} \to \mathbb{R}$ decomposes in a unique way as $f = f_e + f_o$, where f_e is even and f_o is odd. For instance, the equation $e^x = \cosh x + \sinh x$ can be thought of as the decomposition of the exponential function e^x into its even and odd parts.

To motivate the following work, we revisit the definitions of even and odd functions and express the defining equations slightly differently. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and write $\epsilon = -1 \in \mathbb{R}$. Then f is even if

$$f(\epsilon x) = \epsilon^0 f(x) \quad \text{for all } x \in \mathbb{R}, \tag{1}$$

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and f is odd if

$$f(\epsilon x) = \epsilon^1 f(x) \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

In other words, a function is even (or odd) if it satisfies a certain functional equation involving a square root of unity. Note that this definition also makes sense if we replace the field $\mathbb{R}$ with the field $\mathbb{C}$ of complex numbers.

1.2. ***Summary of results.*** In the following, we let $f : \mathbb{C} \to \mathbb{C}$ be a function and fix an integer $n > 1$. If $r \in \mathbb{Z}$ with $0 \le r < n$, we say that the function f is of type r mod n if f satisfies a certain functional equation (depending upon r) for every n-th root of unity. In the special case that $n = 2$, this definition reduces to the usual notions of even and odd functions. In Theorem 5, we show (for arbitrary n) that every function $f : \mathbb{C} \to \mathbb{C}$ decomposes as $f = f_0 + \cdots + f_{n-1}$, where f_r is of type r mod n. Moreover, we show in Theorem 6 that this decomposition is unique. The set of all functions $f : \mathbb{C} \to \mathbb{C}$ may be regarded as a vector space, and the set of all functions of type r mod n may be regarded as a subspace. Thus we also explain how the decomposition $f = f_0 + \cdots + f_{n-1}$ may be thought of in terms of projections from a vector space onto various subspaces.

We show in Section 3 (Corollary 15) that if a given complex function $f : \mathbb{C} \to \mathbb{C}$ is real (i.e., $f(\mathbb{R}) \subseteq \mathbb{R}$), then under certain assumptions, the functions f_r (in the decomposition $f = f_0 + \cdots + f_{n-1}$) are also real. This explains, for example, why the functions $\cosh : \mathbb{C} \to \mathbb{C}$ and $\sinh : \mathbb{C} \to \mathbb{C}$ produce real outputs when the inputs are restricted to real numbers. In the classical setting of even and odd functions, it is well-known that the derivative of an even (resp. odd) function is odd (resp. even), and in Section 4 we prove several analogous results that apply in our setting. In the case of the exponential function e^z, these analytic results lead to solutions to a familiar real differential equation, and we address this connection between our framework and differential equations in Example 20.

In Section 5, we show that some of these results may be explained by working in the algebraic setting of representation theory. We replace the set of n-th roots of unity with a finite group G, and we replace the field of complex numbers with a $\mathbb{C}[G]$-module, where $\mathbb{C}[G]$ denotes the complex group of G. Under certain conditions, a function $f : V \to W$ decomposes as a sum of functions that satisfy various functional equations analogous to those of Section 2. These conditions are easily satisfied in the setting of a function $f : \mathbb{C} \to \mathbb{C}$.

2. Definitions and basic results

Fix an integer $n > 1$. A complex number ϵ is an n-th root of unity if $\epsilon^n = 1$. We now generalize the definitions in (1) and (2) in the setting where $f : \mathbb{C} \to \mathbb{C}$ is a complex-valued function.

Definition 1. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function. Fix an integer $n > 1$ and an integer r with $0 \le r < n$. We say that f is of *type* r mod n if

$$f(\epsilon z) = \epsilon^r f(z) \tag{3}$$

for every $z \in \mathbb{C}$ and every n-th root of unity $\epsilon \in \mathbb{C}$.

If there is no danger of ambiguity regarding n, we may shorten the notation and say that a function $f : \mathbb{C} \to \mathbb{C}$ has type r. If, for example $n = 3$, then a function $f : \mathbb{C} \to \mathbb{C}$ may have type 0, type 1, or type 2. The third roots of unity are $\epsilon = 1$, $e^{2\pi i/3}$, or $e^{4\pi i/3}$. Thus a type-0 function satisfies the equations

$$\begin{aligned} f(1z) &= 1f(z), \\ f(e^{2\pi i/3}z) &= f(z), \\ f(e^{4\pi i/3}z) &= f(z), \end{aligned}$$

a type-1 function satisfies

$$\begin{aligned} f(1z) &= 1f(z), \\ f(e^{2\pi i/3}z) &= e^{2\pi i/3}f(z), \\ f(e^{4\pi i/3}z) &= e^{4\pi i/3}f(z), \end{aligned}$$

and a type-2 function satisfies

$$\begin{aligned} f(1z) &= 1f(z), \\ f(e^{2\pi i/3}z) &= e^{4\pi i/3}f(z), \\ f(e^{4\pi i/3}z) &= e^{2\pi i/3}f(z). \end{aligned}$$

If $\epsilon \in \mathbb{C}$ is an n-th root of unity and $\epsilon^k \neq 1$ for all $1 \le k < n$, we say that ϵ is a *primitive* n-th root of unity. For example, in case $n = 3$ above, the primitive third roots of unity are $e^{2\pi i/3}$ and $e^{4\pi i/3}$, whereas 1 is not a primitive third root of unity. The next lemma shows that we do not need to check that a given function satisfies (3) for every n-th root of unity. Rather, it is enough to know that (3) holds for at least one primitive n-th root ϵ.

Lemma 2. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function. Fix an integer $n > 1$ and an integer $0 \le r < n$. Let $\epsilon \in \mathbb{C}$ be a primitive n-th root of unity. If f has the property that $f(\epsilon z) = \epsilon^r f(z)$ for all $z \in \mathbb{C}$, then $f(\omega z) = \omega^r f(z)$ for every $z \in \mathbb{C}$ and every n-th root of unity $\omega \in \mathbb{C}$.*

Proof. Note that $\omega = \epsilon^k$ for some integer $0 \le k < n$. It follows that $f(\omega z) = f(\epsilon^k z)$, and we see that $f(\epsilon^k z) = \epsilon^r f(\epsilon^{k-1} z) = \epsilon^r \epsilon^r f(\epsilon^{k-2} z) = \cdots = (\epsilon^r)^{k-1} f(\epsilon^1 z) = (\epsilon^r)^k f(z)$. Hence $f(\omega z) = f(\epsilon^k z) = (\epsilon^r)^k f(z) = (\epsilon^k)^r f(z) = \omega^r f(z)$. □

The following construction gives rise to a function of type r mod n defined in terms of some given function $f : \mathbb{C} \to \mathbb{C}$. We shall see in Theorem 5 that this construction leads to a way to decompose f as a sum of functions of types $0, 1, \ldots,$ and $n-1$.

Definition 3. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function. Given an integer $n > 1$, a primitive n-th root of unity $\epsilon \in \mathbb{C}$, and $r \in \mathbb{Z}$ with $0 \le r < n$, define $f_{(r,\epsilon)} : \mathbb{C} \to \mathbb{C}$ by

$$f_{(r,\epsilon)}(z) = \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{-kr} f(\epsilon^k z). \tag{4}$$

Theorem 4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function. Given an integer $n > 1$, a primitive n-th root of unity $\epsilon \in \mathbb{C}$, and $r \in \mathbb{Z}$ with $0 \le r < n$, define $f_{(r,\epsilon)}(z)$ as in Definition 3. Then $f_{(r,\epsilon)}$ is of type r mod n.*

Proof. By Lemma 2, it suffices to show that $f_{(r,\epsilon)}(\epsilon z) = \epsilon^r f_{(r,\epsilon)}(z)$. Note that

$$f_{(r,\epsilon)}(\epsilon z) = \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{-kr} f(\epsilon^{k+1} z) = \frac{1}{n} \sum_{l=1}^{n} \epsilon^{-r(l-1)} f(\epsilon^l z), \quad \text{where } l = k+1.$$

Also note that $\epsilon^{-r(n-1)} f(\epsilon^n z) = \epsilon^{-r(0-1)} f(\epsilon^0 z)$, so

$$f_{(r,\epsilon)}(\epsilon z) = \frac{1}{n} \sum_{l=0}^{n-1} \epsilon^{-r(l-1)} f(\epsilon^l z) = \frac{\epsilon^r}{n} \sum_{l=0}^{n-1} \epsilon^{-rl} f(\epsilon^l z) = \epsilon^r f_{(r,\epsilon)}(z). \qquad \square$$

Theorem 5. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function. Fix an integer $n > 1$, and let $\epsilon \in \mathbb{C}$ be a primitive n-th root of unity. Then $f = \sum_{r=0}^{n-1} f_{(r,\epsilon)}$, where $f_{(r,\epsilon)}$ is given by Definition 3.*

Proof. Note that

$$\sum_{r=0}^{n-1} f_{(r,\epsilon)}(z) = \sum_{r=0}^{n-1} \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{-kr} f(\epsilon^k z) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\sum_{r=0}^{n-1} \epsilon^{-kr} \right) f(\epsilon^k z).$$

Since

$$\sum_{r=0}^{n-1} \epsilon^{-kr} = \sum_{r=0}^{n-1} (\epsilon^{-k})^r = \begin{cases} \dfrac{1-(\epsilon^{-k})^n}{1-\epsilon^{-k}} = 0 & \text{if } 0 < k < n, \\ n & \text{if } k = 0, \end{cases}$$

it follows that

$$\sum_{r=0}^{n-1} f_{(r,\epsilon)}(z) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\sum_{r=0}^{n-1} \epsilon^{-kr} \right) f(\epsilon^k z) = \frac{n}{n} f(\epsilon^0 z) = f(z). \qquad \square$$

Although Theorem 5 asserts that every function can be written as a sum of functions of types 0 through $n-1$, it does not preclude the possibility that this can be done in several ways. Theorem 6 addresses this issue.

Theorem 6. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function, and fix an integer $n > 1$. If $f = f_0 + \cdots + f_{n-1}$ and $f = g_0 + \cdots + g_{n-1}$ where f_r and g_r have type $r \bmod n$ for $0 \le r < n$, then $f_r = g_r$ for all r.*

Proof. Suppose that $f = f_0 + \cdots + f_{n-1}$ and $f = g_0 + \cdots + g_{n-1}$ where f_r and g_r have type r. Then $h_0 + \cdots + h_{n-1} = 0$ where $h_r = f_r - g_r$ has type r for all r. Thus it is sufficient to prove that if $h_0 + \cdots + h_{n-1} = 0$, where h_r has type r, then $h_r = 0$ for all r.

Suppose the result is false. There exists a strictly increasing sequence $r_1, \dots, r_k$, with $r_i \in \{0, 1, \dots, n-1\}$ for all i, along with functions q_{r_i} $(i = 1, \dots, k)$ so that q_{r_i} is a nonzero function of type r_i and

$$q_{r_1} + \cdots + q_{r_k} = 0. \tag{5}$$

Furthermore, we may suppose we have chosen such a counterexample with k minimal.

Let ϵ be a primitive n-th root of unity. Evaluating both sides of (5) at ϵz implies that $0 = (q_{r_1} + \cdots + q_{r_k})(\epsilon z) = \epsilon^{r_1} q_{r_1}(z) + \cdots + \epsilon^{r_k} q_{r_k}(z)$, while multiplying both sides of (5) by ϵ^{r_1} yields $\epsilon^{r_1} q_{r_1} + \epsilon^{r_1} q_{r_2} + \cdots + \epsilon^{r_1} q_{r_k} = 0$. After subtracting these two equations, we see that

$$(\epsilon^{r_2} - \epsilon^{r_1}) q_{r_2} + \cdots + (\epsilon^{r_k} - \epsilon^{r_1}) q_{r_k} = 0.$$

By assumption $q_{r_i} \ne 0$ and $\epsilon^{r_i} - \epsilon^{r_1} \ne 0$ since ϵ is a primitive n-th root of unity and $r_i \ne r_1$. Hence $r_2 \cdots r_k$ is a strictly increasing sequence with $r_2, \dots, r_k \in \{0, 1, \dots, n-1\}$ such that $(\epsilon^{r_i} - \epsilon^{r_1}) q_{r_i}$ $(2 \le i \le n-1)$ is a nonzero function of type r_i with $(\epsilon^{r_2} - \epsilon^{r_1}) q_{r_2} + \cdots + (\epsilon^{r_k} - \epsilon^{r_1}) q_{r_k} = 0$, contradicting the fact that k was minimal. It follows that $q_{r_i} = 0$ for all i. $\square$

Corollary 7. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \le r < n$. Let $\epsilon, \omega \in \mathbb{C}$ be primitive n-th roots of unity. Then $f_{(r,\epsilon)} = f_{(r,\omega)}$.*

Proof. By Theorem 4 and Lemma 2 it follows that $f_{(r,\epsilon)}(\epsilon z) = \epsilon^r f_{(r,\epsilon)}(z)$ and $f_{(r,\omega)}(\epsilon z) = \epsilon^r f_{(r,\omega)}(z)$ for all $z \in \mathbb{C}$. From Theorem 5 we know that $f = \sum_{r=0}^{n-1} f_{(r,\epsilon)}$ and $f = \sum_{r=0}^{n-1} f_{(r,\omega)}$. Theorem 6 implies that $f_{(r,\epsilon)} = f_{(r,\omega)}$ for all r. $\square$

Remark 8. We have shown that $f_{(r,\epsilon)} = f_{(r,\omega)}$ whenever $\epsilon, \omega \in \mathbb{C}$ are primitive n-th roots of unity. Thus it is unambiguous to define the notation f_r by the equation

$$f_r = f_{(r,\epsilon)}, \tag{6}$$

where ϵ is any primitive n-th root of unity and $f_{(r,\epsilon)}$ is given by Definition 3.

An obvious corollary of Theorem 6 is that there is a unique way to write the zero function as a sum of functions of various types. This can essentially be regarded as the statement that functions of differing types (mod n) are linearly independent, and thus it makes sense to phrase these results in terms of linear algebra.

Definition 9. Let F be the vector space of all functions $f : \mathbb{C} \to \mathbb{C}$. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \le r < n$. Define $F_r \subseteq F$ by

$$F_r = \{f \in F \mid f \text{ has type } r \text{ mod n}\}.$$

It is straightforward to show that if $f, g \in F_r$ and $c \in \mathbb{C}$, then $cf + g \in F_r$. Thus the subset F_r is in fact a vector subspace of F. Note that Theorem 5 and Theorem 6 may be summarized by noting that F decomposes as

$$F = F_0 \oplus \cdots \oplus F_{n-1}.$$

Definition 10. Let $f \in F$. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \le r < n$. Define $\pi_r(f)$ to be the unique type-r summand that corresponds to writing f as a sum of functions of types 0 through $n - 1$.

In light of the decomposition $F = F_0 \oplus \cdots \oplus F_{n-1}$, the map π_r is well-defined and may be regarded as the projection from F onto the subspace F_r. From Theorem 5 and Theorem 6, it follows that $\pi_r(f) = f_r$, where f_r is defined as in (6). Although this equation could be used as a definition for $\pi_r : F \to F_r$, Definition 10 has the advantage of that the next results follow almost immediately from this definition.

Lemma 11. *Let $f \in F$. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \le r < n$. Then $(f_r)_r = f_r$.*

Proof. This is equivalent to the assertion that $\pi_r \circ \pi_r = \pi_r$. Since $\pi_r(f)$ is of type r, Definition 10 implies that $\pi_r(\pi_r(f)) = \pi_r(f)$. □

Lemma 12. *Let $f \in F$. Fix an integer $n > 1$, and let $r, s \in \mathbb{Z}$ with $0 \le r, s < n$. Then $(f_r)_s = 0$ if $r \ne s$.*

Proof. If $f_r \in F_r$ is decomposed according to the direct sum $F = F_0 \oplus \cdots \oplus F_{n-1}$, then the r-th component of f_r is f_r, and every other component is 0, so it follows that $(f_r)_s = 0$ when $r \ne s$. □

3. Relationship to real-valued functions

Several important complex-valued functions $f : \mathbb{C} \to \mathbb{C}$ have the property that $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$. For example, the functions e^z, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ have this property, as do all polynomial functions with real coefficients. In this section, we show that this property carries over to the type-r component of f.

Lemma 13. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function with the property that $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \le r < n$. Define π_r as in Definition 10. Then $\overline{\pi_r(f)(z)} = \pi_r(f)(\overline{z})$ for all $z \in \mathbb{C}$.*

Proof. Let ϵ be a primitive n-th root of unity. Since $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ for all $z_1, z_2 \in \mathbb{C}$, it follows that

$$\overline{\pi_r(f)(z)} = \overline{\frac{1}{n}\sum_{k=0}^{n-1} \epsilon^{-kr} f(\epsilon^k z)} = \frac{1}{n}\sum_{k=0}^{n-1} \overline{\epsilon^{-kr}}\, \overline{f(\epsilon^k z)} = \frac{1}{n}\sum_{k=0}^{n-1} \overline{\epsilon^{-kr}}\, f(\overline{\epsilon^k}\overline{z}).$$

Observe that $\overline{\epsilon} = \epsilon^{-1}$ because $|\epsilon| = 1$, and thus

$$\overline{\pi_r(f)(z)} = \frac{1}{n}\sum_{k=0}^{n-1} \overline{\epsilon}^{-kr} f(\overline{\epsilon}^k \overline{z}) = \frac{1}{n}\sum_{k=0}^{n-1} \omega^{-kr} f(\omega^k \overline{z}),$$

where $\omega = \overline{\epsilon}$. Since $\omega = \epsilon^{-1}$, ω is also a primitive n-th root of unity, whence $\overline{\pi_r(f)(z)} = f_{(r,\omega)}(\overline{z}) = \pi_r(f)(\overline{z})$ for all $z \in \mathbb{C}$ by Remark 8. $\square$

Recall that a complex function $f : \mathbb{C} \to \mathbb{C}$ is said to be *real* if $f(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$. The next lemma provides a criterion to show that a function is real, and a proof can be found in [Churchill and Brown 2008, page 87].

Lemma 14. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function. If f has the property that $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$, then f is real.*

The following result is now obvious in light of Lemma 14 and Lemma 13.

Corollary 15. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function with the property that $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \le r < n$. If $z \in \mathbb{R}$, then $f_r(z) \in \mathbb{R}$.*

Since $\cosh z$ and $\sinh z$ may be regarded as $\pi_0(e^z)$ and $\pi_1(e^z)$ (with $n = 2$), we recover the obvious facts that $\cosh x, \sinh x \in \mathbb{R}$ if $x \in \mathbb{R}$. More interestingly ($n = 3$), we see for example that if $\epsilon = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $r \in \{0, 1, 2\}$, then $\frac{1}{3}\left(e^x + \epsilon^{-r} e^{\epsilon x} + \epsilon^{-2r} e^{\epsilon^2 x}\right) \in \mathbb{R}$ for all $x \in \mathbb{R}$.

It is not immediately obvious whether the condition that $f : \mathbb{C} \to \mathbb{C}$ is real is sufficient to guarantee that f_r is real whenever $0 \le r < n$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} 0 & \text{if } z \in \mathbb{R}, \\ i & \text{if } z \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

Then, if $n = 3$, it is straightforward to compute that

$$f_0(1) = \frac{2i}{3} \quad \text{and} \quad f_1(1) = f_2(1) = \frac{i}{3}\left(e^{2\pi i/3} + e^{4\pi i/3}\right) = -\frac{i}{3},$$

which implies that $f_r(1) \notin \mathbb{R}$ for $r = 0, 1, 2$. In particular, this shows that f must satisfy a stronger condition (than the condition that f is real) in order to guarantee that f_r is real for $0 \leq r < n$.

4. Relationship to the derivative

Recall that F denotes the space of all functions $f : \mathbb{C} \to \mathbb{C}$.

Definition 16. Define the vector space $\mathcal{F}$ by

$$\mathcal{F} = \{f \in F \mid f \text{ is holomorphic}\}.$$

Definition 17. Let $f \in F$. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \leq r < n$. Define the subspace $\mathcal{F}_r$ by

$$\mathcal{F}_r = \mathcal{F} \cap F_r.$$

If $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, the following theorem establishes a relationship between the projection maps π_r and the differentiation operator.

Theorem 18. *Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \leq r < n$. Define π_r and π_{r-1} as in Definition 10, and let $\frac{d}{dz} : \mathcal{F} \to \mathcal{F}$ denote the differentiation operator. Then, for $f \in \mathcal{F}$, we have*

$$\left(\frac{d}{dz} \circ \pi_r\right)(f) = \left(\pi_{r-1} \circ \frac{d}{dz}\right)(f),$$

where we read the integer $r - 1$ modulo n.

Proof. Let $f \in \mathcal{F}$ and fix a primitive n-th root of unity $\epsilon \in \mathbb{C}$. Note that by definition

$$\left(\pi_{r-1} \circ \frac{d}{dz}\right)(f)(z) = \pi_{r-1}(f')(z) = \sum_{k=0}^{n-1} \epsilon^{-k(r-1)} f'(\epsilon^k z).$$

From the chain rule, the derivative of the function $z \mapsto f(\epsilon^k z)$ is the function $z \mapsto \epsilon^k f'(\epsilon^k z)$, so we have

$$\left(\frac{d}{dz} \circ \pi_r\right)(f)(z) = \sum_{k=0}^{n-1} \epsilon^k \epsilon^{-kr} f'(\epsilon^k z) = \sum_{k=0}^{n-1} \epsilon^{-k(r-1)} f'(\epsilon^k z). \qquad \square$$

The following corollary generalizes the fact that the derivative of an odd (resp. even) function is even (resp. odd). Although it can be demonstrated directly from the definition [Ahlfors 1979, page 24] of the complex derivative, we prove the result using Theorem 18.

Corollary 19. *Let $f \in \mathcal{F}$. Fix an integer $n > 1$, and let $r \in \mathbb{Z}$ with $0 \leq r < n$. If $f \in \mathcal{F}_r$, then $f' \in \mathcal{F}_{r-1}$, where we read the integer $r - 1$ modulo n.*

Proof. By Theorem 18, $\displaystyle \frac{d}{dz}(f) = \frac{d}{dz}(\pi_r(f)) = \pi_{r-1}\left(\frac{d}{dz}(f)\right) \in \mathcal{F}_{r-1}$. $\square$

Example 20. Fix an integer $n > 1$, and let $f(z) = e^z$. We saw in Corollary 15 that for $0 \le k < n$, $f_k(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$. Moreover Theorem 18 implies that $df_r/dz = \pi_{r-1}(df/dz) = f_{r-1}$. Thus if we let $f_k|_{\mathbb{R}}$ denote the restriction of f_k to the real numbers, it follows that

$$\frac{d}{dx}(f_r|_{\mathbb{R}}) = f_{r-1}|_{\mathbb{R}},$$

where d/dx denotes the real differentiation operator and the subscripts r and $r-1$ are read modulo n. Thus the function $f_r|_{\mathbb{R}}$ is a solution to the (real) differential equation $d^n y/dx^n = y$. If, for example, $n = 3$, it is straightforward to check that the functions $f_0|_{\mathbb{R}}$, $f_1|_{\mathbb{R}}$, and $f_2|_{\mathbb{R}}$ form a basis for the solution space of $d^n y/dx^n = y$.

5. Relationship to representation theory

The previous setting can be generalized considerably. For a fixed integer $n > 1$, the set G of all n-th roots of unity in $\mathbb{C}$ forms a multiplicative group. This group acts on the space $\mathbb{C}$ as follows. For $g \in G \subseteq \mathbb{C}$ and $z \in \mathbb{C}$, the action is given by $g.z = gz$. (Here, we use the dot notation for group actions, as in [Fulton and Harris 1991].) Thus the domain and codomain of a function $f : \mathbb{C} \to \mathbb{C}$ are G-modules. Because of this, it is natural to conjecture that the above results can explained module-theoretically. Indeed, many of the previous concepts may be regarded as special cases of module-theoretic results. For example, Definition 21 is a module-theoretic analogue of Definition 3, and Corollary 28 yields Theorem 5 as a special case.

If G is a finite group, we define the group algebra $\mathbb{C}[G]$ as in [Isaacs 1976]. We define the notions of a module, a simple module, and a module homomorphism as in [Isaacs 1993] or any other standard text. Note that the function $f : V \to W$ in Definition 21 need not be linear.

Definition 21. Let G be a finite group, and V and W be $\mathbb{C}[G]$-modules. Suppose $f : V \to W$ is a function and $\phi : G \to G$ is a homomorphism. Then define $f_\phi : V \to W$ by

$$f_\phi(v) = \frac{1}{|G|} \sum_{h \in G} \phi(h^{-1}).f(h.v).$$

Note that if G is the group of n-th roots of unity in $\mathbb{C}$ and $\phi : G \to G$ is given by $\phi(g) = g^r$, then the function f_ϕ is exactly the function $f_{(r,\epsilon)}$ given in Definition 3. The following theorem states that not only does f_ϕ generalize $f_{(r,\epsilon)}$, but it behaves in a manner that generalizes Theorem 4.

Theorem 22. *Let G be a finite group and V and W be $\mathbb{C}[G]$-modules. Suppose $f : V \to W$ is a function and $\phi : G \to G$ is a homomorphism. Then $f_\phi(g.v) = \phi(g).f_\phi(v)$.*

Proof. From the definition of f_ϕ, we have, with $u = hg$,

$$f_\phi(g.v) = \frac{1}{|G|}\sum_{h\in G}\phi(h^{-1}).f(hg.v) = \frac{1}{|G|}\sum_{u\in G}\phi(gu^{-1}).f(u.v)$$
$$= \phi(g)\frac{1}{|G|}\sum_{u\in G}\phi(u^{-1}).f(u.v) = \phi(g).f_\phi(v). \qquad \square$$

In the case where G is the n-th roots of unity in $\mathbb{C}$ and $\phi(g) = g^r$ then the properties of the homomorphism f_ϕ are identical to those of the function $f_{(r,\epsilon)}$. The following Theorem shows that the property of $f_{(r,\epsilon)}$ shown in Lemma 11 not only holds under these conditions, but also in the more abstract setting of Theorem 22.

Theorem 23. *Let G be a finite group, and V and W be $\mathbb{C}[G]$-modules. Suppose $f : V \to W$ is a function and $\phi : G \to G$ is a homomorphism. Then $(f_\phi)_\phi = f_\phi$.*

Proof. By definition $f_\phi(v) = \frac{1}{|G|}\sum_{h\in G}\phi(h^{-1}).f(h.v)$. It follows that

$$((f_\phi)_\phi)(v) = \frac{1}{|G|}\sum_{h\in G}\phi(h^{-1}).f_\phi(h.v) = \frac{1}{|G|}\sum_{h\in G}\phi(h^{-1})\phi(h).f_\phi(v)$$
$$= \frac{1}{|G|}\sum_{h\in G}\phi(h^{-1}h).f_\phi(v) = \frac{1}{|G|}(nf_\phi(v)) = f_\phi(v). \qquad \square$$

When G is cyclic of order n, every homomorphism from G to G is determined by the image of some generator of G. For $0 \le r < n$, define $\phi_r : G \to G$ by $\phi_r(x) = x^r$ for all $x \in G$. Then the set of homomorphisms $G \to G$ is $\{\phi_r \mid 0 \le r < n\}$. As Corollary 24 shows, this new setting allows us to generalize the property of $f_{(r,\epsilon)}$ from Theorem 4 in slightly more specific terms than those of Theorem 22.

Corollary 24. *Let G be a finite cyclic group and V and W be $\mathbb{C}[G]$-modules. Suppose $f : V \to W$ is a function, and let $\phi_r : G \to G$ be the homomorphism given by $\phi_r(x) = x^r$. Then $f_{\phi_r}(x.v) = x^r.f_{\phi_r}(v)$ for all $v \in V$ and $x \in G$.*

If G is cyclic and V and W are $\mathbb{C}[G]$-modules with W simple, then it is possible to generalize Theorem 5. To demonstrate this, we rely on the following well-known fact, whose proof can be found in [Isaacs 1976].

Lemma 25 (Schur's Lemma). *Let G be a finite group, and suppose V and W are simple $\mathbb{C}[G]$-modules and $\phi : V \to W$ is a module homomorphism.*

(1) *Either ϕ is an isomorphism or $\phi = 0$.*

(2) *If $V = W$ then $\phi : W \to W$ is a scalar multiple of the identity function.*

If $x \in G$ is central in G, then the function $f_x : W \to W$ defined by $f_x(w) = x.w$ is a $\mathbb{C}[G]$-module homomorphism. Thus Schur's Lemma implies that every central element of G acts by a scalar on any simple module W. With G cyclic, every element in G is central. In particular, the generator $g \in G$ is central, so there must be some scalar ξ by which g acts on the elements of simple modules. Furthermore, G is finite so $|G| = n$ for some integer n. The next lemma shows that this integer allows us to be somewhat precise about the value of $\xi \in \mathbb{C}$.

Lemma 26. *Let G be a finite cyclic group with generator g and $|G| = n$. If W is a simple $\mathbb{C}[G]$-module, then g acts on W as multiplication by an n-th root of unity.*

Proof. The group G is abelian, so by Schur's Lemma, there exists $\xi \in \mathbb{C}$ so that $g.w = \xi w$ for all $w \in W$. This implies that an arbitrary element $g^k \in G$ acts by the scalar ξ^k. Since $|G| = n$, g^n is the identity element of G, and it follows that for $w \in W$, $w = g^n.w = \xi^n w$, which forces $\xi^n = 1$. □

In light of Theorem 23 and Corollary 24, it is reasonable to conjecture that there is some module-theoretic analogue of Theorem 5. The following theorem establishes a formula for the sum of the functions $f_{\phi_0}, f_{\phi_1}, \dots, f_{\phi_{n-1}}$. As a consequence of working in this more general setting, the resulting formula is more complicated than the formula in Theorem 5.

Theorem 27. *Let G be a finite cyclic group with generator g and $|G| = n$. Let V, W be $\mathbb{C}[G]$-modules with W simple, and let $f : V \to W$. If g acts on all $w \in W$ by the scalar ξ having multiplicative order d, then for all $v \in V$,*

$$\sum_{r=0}^{n-1} f_{\phi_r}(v) = \sum_{\substack{0 \le k < n \\ d \mid k}} f(g^k.v).$$

Proof. For $v \in V$,

$$\sum_{r=0}^{n-1} f_{\phi_r}(v) = \frac{1}{n}\sum_{r=0}^{n-1}\sum_{k=0}^{n-1}(g^{-k})^r.f(g^k.v) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{r=0}^{n-1}(\xi^{-k})^r\right) f(g^k.v).$$

Observe that

$$\sum_{r=0}^{n-1}(\xi^{-k})^r = \begin{cases} \dfrac{1-(\xi^-k)^n}{1-\xi^{-k}} = 0 & \text{if } d \nmid k \\ n & \text{if } d \mid k. \end{cases}$$

Hence

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{r=0}^{n-1}(\xi^{-k})^r\right) f(g^k.v) = \frac{1}{n}\sum_{\substack{0 \le k < n \\ d \mid k}} nf(g^k.v) = \sum_{\substack{0 \le k < n \\ d \mid k}} f(g^k.v),$$

and the desired result follows. □

Lemma 26 does not make it clear which n-th root of unity ξ is. If ξ happens to be primitive, then $|\xi| = |G| = n$. Applying this reasoning to Theorem 27 leads directly to the following module-theoretic generalization of Theorem 5.

Corollary 28. *Let G be a finite cyclic group with generator g and $|G| = n$. Let V and W be $\mathbb{C}[G]$-modules with W simple, and $f : V \to W$. Let $\xi \in \mathbb{C}$ be the n-th root of unity with the property that $g.w = \xi w$ for all $w \in W$. If ξ is a primitive n-th root of unity, then $f = \sum_{r=0}^{n-1} f_{\phi_r}$.*

Proof. Theorem 27 implies that $\sum_{r=0}^{n-1} f_{\phi_r}(v) = \sum_{k\in\Delta} f(g^k.v)$, where

$$\Delta = \{0 \le k < n \mid n \text{ divides } k\}.$$

But $\Delta = \{0\}$, so it follows that $\sum_{r=0}^{n-1} f_{\phi_r}(v) = f(v)$. □

This framework obviously applies in the setting of a function $f : \mathbb{C} \to \mathbb{C}$, and thus many of the results of Section 2 may be regarded as consequences of representation theory. With the current perspective, it is, for example, possible to decompose functions of the form $f : V \to \mathbb{C}$, where V is any module for the group G of complex n-th roots of unity. For instance, if V is the set of all $m \times m$ matrices with complex entries, then G acts on V by entry-wise multiplication. Alternatively, if V is taken to be the group algebra $\mathbb{C}[G]$, then G acts on V via the regular action, and this setting applies to functions $f : \mathbb{C}[G] \to \mathbb{C}$.

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References

[Ahlfors 1979] L. V. Ahlfors, *Complex analysis: an introduction to the theory of analytic functions of one complex variable*, 3rd ed., McGraw-Hill, New York, 1979. MR 80c:30001 Zbl 0395.30001

[Churchill and Brown 2008] R. V. Churchill and J. W. Brown, *Complex variables and applications*, 8th ed., McGraw-Hill, New York, 2008.

[Fulton and Harris 1991] W. Fulton and J. Harris, *Representation theory: a first course*, Grad. Texts in Math. **129**, Springer, New York, 1991. MR 93a:20069

[Isaacs 1976] I. M. Isaacs, *Character theory of finite groups*, Pure and Applied Math. **69**, Academic Press, New York, 1976. Reprinted Dover, New York, 1994. MR 57 #417 Zbl 0337.20005

[Isaacs 1993] I. M. Isaacs, *Algebra: a graduate course*, Brooks/Cole, Pacific Grove, CA, 1993. MR 95k:00003 Zbl 1157.00004

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