involve

a journal of mathematics

Continuous p-Bessel mappings and continuous p-frames in Banach spaces

Mohammad Hasan Faroughi and Elnaz Osgooei

msp

2011 vol. 4, no. 2

INVOLVE 4:2 (2011)
https://doi.org/10.2140/involve.2011.4.167

Continuous p-Bessel mappings and continuous p-frames in Banach spaces

Mohammad Hasan Faroughi and Elnaz Osgooei

(Communicated by David R. Larson)

We define the concept of continuous p-frames (cp-frames) for Banach spaces, generalizing discrete p-frames. We prove that under certain conditions the direct sum of a finite number of cp-frames is again a cp-frame. We obtain equivalent conditions for duals of cp-Bessel mappings and show existence and uniqueness of duals of independent cp-frames. Lastly we discuss perturbation of these frames.

1. Introduction

Frames were first introduced in the context of nonharmonic Fourier series [Duffin and Schaeffer 1952]. Outside of signal processing, frames did not seem to generate much interest until the groundbreaking work [Daubechies et al. 1986]. Today, the theory of discrete frames plays an important role not just in digital signal processing and scientific computation, but also in pure and applied mathematics. The interested reader is referred to [Han and Larson 2000; Heil and Walnut 1989] for theory and applications of frames.

A discrete frame is a countable family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of the frame elements. This concept was generalized in [Ali et al. 1993] to families indexed by some locally compact space endowed with a Radon measure; these frames are known as continuous frames. For more studies about frame theory and continuous frames we refer to [Christensen 2003; Ali et al. 1993; Gabardo and Han 2003; Rahimi et al. 2006].

Various generalizations of frames have been proposed recently, such as frames of subspaces [Asgari and Khosravi 2005], p-frames [Aldroubi et al. 2001; Cao et al. 2008; Christensen and Stoeva 2003], p-frames of subspaces [Najati and Faroughi 2007], g-frames [Sun 2006], and continuous g-frames [Abdollahpour and Faroughi

MSC2010: primary 42C99, 42C15; secondary 42C40.
Keywords: frames, continuous p-frames, Schauder basis, reflexive space.
This is part of Osgooei's Ph.D. thesis at Tabriz University.

2008; Joveini and Amini 2009]. We take as our starting point the generalization presented in [Christensen and Stoeva 2003].

Throughout this paper, (Ω, μ) will be a measure space and μ a positive, σ-finite measure. X is a Banach space with dual X^*. We choose $1 < p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. The normed dual X^* of a Banach space X is itself a Banach space and hence has a normed dual of its own, denoted by X^{**}. A mapping $\Lambda_X : X \mapsto X^{**}$ is well defined by the equation $\langle x, x^* \rangle = \langle x^*, \Lambda_X x \rangle$ for each $x^* \in X^*$; also, $\|\Lambda_X x\| = \|x\|$ for each $x \in X$. So $\Lambda_X : X \to X^{**}$ is an isometric isomorphism of X onto a closed subspace of X^{**}. If X is a reflexive Banach space then Λ_X is an isometric isomorphism of X onto X^{**}.

Definition 1.1. A countable family $\{g_i\}_{i=1}^{\infty} \subset X^*$ is a p-frame for X if there exist constants $A, B > 0$ such that

$$A\|f\| \le \left(\sum_{i=1}^{\infty} |g_i(f)|^p \right)^{1/p} \le B\|f\|. \tag{1-1}$$

If at least the second of these inequalities, called the upper p-frame condition, is satisfied, we say that $\{g_i\}$ is a p-Bessel sequence.

Definition 1.2. Let H be a complex Hilbert space and (Ω, μ) a measure space. A map $F : \Omega \to H$ is called weakly measurable if, for each $f \in H$, the function on Ω defined by $\omega \mapsto \langle f, F(\omega) \rangle$ is measurable. F is called a continuous frame for H with respect to (Ω, μ) if F is weakly measurable and there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \int_{\Omega} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \le B\|f\|^2, \quad f \in H. \tag{1-2}$$

In the next results, $R(\cdot)$ denotes the range of a map.

Lemma 1.3 [Rudin 1973]. *Suppose X and Y are Banach spaces and $T \in B(X, Y)$. Then $R(T) = Y$ if and only if $\|T^* y^*\| \ge c\|y^*\|$ for some constant $c > 0$ and for each $y^* \in Y^*$.*

Theorem 1.4 [Rudin 1974]. *$L^p(\Omega, \mu)$ is isometricly isomorphism to the dual space of $L^q(\Omega, \mu)$ via the mapping $K^p : L^p(\Omega, \mu) \to L^q(\Omega, \mu)^*$ give by*

$$K^p \psi(\phi) = \int_{\Omega} \psi(\omega)\phi(\omega)\, d\mu(\omega)$$

for all $\psi \in L^p(\Omega, \mu)$ and $\phi \in L^q(\Omega, \mu)$. We can define an isometric isomorphism $K^q = (K^p)^* \Lambda_q : L^q(\Omega, \mu) \to L^p(\Omega, \mu)^*$ for which Λ_q is the isometric isomorphism of $L^q(\Omega, \mu)$ onto $L^q(\Omega, \mu)^{**}$.

Lemma 1.5 [Heuser 1982]. *Given a bounded operator $U : X \to Y$, the adjoint $U^* : Y^* \to X^*$ is surjective if and only if U has a bounded inverse on $R(U)$.*

Theorem 1.6 [Douglas 1972]. *Let X and Y be Banach spaces. For all $x \in X$ and $y \in Y$, define the* 1*-norm*, $\|(x, y)\|_1 = \|x\|_X + \|y\|_Y$ *and the* ∞*-norm* $\|(x, y)\|_\infty = \sup\{\|x\|_X, \|y\|_Y\}$ *on the algebraic direct sum $X \oplus Y$. Then $X \oplus Y$ is a Banach space with respect to both norms and these two norms are equivalent.*

In Section 2, we define the concept of cp-Bessel mappings and cp-frames in Banach spaces and show that under some conditions the direct sum of a finite number of cp-frames is again a cp-frame. In Section 3, we define the concept of a cq-Riesz basis and study some relations between cp-frames and cq-Riesz bases. In Section 4, we present a cp-frame mapping $S_F : X \to X^*$ and show that two cp-frames are similar if and only if their analysis operators have the same range. We obtain some equivalent conditions for duals of cp-Bessel mappings and show existence and uniqueness of duals of independent cp-frames in Section 5 and finally in Section 6 we discuss the perturbation of these frames.

2. Continuous p-frames

Definition 2.1. A mapping $F : \Omega \to X^*$ is called a cp-frame for X with respect to (Ω, μ) if F is weakly measurable (Definition 1.2) and there exist positive constants A and B such that

$$A\|x\| \le \left(\int_\Omega |\langle x, F(\omega)\rangle|^p \, d\mu(\omega)\right)^{1/p} \le B\|x\|, \quad x \in X. \tag{2-1}$$

The constants A and B are called the lower and upper cp-frame bounds, respectively. F is called a tight cp-frame if A and B can be chosen such that $A = B$, and a Parseval cp-frame if A and B can be chosen such that $A = B = 1$.

F is called a cp-Bessel mapping for X with respect to (Ω, μ) if it is weakly measurable and the second inequality in (2-1) holds. In this case B is called a cp-Bessel constant.

If, in the definition of a cp-frame, we take $\Omega = \mathbb{N}$ and let μ be the counting measure, then our cp-frame will be a p-frame; thus we expect that some properties of p-frames can be satisfied in cp-frames.

Throughout this paper, we simply say F is a cp-frame for X and F is a cp-Bessel mapping for X, instead of F is a cp-frame for X with respect to (Ω, μ) and F is a cp-Bessel mapping for X with respect to (Ω, μ), respectively.

Our study of a cp-frame is based on analysis of two operators,

$$U_F : X \to L^p(\Omega, \mu) \quad \text{and} \quad T_F : L^q(\Omega, \mu) \to X^*.$$

The first is defined by

$$U_F x(\omega) = \langle x, F(\omega)\rangle, \quad x \in X, \ \omega \in \Omega, \tag{2-2}$$

and the second is weakly defined by

$$T_F\phi(x)=\langle x,T_F\phi\rangle=\int_\Omega \phi(\omega)\langle x,F(\omega)\rangle\,d\mu(\omega),\quad \phi\in L^q(\Omega,\mu),\ x\in X. \tag{2-3}$$

It is clear that if F is a cp-Bessel mapping, then U_F is well defined and bounded operator. U_F is called the analysis and T_F is called the synthesis operator of F.

Lemma 2.2. *Let F be a* cp*-frame for X. Then the operator $U_F : X \to L^p(\Omega,\mu)$, given by (2-2), has a closed range and X is reflexive.*

Proof. It is easy to verify that U_F has a closed range. By the cp-frame condition, X is isomorphic to $R(U_F)$, but $R(U_F)$ is reflexive because it is a closed subspace of the reflexive space $L^p(\Omega,\mu)$ and therefore X is reflexive. □

Theorem 2.3. *Let $F:\Omega\to X^*$ be a* cp*-Bessel mapping for X with Bessel bound B. Then the operator $T_F : L^q(\Omega,\mu)\to X^*$, weakly defined in (2-3), is well defined, linear and $\|T_F\|\le B$.*

Proof. It is straightforward. □

Lemma 2.4. *Let $F:\Omega\to X^*$ be a* cp*-Bessel mapping for X.*

(i) $U_F^*=T_F(K^q)^{-1}$.

(ii) *If X is reflexive, then $T_F^*=K^pU_F\Lambda_X^{-1}$.*

Proof. (i) Since F is a cp-Bessel mapping for X, there exists a unique operator $U_F^*: L^p(\Omega,\mu)^*\to X^*$ such that

$$\langle x,U_F^*\psi\rangle=\langle U_Fx,\psi\rangle,\quad x\in X,\ \psi\in L^p(\Omega,\mu)^*.$$

Using Theorem 1.4, we can find $\phi\in L^q(\Omega,\mu)$ such that $K^q(\phi)=\psi$. So, for all $x\in X$ and $\psi\in L^p(\Omega,\mu)^*$,

$$\begin{aligned}\langle x,U_F^*\psi\rangle=\langle U_Fx,\psi\rangle=\langle U_Fx,K^q(\phi)\rangle&=\int_\Omega\phi(\omega)\langle x,F(\omega)\rangle\,d\mu(\omega)\\&=\langle x,T_F(\phi)\rangle=\langle x,T_F(K^q)^{-1}\psi\rangle.\end{aligned}$$

Therefore $U_F^*=T_F(K^q)^{-1}$.

(ii) By Theorem 2.3, T_F is well defined and bounded. So for all $f\in X^{**}$ and $\phi\in L^q(\Omega,\mu)$ we have $\langle\phi,T_F^*f\rangle=\langle T_F\phi,f\rangle$. Since X is reflexive, for each $f\in X^{**}$ we can find $x\in X$ such that $\Lambda_Xx=f$. Therefore

$$\begin{aligned}\langle\phi,T_F^*f\rangle=\langle T_F\phi,f\rangle=\langle T_F\phi,\Lambda_Xx\rangle=\langle x,T_F\phi\rangle&=\int_\Omega\phi(\omega)\langle x,F(\omega)\rangle\,d\mu(\omega)\\&=K^p(\langle x,F\rangle)(\phi)=K^p(\langle\Lambda_X^{-1}f,F\rangle)(\phi)=\langle\phi,K^pU_F\Lambda_X^{-1}f\rangle.\end{aligned}$$

So $T_F^*=K^pU_F\Lambda_X^{-1}$. □

Theorem 2.5. *Let X be a reflexive Banach space and $F : \Omega \to X^*$ be weakly measurable. If the mapping $T_F : L^q(\Omega, \mu) \to X^*$ weakly defined by*

$$\langle x, T_F\phi\rangle = \int_\Omega \phi(\omega)\langle x, F(\omega)\rangle d\mu(\omega), \quad \phi \in L^q(\Omega, \mu),\ x \in X,$$

is a bounded operator and $\|T_F\| \le B$, then F is a cp*-Bessel mapping for X.*

Proof. Since T_F is well defined and bounded, we have for all $f \in X^{**}$ and $\phi \in L^q(\Omega, \mu)$

$$\langle \phi, T_F^* f\rangle = \langle T_F\phi, f\rangle = \int_\Omega \phi(\omega)\langle \Lambda_X^{-1} f, F(\omega)\rangle\, d\mu(\omega).$$

For each $f \in X^{**}$, we define $\psi_f : \Omega \to \mathbb{C}$ by $\psi_f(\omega) = \langle \Lambda_X^{-1} f, F(\omega)\rangle$. Since ψ_f is measurable and

$$\left|\int_\Omega \phi(\omega)\psi_f(\omega)\, d\mu(\omega)\right| < \infty \quad \text{for all } \phi \in L^q(\Omega, \mu),$$

we obtain $\psi_f \in L^p(\Omega, \mu)$. By Theorem 1.4, we have

$$\psi_f(\omega) = (K^p)^{-1}(T_F^* f)(\omega), \quad \omega \in \Omega.$$

Hence, for each $x \in X$,

$$\begin{aligned}\left(\int_\Omega |\langle x, F(\omega)\rangle|^p d\mu(\omega)\right)^{1/p} &= \|(K^p)^{-1}T_F^*\Lambda_X x\| = \|T_F^*\Lambda_X x\| \\ &\le \|T_F^*\|\|x\| \le B\|x\|. \end{aligned}$$

□

Theorem 2.6. *Let X be a reflexive Banach space and $F : \Omega \to X^*$ be a weakly measurable mapping. Then F is a* cp*-frame for X if and only if T_F is a well defined and bounded operator of $L^q(\Omega, \mu)$ onto X^*. In this case, the frame bounds are $\|(T_F^*)^{-1}\|^{-1}$ and $\|T_F\|$.*

Proof. By Theorems 2.3 and 2.5, the upper cp-frame condition satisfies if and only if T_F is well defined and bounded operator of $L^q(\Omega, \mu)$ into X^*. Now suppose that F is a cp-frame for X. Then U_F has a bounded inverse on its range $R(U_F)$ and by Lemma 1.5, U_F^* is surjective and therefore T_F is surjective by Lemma 2.4.

Conversely, suppose that T_F is a well defined and bounded operator of $L^q(\Omega, \mu)$ onto X^*. By Lemma 2.4, for each $x \in X$,

$$\|U_F x\| = \|(K^P)^{-1}T_F^*\Lambda_X x\| = \|T_F^*\Lambda_X x\| \le \|T_F\|\|x\|.$$

On the other hand since T_F is bounded and surjective, T_F^* is one to one, hence T_F^* has a bounded inverse on $R(T_F^*)$. So, by Lemma 2.4, for each $x \in X$ we have

$$\|x\| = \|\Lambda_X x\| = \|(T_F^*)^{-1}T_F^*\Lambda_X x\| \le \|(T_F^*)^{-1}\|\|U_F x\|.$$

□

Corollary 2.7. *Let $G : \Omega \to X^{**}$ be a weakly measurable mapping. Then the following assertions are equivalent*:

(i) *There exist positive constants A and B such that*

$$A\|g\| \leq \left(\int_\Omega |\langle g, G(\omega)\rangle|^p d\mu(\omega)\right)^{1/p} \leq B\|g\|, \quad g \in X^*.$$

(ii) *X is reflexive and $T_G : L^q(\Omega, \mu) \to X^{**}$ is a well defined*, *bounded operator of $L^q(\Omega, \mu)$ onto X^{**}.*

Proof. (i) means that $G : \Omega \to X^{**}$ constitutes a cp-frame for X^*. Therefore X^* is reflexive by Lemma 2.2, and thus X is reflexive. The converse is evident by Theorem 2.6. □

Theorem 2.8. *Let X and Y be reflexive Banach spaces. Suppose that $F : \Omega \to X^*$ is a* cp*-Bessel mapping for X and $W : Y \to X$ is a bounded operator.*

(i) *$W^*F : \Omega \to Y^*$ is a* cp*-Bessel mapping for Y and $W^*T_F = T_{W^*F}$.*

(ii) *Let $F : \Omega \to X^*$ be a* cp*-frame for X. Then*, *W^*F is a* cp*-frame for Y if and only if W^* is surjective.*

Proof. (i) For each $y \in Y$, the function $\omega \mapsto \langle y, W^*F(\omega)\rangle = \langle Wy, F(\omega)\rangle$ is measurable. Let B be an upper frame bound for F. Then, for each $y \in Y$,

$$\begin{aligned}\left(\int_\Omega |\langle y, W^*F(\omega)\rangle|^p d\mu(\omega)\right)^{1/p} &= \left(\int_\Omega |\langle Wy, F(\omega)\rangle|^p d\mu(\omega)\right)^{1/p}\\ &\leq B\|Wy\| \leq B\|W\|\|y\|.\end{aligned}$$

Therefore W^*F is a cp-Bessel mapping for Y. For all $y \in Y$ and $\phi \in L^q(\Omega, \mu)$,

$$\begin{aligned}\langle y, T_{W^*F}\phi\rangle &= \int_\Omega \phi(\omega)\langle y, W^*F(\omega)\rangle \, d\mu(\omega) = \int_\Omega \phi(\omega)\langle Wy, F(\omega)\rangle \, d\mu(\omega)\\ &= \langle Wy, T_F\phi\rangle = \langle y, W^*T_F\phi\rangle.\end{aligned}$$

(ii) If W^* is surjective, then by Theorem 2.6, W^*T_F is surjective. So W^*F is a cp-frame for Y. Conversely, if W^*F is a cp-frame for Y then T_{W^*F} is surjective and so W^* is surjective. □

Proposition 2.9 [Fabian et al. 2001]. *Let Y be a closed subspace of a Banach space Z. If Y is complemented and X is a complement of Y in Z*, *then Z/Y is isomorphic to X. The dual Z^* is then isomorphic to $Y^* \oplus X^*$*; *in short*, $(Y \oplus X)^* = Y^* \oplus X^*$.

Theorem 2.10. *Let X and Y be reflexive Banach spaces. Suppose that $F : \Omega \to X^*$ and $G : \Omega \to Y^*$ are* cp*-Bessel mappings. Then $\psi : \Omega \to X^* \oplus Y^* \cong (X \oplus Y)^*$, $\psi(\omega) = (F(\omega), G(\omega))$ is a* cp*-Bessel mapping for $X \oplus Y$. The mapping*

$$T_\psi : L^q(\Omega, \mu) \to (X \oplus Y)^* \cong X^* \oplus Y^*$$

is well defined and bounded, and $T_\psi\phi = (T_F\phi, T_G\phi)$ *for all* $\phi \in L^q(\Omega, \mu)$. *Also,*

$$T_\psi^* : (X \oplus Y)^{**} \cong X^{**} \oplus Y^{**} \to L^q(\Omega, \mu)^*$$

is well defined, linear and bounded and $T_\psi^*(f, g) = T_F^* f + T_G^* g$ *for all* (f, g) *in* $X^{**} \oplus Y^{**}$.

Proof. Using Theorem 1.6 and Proposition 2.9, the proof is evident. □

Theorem 2.11. *Let X and Y be reflexive Banach spaces. Suppose that $F : \Omega \to X^*$ and $G : \Omega \to Y^*$ are* cp*-frames for X and Y, respectively. If $R(T_F^*) \cap R(T_G^*) = 0$ and $R(T_F^*) + R(T_G^*)$ is a closed subspace of $L^q(\Omega, \mu)^*$, then $\psi : \Omega \to (X \oplus Y)^*$ is a* cp*-frame for $X \oplus Y$.*

Proof. We define $L : R(T_F^*) \oplus R(T_G^*) \to R(T_F^*) + R(T_G^*)$ by $L(\eta, \gamma) = \eta + \gamma$. Clearly L is well defined, linear and bijective. We have $\|L(\eta, \gamma)\| = \|\eta + \gamma\| \leq (\|\eta\| + \|\gamma\|) = \|(\eta, \gamma)\|_1$. By Theorem 1.6, L is continuous. By the open mapping theorem, L^{-1} is well defined and bounded, since $R(T_F^*) + R(T_G^*)$ is a closed subspace of $L^q(\Omega, \mu)^*$. Therefore by Theorem 1.6, there exists $M > 0$ such that

$$\|(\eta, \gamma)\|_\infty \leq M\|\eta + \gamma\|. \tag{2-4}$$

Let A_1 and A_2 be lower cp-frame bounds for F and G, and set $K = \min\{A_1, A_2\}$. By Theorem 1.6, there exists $M_1 > 0$ such that, for all $(x, y) \in X \oplus Y$,

$$\begin{aligned} K^p\|(x, y)\|_\infty^p &\leq K^p M_1^p(\|x\| + \|y\|)^p \leq K^p M_1^p 2^p(\|x\|^p + \|y\|^p) \\ &\leq 2^p M_1^p \int_\Omega |\langle x, F(\omega)\rangle|^p d\mu(\omega) + 2^p M_1^p \int_\Omega |\langle y, G(\omega)\rangle|^p d\mu(\omega) \\ &\leq 2^p M_1^p \|(K^p)^{-1} T_F^* \Lambda_X x\| + 2^p M_1^p \|(K^p)^{-1} T_G^* \Lambda_Y y\| \\ &= 2^p M_1^p \|T_F^* \Lambda_X x\| + 2^p M_1^p \|T_G^* \Lambda_Y y\| \\ &= 2^p M_1^p \|(T_F^* \Lambda_X x, T_G^* \Lambda_Y y)\|_1, \end{aligned} \tag{2-5}$$

where $\Lambda_X : X \to X^{**}$ and $\Lambda_Y : Y \to Y^{**}$ are isometric isomorphisms of X onto X^{**} and of Y onto Y^{**}, respectively. Again by using Theorem 1.6, there is $M_2 > 0$ such that

$$\|(T_F^* \Lambda_X x, T_G^* \Lambda_Y y)\|_1 \leq M_2\|(T_F^* \Lambda_X x, T_G^* \Lambda_Y y)\|_\infty. \tag{2-6}$$

By (2-4), (2-5) and (2-6)

$$\begin{aligned} K^p\|(x, y)\|_\infty^p &\leq 2^p M_1^p M_2 M\|T_F^* \Lambda_X x + T_G^* \Lambda_Y y\| = 2^p M_1^p M_2 M\|T_\psi^*(\Lambda_X x, \Lambda_Y y)\| \\ &= 2^p M_1^p M_2 M\|(K^p)^{-1} T_\psi^*(\Lambda_X x, \Lambda_Y y)\| \\ &= 2^p M_1^p M_2 M\|(K^p)^{-1} T_\psi^* \Lambda_{X\oplus Y}(x, y)\| \\ &= 2^p M_1^p M_2 M \int_\Omega |\langle (x, y), \psi(\omega)\rangle|^p d\mu(\omega). \end{aligned}$$

□

Corollary 2.12. *Let $X_1, \cdots, X_n$ be reflexive Banach spaces. Suppose that $F_i : \Omega \to X_i^*$, are* c*p-frames for X_i for all $i \in \mathbb{N}$. If $R(T^*_{F_j}) \cap \left(\sum_{i=1_{i \neq j}}^{n} R(T^*_{F_i})\right) = 0$ for each $j \in \mathbb{N}$ and $\sum_{i=1}^{n} R(T^*_{F_i})$ is a closed subspace of $L^q(\Omega, \mu)^*$, then the map $\eta : \Omega \to \left(\bigoplus_{i=1}^{n} X_i\right)^*$ defined by $\eta(\omega) = (F_1(\omega), \cdots, F_n(\omega))$ is a* c*p-frame for $\bigoplus_{i=1}^{n} X_i$.*

3. Continuous q-Riesz bases

Throughout this paper X is a reflexive Banach space.

Definition 3.1. Let $1 < q < \infty$. A mapping $F : \Omega \to X^*$ is called a cq-Riesz basis for X^* if

(i) $\{x : \langle x, F(\omega)\rangle = 0, w \in \Omega\} = \{0\}$,

(ii) F is weakly measurable, and

(iii) the operator $T_F : L^q(\Omega, \mu) \to X^*$ weakly defined by

$$\langle x, T_F \phi\rangle = \int_\Omega \phi(\omega)\langle x, F(\omega)\rangle \, d\mu(\omega), \quad x \in X, \quad \phi \in L^q(\Omega, \mu),$$

is well defined and there are positive constants A and B such that

$$A\|\phi\|_q \leq \|T_F \phi\|_{X^*} \leq B\|\phi\|_q, \quad \phi \in L^q(\Omega, \mu).$$

A and B are called, respectively, the lower and upper cq-Riesz basis bounds of F.

Theorem 3.2. *Let $F : \Omega \to X^*$ be a* c*q-Riesz basis for X^* with* c*q-Riesz basis bounds A and B. Then F is a* c*p-frame for X with* c*p-frame bounds A and B.*

Proof. Since F is a cq-Riesz basis for X^*, the operator T_F is well defined, bounded and surjective. By Theorem 2.6, F is a cp-frame for X. The upper cq-Riesz basis bound coincide with the upper cp-frame bound by Theorem 2.5. The analogue statement for the lower bound follows from [Dunford and Schwartz 1958, p. 479] and Theorem 2.6. □

Theorem 3.3. *Let $F : \Omega \to X^*$ be a* c*p-frame for X. Then the following statements are equivalent*:

(i) *F is a* c*q-Riesz basis for X^*.*

(ii) *T_F is injective.*

(iii) $R(U_F) = L^p(\Omega, \mu)$.

Proof. (i) $\Longrightarrow$ (ii) By the definition of cq-Riesz basis the proof is evident.

(ii) $\Longrightarrow$ (i) T_F is well defined, bounded and onto by Theorem 2.6, and is injective by (ii), so it has a bounded inverse. Therefore F is a cq-Riesz basis for X^*.

(i) $\Longrightarrow$ (iii) By assumption, T_F has a bounded inverse on $R(T_F) = X^*$. By Lemma 1.5, T_F^* is surjective and Lemma 2.4, implies that $R(U_F) = L^p(\Omega, \mu)$.

(iii) $\Longrightarrow$ (i) is clear. □

4. Maps of cp-frames and their invertibility

In this section, we need a mapping from the Banach space $L^p(\Omega, \mu)$ into its dual space, $L^q(\Omega, \mu)$. For this we use the concept of duality mapping.

First recall that a Banach space X is said to be:

- strictly convex if, whenever $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$, then $\|\lambda x + (1-\lambda)y\| < 1$ for $\lambda \in (0, 1)$;
- uniformly convex if the conditions $\{x_i\} \subseteq X$, $\{y_i\} \subseteq X$, $\|x_i\| \leq 1$, $\|y_i\| \leq 1$, $\lim_{i\to\infty} \|x_i + y_i\| = 2$, imply that $\lim_{i\to\infty} \|x_i - y_i\| = 0$.

Definition 4.1. The mapping ϕ_X of X into the set of subsets of X^*, defined by

$$\phi_X x = \{x^* \in X^* : x^*(x) = \|x\|\|x^*\|, \|x^*\| = \|x\|\}$$

is called the duality mapping on X.

By the Hahn–Banach theorem $\phi_X x$ is nonempty for all $x \in X$ and $\phi_X 0 = 0$. In general the duality mapping is set-valued, but for certain spaces it is single-valued and such spaces are called smooth.

Proposition 4.2 [Dragomir 2004]. (i) *If X^* is strictly convex then for each $x \in X$, $\phi_X x$ consists of unique element $x^* \in X^*$.*

(ii) *If X and X^* are strictly convex and X is reflexive then ϕ_X is bijective.*

(iii) *If H is a Hilbert space then $\phi_H x = x$ for each $x \in H$.*

Remark 4.3. We can deduce by [Carothers 2005, Corollary 11.13] and [Martin 1976, p. 12] that $L^q(\Omega, \mu)$ is strictly convex.

The next statement is clear from the definition of duality mapping on $L^p(\Omega, \mu)$:

Proposition 4.4. *For all nonzero $\psi \in L^p(\Omega, \mu)$ we have $\phi_{L^p(\Omega,\mu)}\psi = \dfrac{\overline{\psi}|\psi|^{p-2}}{\|\psi\|_p^{p-2}}$.*

Definition 4.5. Let $F : \Omega \to X^*$ be a cp-frame for X. The bounded mapping $S_F : X \to X^*$ defined by $S_F = T_F(K^q)^{-1}\phi_{L^p(\Omega,\mu)}U_F$ will be called a cp-frame mapping of F.

Proposition 4.6. *Suppose that $F : \Omega \to X^*$ is a cp-frame for X with frame bounds A and B. Then S_F has the following properties*:

(i) $S_F = U_F^*\phi_{L^p(\Omega,\mu)}U_F$.

(ii) $A^2\|x\|^2 \leq S_F x(x) \leq B^2\|x\|^2, \quad x \in X.$

Proof. Clear from the definition of S_F and of the duality mapping on $L^p(\Omega, \mu)$. $\square$

Definition 4.7. A mapping $[\cdot, \cdot]$ from $X \times X$ into $\mathbb{R}$ is said to be a semi-inner product on X if it has these properties:

(i) $[x, x] \geq 0$ for all $x \in X$ and $[x, x] = 0$ if and only if $x = 0$.

(ii) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$ for all $\alpha, \beta \in \mathbb{R}$ and for all $x, y, z \in X$.

(iii) $|[x, y]|^2 \leq [x, x][y, y]$ for all $x, y \in X$.

If X^* is strictly convex, then there is a unique semi-inner product on X such that $\|x\|_X = [x, x]^{1/2}$ for all $x \in X$ and $\phi_X x(y) = [y, x]$ for all $x, y \in X$ [Dragomir 2004], where ϕ_X is the duality mapping on X. In this case an operator $A : X \to X$ is said to be adjoint abelian if $[Ax, y] = [x, Ay]$ for all $x, y \in X$ or equivalently $A^* \phi_X = \phi_X A$ [Stampfli 1969].

An element $x \in X$ is called (Giles-)orthogonal to $y \in X$, and we write $x \perp y$, if $[y, x] = 0$. If M is a linear subspace of X, the orthogonal complement of M in the Giles sense is denoted by $M^{\perp} = \{x \in X;\ x \perp y,\ y \in M\}$.

Remark 4.8. Let $F : \Omega \to X^*$ be a cp-frame for X. Suppose that $\operatorname{Ker}(T_F)$ and $(\operatorname{Ker}(T_F))^{\perp}$ are topologically complementary in $L^q(\Omega, \mu)$, then clearly the operator $T_F|_{(\operatorname{Ker}(T_F))^{\perp}}$ is invertible and $T_F^{\perp} = (T_F|_{(\operatorname{Ker}(T_F))^{\perp}})^{-1}$ is a bounded right inverse of T_F.

Definition 4.9. Let $F : \Omega \to X^*$ be a cp-frame for X. Suppose that $\operatorname{Ker}(T_F)$ and $(\operatorname{Ker}(T_F))^{\perp}$ are topologically complementary in $L^q(\Omega, \mu)$, we define the mapping $K : X^* \to X$ by $K = \Lambda_X^{-1}(T_F^{\perp})^* \phi_{L^q(\Omega,\mu)} T_F^{\perp}$.

Lemma 4.10. *Let $F : \Omega \to X^*$ be a* cp*-frame for X. Suppose that* $\operatorname{Ker}(T_F)$ *and* $(\operatorname{Ker}(T_F))^{\perp}$ *are topologically complementary in $L^q(\Omega, \mu)$.*

(i) *$K(g)(g) \geq \|g\|_{X^*}^2 / B^2$, where B denotes an upper* cp*-frame bound for F.*

Moreover, when the operator $T_F^{\perp} T_F$ is adjoint abelian, the following assertions hold:

(ii) *S_F is invertible and $S_F^{-1} = K$.*

(iii) $S_F^{-1} = U_F^{-1}(K^p)^{-1} \phi_{L^q(\Omega,\mu)} T_F^{\perp}$.

Proof. The proof is similar to that of [Stoeva 2008, Theorem 5.1]. $\square$

Definition 4.11. Two cp-frames $F : \Omega \to X^*$ and $G : \Omega \to X^*$ for X are similar if there exists an invertible operator $V : X \to X$ such that $F(\omega) = V^* G(\omega)$ for each $\omega \in \Omega$.

Theorem 4.12. *Let the assumptions in Definition 4.9 be satisfied for $F : \Omega \to X^*$ and $G : \Omega \to X^*$. Suppose that $T_F^{\perp} T_F$ and $T_G^{\perp} T_G$ are adjoint abelian operators. Then F and G are similar if and only if their analysis operators have same ranges.*

Proof. Suppose F and G are similar. Then there exists an invertible operator $V : X \to X$ such that $F(\omega) = V^*G(\omega)$, $\omega \in \Omega$. Let $\phi \in R(U_F)$. Then there exists $x \in X$, such that

$$\phi(\omega) = U_F x(\omega) = \langle x, F(\omega)\rangle = \langle x, V^*G(\omega)\rangle = U_G(Vx)(\omega), \quad \omega \in \Omega.$$

So $\phi \in R(U_G)$. By a similar argument, $R(U_G) \subseteq R(U_F)$.

Conversely, assume $R(U_F) = R(U_G)$. For each $x \in X$, there is $y \in X$ such that $U_F(x) = U_G(y)$ or $\langle x, F(\omega)\rangle = \langle y, G(\omega)\rangle$, $\omega \in \Omega$. We define the operator $V : X \to X$ by $Vx = y$. Since the cp-frame mappings for F and G are invertible, y is uniquely determined by V and V is linear, one to one and surjective. □

5. Duals of cp-Bessel mappings

In this section, X is an infinite-dimensional, reflexive Banach space.

Definition 5.1 [Fabian et al. 2001]. A sequence $\{e_i\}_{i=1}^{\infty}$ in X is called a Schauder basis of X, if for each $x \in X$ there is a unique sequence of scalars $(a_i)_{i=1}^{\infty}$, called the coordinates of x, such that $x = \sum_{i=1}^{\infty} a_i e_i$.

Let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of a Banach space X. For $j \in \mathbb{N}$ and $x = \sum_{i=1}^{\infty} a_i e_i$, denote $f_j(x) = a_j$. Using [Fabian et al. 2001, Theorem 6.5], $f_j \in X^*$. The functionals $\{f_i\}_{i=1}^{\infty}$ are called the associated biorthogonal functionals (coordinate functionals) to $\{e_i\}_{i=1}^{\infty}$ and for each $x \in X$, we have $x = \sum_{i=1}^{\infty} f_i(x) e_i$.

We will denote the biorthogonal functionals $\{f_i\}$ by $\{e_i^*\}$, and say that $\{e_i, e_i^*\}$ is a Schauder basis of X. Such a Schauder basis is called shrinking if $\overline{\operatorname{span}}\{e_i^*\} = X^*$. It is called boundedly complete if $\sum_{i=1}^{\infty} a_i e_i$ converges whenever the scalars a_i are such that $\sup_n \|\sum_{i=1}^{n} a_i e_i\| < \infty$.

Theorem 5.2 [Fabian et al. 2001]. *Let $\{e_i, e_i^*\}$ be a Schauder basis of a Banach space X with the canonical projections $p_n : X \to X$, $p_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{n} a_i e_i$ for each $n \in \mathbb{N}$. Then the following assertions are equivalent*:

(i) *$\{e_i, e_i^*\}$ is shrinking.*

(ii) *$\{e_i^*, e_i\}$ is a Schauder basis of X^*.*

Theorem 5.3 [Fabian et al. 2001]. *Let X be a Banach space with a Schauder basis $\{e_i, e_i^*\}_{i=1}^{\infty}$. Then X is reflexive if and only if $\{e_i, e_i^*\}$ is both shrinking and boundedly complete.*

Theorem 5.4. *Let $F : \Omega \to X^*$ be a cp-Bessel mapping for X and $G : \Omega \to X^{**}$ be a cq-Bessel mapping for X^*. Then the following assertions are equivalent*:

(i) *For each $x \in X$, $x = \Lambda_X^{-1} T_G (K^p)^{-1} T_F^* \Lambda_X x$.*

(ii) *For each $g \in X^*$, $g = T_F (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g$.*

(iii) *For each $x \in X$ and $g \in X^*$, $\langle x, g\rangle = \int_{\Omega} \langle x, F(\omega)\rangle \langle g, G(\omega)\rangle \, d\mu(\omega)$.*

(iv) *For each Schauder basis* $\{e_i, e_i^*\}$ *of* X,

$$\langle e_i, e_j^* \rangle = \int_\Omega \langle e_i, F(\omega)\rangle \langle e_j^*, G(\omega)\rangle \, d\mu(\omega), \quad i, j \in \mathbb{N}.$$

Proof. (i) $\Longrightarrow$ (ii) Let $x \in X$ and $g \in X^*$. We have

$$\begin{aligned}
\langle x, g\rangle &= \langle \Lambda_X^{-1} T_G (K^p)^{-1} T_F^* \Lambda_X x, g\rangle = \langle T_G (K^p)^{-1} T_F^* \Lambda_X x, (\Lambda_X^*)^{-1} g\rangle \\
&= \langle (K^p)^{-1} T_F^* \Lambda_X x, T_G^* (\Lambda_X^*)^{-1} g\rangle = \langle T_F^* \Lambda_X x, \Lambda_q (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle \\
&= \langle \Lambda_X x, T_F^{**} \Lambda_q (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle \\
&= \langle \Lambda_X x, (\Lambda_X^{-1})^* T_F (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle \\
&= \langle x, T_F (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle.
\end{aligned}$$

So, for each $g \in X^*$,

$$g = T_F (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g.$$

(ii) $\Longrightarrow$ (iii) For all $x \in X$ and $g \in X^*$,

$$\begin{aligned}
\langle x, g\rangle &= \langle x, T_F (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle \\
&= \int_\Omega \langle x, F(\omega)\rangle (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g(\omega) \, d\mu(\omega).
\end{aligned} \tag{5-1}$$

But for all $\psi \in L^p(\Omega, \mu)$ and $h \in X^{***}$ (the dual of X^{**}),

$$\langle \psi, T_G^* h\rangle = \langle T_G \psi, h\rangle = \int_\Omega \psi(\omega) \langle \Lambda_X^* h, G(\omega)\rangle \, d\mu(\omega) = K^q(\langle \Lambda_X^* h, G\rangle)(\psi).$$

So

$$T_G^* h = K^q(\langle \Lambda_X^* h, G\rangle). \tag{5-2}$$

Therefore, by (5-1) and (5-2), we have

$$\begin{aligned}
\langle x, g\rangle &= \int_\Omega \langle x, F(\omega)\rangle (K^q)^{-1} K^q (\langle \Lambda_X^* (\Lambda_X^*)^{-1} g, G(\omega)\rangle) \, d\mu(\omega) \\
&= \int_\Omega \langle x, F(\omega)\rangle \langle g, G(\omega)\rangle \, d\mu(\omega).
\end{aligned}$$

(iii) $\Longrightarrow$ (ii) This is clear from the proof of (ii) $\Longrightarrow$ (iii).

(ii) $\Longrightarrow$ (i) For all $x \in X$ and $g \in X^*$, we have

$$\begin{aligned}
\langle x, g\rangle &= \langle x, T_F (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle = \langle\, x, \Lambda_X^* T_F^{**} \Lambda_q (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle \\
&= \langle T_F^* (\Lambda_X x), \Lambda_q (\Lambda_q)^{-1} ((K^p)^*)^{-1} T_G^* (\Lambda_X^*)^{-1} g\rangle \\
&= \langle T_G (K^p)^{-1} T_F^* (\Lambda_X x), (\Lambda_X^*)^{-1} g\rangle = \langle \Lambda_X^{-1} T_G (K^p)^{-1} T_F^* (\Lambda_X x), g\rangle.
\end{aligned}$$

Since X^* separates the points of X, we get

$$x = \Lambda_X^{-1} T_G (K^p)^{-1} T_F^* (\Lambda_X x), \qquad x \in X.$$

(iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (iii) For all $x \in X$ and $g \in X^*$,

$$\int_\Omega \langle x, F(\omega)\rangle \langle g, G(\omega)\rangle \, d\mu(\omega) = K^p(\langle x, F\rangle)(\langle g, G\rangle). \tag{5-3}$$

By Theorem 5.2 and 5.3, $\{e_i^*, e_i\}$ and $\{\Lambda e_i, e_i^*\}$ are Schauder basis of X^* and X^{**}, respectively. Therefore

$$\begin{aligned}
K^p(\langle x, F\rangle)(\langle g, G\rangle) &= K^p\Bigl(\Bigl\langle x, \sum_{i=1}^{\infty} \langle e_i, F\rangle e_i^* \Bigr\rangle\Bigr)\Bigl(\Bigl\langle g, \sum_{j=1}^{\infty} \langle e_j^*, G\rangle \Lambda_X e_j \Bigr\rangle\Bigr) \\
&= \Bigl(\sum_{i,j=1}^{\infty} \langle x, e_i^*\rangle \langle g, \Lambda_X e_j\rangle \Bigr) K^p(\langle e_i, F\rangle)(\langle e_j^*, G\rangle) \\
&= \Bigl(\sum_{i,j=1}^{\infty} \langle x, e_i^*\rangle \langle g, \Lambda_X e_j\rangle \Bigr) \int_\Omega \langle e_i, F(\omega)\rangle \langle e_j^*, G(\omega)\rangle \, d\mu(\omega) \\
&= \sum_{i,j=1}^{\infty} \langle x, e_i^*\rangle \langle e_j, g\rangle \langle e_i, e_j^*\rangle \\
&= \Bigl\langle \sum_{i=1}^{\infty} \langle x, e_i^*\rangle e_i, \sum_{j=1}^{\infty} \langle e_j, g\rangle e_j^* \Bigr\rangle = \langle x, g\rangle.
\end{aligned}$$

So, by (5-3),

$$\int_\Omega \langle x, F(\omega)\rangle \langle g, G(\omega)\rangle \, d\mu(\omega) = \langle x, g\rangle. \qquad \square$$

Definition 5.5. Let $F : \Omega \to X^*$ be a cp-Bessel mapping for X and $G : \Omega \to X^{**}$ be a cq-Bessel mapping for X^*. We say that (F, G) is a c-dual pair if one of the assertions of Theorem 5.4 is satisfied.

In this case F is called a cp-dual of G and by Theorem 5.4, we can say that G is a cq-dual of F.

Theorem 5.6. *Let (F, G) be a c-dual pair. Then F is a* cp*-frame for X and G is a* cq*-frame for X^*.*

Proof. For each $x \in X$, we have

$$\begin{aligned}
\|x\| = \|\Lambda_X^{-1} T_G (K^p)^{-1} T_F^* \Lambda_X x\| &= \|T_G (K^p)^{-1} T_F^* \Lambda_X x\| \\
&\le \|T_G\| \|(K^p)^{-1} T_F^* \Lambda_X x\| = \|T_G\| \int_\Omega |\langle x, F(\omega)\rangle|^p d\mu(\omega).
\end{aligned}$$

Since (F, G) is a c-dual pair, $\|T_G\|$ is nonzero. Thus

$$\frac{\|x\|}{\|T_G\|} \leq \left(\int_\Omega |\langle x, F(\omega)\rangle|^p d\mu(\omega) \right)^{1/p}.$$

Hence F is a cp-frame for X. We prove similarly that G is a cq-frame for X^*. □

Definition 5.7. Let $F : \Omega \to X^*$ be a cp-frame for X. We say that F is independent if, for every measurable function $\phi : \Omega \to \mathbb{C}$ and every $x \in X$, the condition

$$\int_\Omega \langle x, F(\omega)\rangle \phi(\omega)\, d\mu(\omega) = 0$$

implies that $\phi = 0$.

Theorem 5.8. *Let $F : \Omega \to X^*$ be a cp-frame for X and $\mu(E) \geq k > 0$ for each measurable set E except $E = \varnothing$.*

(i) *If F is an independent cp-frame for X, there exists a unique cq-frame, $G : \Omega \to X^{**}$ for X^*, such that (F, G) is a c-dual pair.*

(ii) *If $\mathrm{Ker}(T_F)$ and $(\mathrm{Ker}(T_F))^\perp$ are topologically complementary in $L^q(\Omega, \mu)$, then there exists a cq-frame $G : \Omega \to X^{**}$ for X^*, such that (F, G) is a c-dual pair.*

Proof. (i) Let F be an independent cp-frame for X. Then $T_F : L^q(\Omega, \mu) \to X^*$ is invertible. We define $G(\omega) = p(\omega)(T_F)^{-1}$, $w \in \Omega$, where $p(\omega) : L^q(\Omega, \mu) \to \mathbb{C}$, defined by $p(\omega)(\phi) = \phi(\omega)$. Now we show that for a fix $\omega_0 \in \Omega$, $p(\omega_0)$ is bounded.

For each $\phi \in L^q(\Omega, \mu)$, $\|\phi\| \leq 1$, put $\Delta = \{\omega \in \Omega : |\phi(\omega)| \geq |\phi(\omega_0)|\}$. Clearly Δ is nonempty and measurable. Since

$$\|\phi\|^q = \int_\Omega |\phi(\omega)|^q\, d\mu(\omega) \geq \int_\Delta |\phi(\omega)|^q\, d\mu(\omega) \geq \mu(\Delta)|\phi(\omega_0)|^q \geq k|\phi(\omega_0)|^q,$$

and

$$\|p(\omega_0)\| = \sup_{\|\phi\|\leq 1} |p(\omega_0)(\phi)| = \sup_{\|\phi\|\leq 1} |\phi(\omega_0)| \leq \sup_{\|\phi\|\leq 1} \left(\frac{1}{k}\right)^{1/q} \|\phi\| = \left(\frac{1}{k}\right)^{1/q},$$

for each $\omega \in \Omega$, $p(\omega)$ is bounded. Therefore $G(\omega) \in X^{**}$. By the definition of $G(\omega)$, for each $g \in X^*$, the mapping $\omega \to \langle g, G(\omega)\rangle$ is measurable and

$$\frac{\|g\|}{\|T_F\|} \leq \left(\int_\Omega |\langle g, G(\omega)\rangle|^q d\mu(\omega) \right)^{1/q} = \|(T_F)^{-1} g\| \leq \|(T_F)^{-1}\| \|g\|.$$

Therefore, G is a cq-frame for X^* with bounds $\|T_F\|^{-1}$ and $\|(T_F)^{-1}\|$.

By the definition of G, $T_G^* = K^q T_F^{-1} \Lambda_X^*$. So, for each $g \in X^*$, we have $g = T_F T_F^{-1}(g) = T_F (K^q)^{-1} T_G^* (\Lambda_X^*)^{-1} g$. Therefore (F, G) is a c-dual pair by Theorem 5.4.

Now we will show the uniqueness of G. Let (F, W) be another c-dual pair. Then

$$T_F(K^q)^{-1}T_G^*(\Lambda_X^*)^{-1} = T_F(K^q)^{-1}T_W^*(\Lambda_X^*)^{-1} = I_{X^*}.$$

Thus $T_G^* = T_W^*$. So $W = G$.

(ii) Since $R(T_F) = X^*$, by Remark 4.8, there is an operator $T_F^{\perp} : X^* \to L^q(\Omega, \mu)$ such that $T_F T_F^{\perp} = I_{X^*}$. For each $g \in X^*$, let $\phi = T_F^{\perp} g$. Therefore for all $x \in X$ and $g \in X^*$

$$\langle x, g\rangle = \langle x, T_F\phi\rangle = \int_\Omega \phi(\omega)\langle x, F(\omega)\rangle\, d\mu(\omega) = \int_\Omega T_F^{\perp} g(\omega)\langle x, F(\omega)\rangle\, d\mu(\omega).$$

For each $\omega \in \Omega$, define $G(\omega) : X^* \to \mathbb{C}$, $G(\omega)(g) = (T_F^{\perp} g)(\omega)$. Then

$$|G(\omega)g| = |p(\omega)(T_F^{\perp} g)| \le \left(\frac{1}{k}\right)^{1/q} \|T_F^{\perp}\|\|g\|,$$

where $p(\omega)$ is defined in the proof of (i). Therefore G is weakly measurable and $G(\omega) \in X^{**}$. Since $T_F T_F^{\perp} = I_{X^*}$, we have, for each $g \in X^*$,

$$\frac{\|g\|}{\|T_F\|} \le \left(\int_\Omega |\langle g, G(\omega)\rangle|^q d\mu(\omega)\right)^{1/q} = \|T_F^{\perp} g\|_q \le \|T_F^{\perp}\|\|g\|. \qquad \square$$

Theorem 5.9. *Let $F : \Omega \to X^*$ be an independent cp-frame for X. Suppose that $\mu(E) \ge k > 0$ for each measurable set E except $E = \varnothing$. Let $\omega_0 \in \Omega$ be such that*

$$\mu(\{\omega_0\}) \ne \frac{1}{\langle F(\omega_o), G(\omega_o)\rangle},$$

where $G : \Omega \to X^{**}$ is the unique cq-dual of F, obtained in Theorem 5.8. Then $F : \Omega \setminus \{\omega_0\} \to X^*$ is a cp-frame for X.

Proof. It is clear that the upper frame condition holds. For the lower frame bound, we have

$$\langle x, F(\omega_0)\rangle = \int_\Omega \langle x, F(\omega)\rangle\langle F(\omega_0), G(\omega)\rangle\, d\mu(\omega), \qquad x \in X.$$

Therefore $\langle x, F(\omega_0)\rangle$ is given by

$$\int_{\Omega\setminus\{\omega_0\}} \langle x, F(\omega)\rangle\langle F(\omega_0), G(\omega)\rangle\, d\mu(\omega) + \langle x, F(\omega_0)\rangle\langle F(\omega_0), G(\omega_0)\rangle\mu(\{\omega_0\}),$$

that is,

$$\langle x, F(\omega_0)\rangle = \frac{1}{1 - \mu(\{\omega_0\})\langle F(\omega_0), G(\omega_0)\rangle} \int_{\Omega\setminus\{\omega_0\}} \langle x, F(\omega)\rangle\langle F(\omega_0), G(\omega)\rangle\, d\mu(\omega).$$

Let A be the lower frame bound of F. For each $x \in X$,

$$|\langle x, F(\omega_0)\rangle|^p \le K \int_{\Omega\setminus\{\omega_o\}} |\langle x, F(\omega)\rangle|^p d\mu(\omega),$$

where

$$K = \left(\frac{1}{1-\mu(\{\omega_0\})\langle F(\omega_0), G(\omega_0)\rangle}\right)^p \left(\int_{\Omega\setminus\{\omega_0\}} |\langle F(\omega_0), G(\omega)\rangle|^q d\mu(\omega)\right)^{p/q}.$$

Therefore, for each $x \in X$,

$$\begin{aligned}
A\|x\|_X &\le \left(\int_{\Omega\setminus\{\omega_o\}} |\langle x, F(\omega)\rangle|^p d\mu(\omega)\right)^{1/p} + \big(|\langle x, F(\omega_0)\rangle|^p \mu(\{\omega_0\})\big)^{1/p} \\
&\le \left(\int_{\Omega\setminus\{\omega_o\}} |\langle x, F(\omega)\rangle|^p d\mu(\omega)\right)^{1/p} \\
&\qquad + \left(\int_{\Omega\setminus\{\omega_o\}} |\langle x, F(\omega)\rangle|^p d\mu(\omega)\right)^{1/p} K^{1/p}(\mu(\{\omega_o\}))^{1/p} \\
&= \big(1 + K^{1/p}(\mu(\{\omega_o\}))^{1/p}\big)\left(\int_{\Omega\setminus\{\omega_o\}} |\langle x, F(\omega)\rangle|^p d\mu(\omega)\right)^{1/p}.
\end{aligned}$$

Therefore $F : \Omega \setminus \{\omega_0\} \to X^*$ is a cp-frame for X with lower frame bound

$$\frac{A}{1 + K^{1/p}(\mu(\{\omega_o\}))^{1/p}}. \qquad \square$$

Corollary 5.10. *Let $F : \Omega \to X^*$ be a* cp*-frame for X and assume $\mu(E) \ge k > 0$ for each measurable set E except $E = \varnothing$. Let $\omega_0 \in \Omega$ be such that*

$$\mu(\{\omega_0\}) \ne \frac{1}{\langle F(\omega_o), G(\omega_o)\rangle}.$$

Suppose $\mathrm{Ker}(T_F)$ *and* $(\mathrm{Ker}(T_F))^\perp$ *are topologically complementary in $L^q(\Omega, \mu)$. Then $F : \Omega \setminus \{\omega_0\} \to X^*$ is a* cp*-frame for X.*

6. Perturbation of cp-frames

Perturbation of discrete frames has been discussed in [Cazassa and Christensen 1997]. The proof of the following theorem is based on the following lemma, which was proved in [Cazassa and Christensen 1997].

Lemma 6.1. *Let U be a linear operator on a Banach space X and assume that there exist $\lambda_1, \lambda_2 \in [0, 1)$ such that for each $x \in X$,*

$$\|x - Ux\| \le \lambda_1\|x\| + \lambda_2\|Ux\|.$$

Then U is bounded and invertible. Moreover, for each $x \in X$,

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \leq \|Ux\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|x\|$$

and

$$\frac{1-\lambda_2}{1+\lambda_1}\|x\| \leq \|U^{-1}x\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|x\|.$$

Theorem 6.2. *Let F be an independent* cp*-frame for X and $\mu(E) \geq k > 0$, for each measurable set E, except $E = \varnothing$. Suppose that $G : \Omega \to X^*$ is weakly measurable and assume that there exist constants $\lambda_1, \lambda_2, \gamma \geq 0$ with* $\max(\lambda_1 + \gamma/A, \lambda_2) < 1$. *Suppose also that, for all $\phi \in L^q(\Omega, \mu)$ and x in the unit sphere of X,*

$$\begin{aligned}\Big|\int_\Omega \phi(\omega)\langle x, F(\omega) - G(\omega)\rangle\, d\mu(\omega)\Big| \\ \leq \lambda_1\Big|\int_\Omega \phi(\omega)\langle x, F(\omega)\rangle\, d\mu(\omega)\Big| + \lambda_2\Big|\int_\Omega \phi(\omega)\langle x, G(\omega)\rangle\, d\mu(\omega)\Big| + \gamma\|\phi\|.\end{aligned}$$

Then $G : \Omega \to X^*$ is a cp*-frame for X with bounds*

$$A\,\frac{1-(\lambda_1+\gamma/A)}{1+\lambda_2} \quad \textit{and} \quad B\,\frac{1+\lambda_1+\gamma/B}{1-\lambda_2},$$

where A and B are the frame bounds of F.

Proof. Let $X_1 = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. We first prove that G is a cp-Bessel mapping for X. By assumption, for all $x \in X$ and $\phi \in L^q(\Omega, \mu)$,

$$\begin{aligned}&\Big|\int_\Omega \phi(\omega)\langle x, G(\omega)\rangle\, d\mu(\omega)\Big| \\ &\leq \Big|\int_\Omega \phi(\omega)\langle x, F(\omega) - G(\omega)\rangle\, d\mu(\omega)\Big| + \Big|\int_\Omega \phi(\omega)\langle x, F(\omega)\rangle\, d\mu(\omega)\Big| \\ &\leq (1+\lambda_1)\Big|\int_\Omega \phi(\omega)\langle x, F(\omega)\rangle\, d\mu(\omega)\Big| + \lambda_2\Big|\int_\Omega \phi(\omega)\langle x, G(\omega)\rangle\, d\mu(\omega)\Big| + \gamma\|\phi\|,\end{aligned}$$

which implies that

$$\begin{aligned}\Big|\int_\Omega \phi(\omega)\langle x, G(\omega)\rangle\, d\mu(\omega)\Big| &\leq \frac{1+\lambda_1}{1-\lambda_2}\Big|\int_\Omega \phi(\omega)\langle x, F(\omega)\rangle\, d\mu(\omega)\Big| + \frac{\gamma}{1-\lambda_2}\|\phi\| \\ &\leq \Big(\frac{1+\lambda_1}{1-\lambda_2}B + \frac{\gamma}{1-\lambda_2}\Big)\|\phi\|.\end{aligned}$$

Let $K : L^q(\Omega, \mu) \to X^*$ be defined by

$$\langle x, K\phi\rangle = \int_\Omega \phi(\omega)\langle x, G(\omega)\rangle\, d\mu(\omega), \quad x \in X, \phi \in L^q(\Omega, \mu).$$

Then

$$\|K\phi\| = \sup_{\|x\|=1} |\langle x, K\phi\rangle| = \sup_{\|x\|=1} \left| \int_\Omega \phi(\omega)\langle x, G(\omega)\rangle \, d\mu(\omega) \right|$$
$$\leq \left(\frac{1+\lambda_1}{1-\lambda_2} B + \frac{\gamma}{1-\lambda_2} \right) \|\phi\|.$$

Therefore K is well defined and bounded. So by Theorem 2.5, G is a cp-Bessel mapping for X with upper bound $B(1+\lambda_1+\gamma/B)/(1-\lambda_2)$.

We define $V = K(K^q)^{-1}T_W^*(\Lambda_X^*)^{-1}$, for which W is the unique cq-dual of F which is obtained in Theorem 5.8. Then, for all $x \in X$ and $g \in X^*$,

$$\langle x, Vg\rangle = \langle x, K(K^q)^{-1}T_W^*(\Lambda_X^*)^{-1}g\rangle = \int_\Omega \langle g, W(\omega)\rangle\langle x, G(\omega)\rangle \, d\mu(\omega)$$

and

$$\langle x, g\rangle = \int_\Omega \langle x, F(\omega)\rangle\langle g, W(\omega)\rangle \, d\mu(\omega).$$

Let $\phi_g : \Omega \to \mathbb{C}$ be defined by $\phi_g(\omega) = \langle g, W(\omega)\rangle$. Clearly $\phi_g \in L^q(\Omega, \mu)$. Therefore, by assumption, we deduce that for all $x \in X_1$ and $g \in X^*$,

$$|\langle x, g - Vg\rangle| \leq \lambda_1|\langle x, g\rangle| + \lambda_2|\langle x, Vg\rangle| + \gamma\|\phi_g\|.$$

Hence

$$\|g - Vg\| = \sup_{\|x\|=1} |\langle x, g - Vg\rangle| \leq \lambda_1\|g\| + \lambda_2\|Vg\| + \gamma\|\phi_g\|$$
$$\leq \left(\lambda_1 + \frac{\gamma}{A}\right)\|g\| + \lambda_2\|Vg\|.$$

By Lemma 6.1, V is invertible and

$$\|V\| \leq \frac{1+\lambda_1+\gamma/A}{1-\lambda_2}, \qquad \|V^{-1}\| \leq \frac{1+\lambda_2}{1-(\lambda_1+\gamma/A)}.$$

Then

$$\langle x, g\rangle = \langle x, VV^{-1}g\rangle = \int_\Omega \langle V^{-1}g, W(\omega)\rangle\langle x, G(\omega)\rangle \, d\mu(\omega),$$

and we obtain

$$\|x\| = \|\Lambda_X x\| = \sup_{\|g\|=1} |\langle g, \Lambda_X x\rangle| = \sup_{\|g\|=1} |\langle x, g\rangle|$$
$$= \sup_{\|g\|=1} \left| \int_\Omega \langle V^{-1}g, W(\omega)\rangle\langle x, G(\omega)\rangle \, d\mu(\omega)\right|$$
$$\leq \sup_{\|g\|=1} \left(\int_\Omega |\langle V^{-1}g, W(\omega)\rangle|^q d\mu(\omega) \right)^{1/q} \left(\int_\Omega |\langle x, G(\omega)\rangle|^p d\mu(\omega) \right)^{1/p}.$$

Therefore, for each $x \in X$,

$$A\,\frac{1-(\lambda_1+\gamma/A)}{1+\lambda_2}\,\|x\| \le \left(\int_\Omega |\langle x, G(\omega)\rangle|^p \, d\mu(\omega)\right)^{1/p}. \qquad \square$$

Acknowlegement

The authors would like to thank the referee for useful comments and suggestions.

References

[Abdollahpour and Faroughi 2008] M. R. Abdollahpour and M. H. Faroughi, "Continuous G-frames in Hilbert spaces", *Southeast Asian Bull. Math.* **32**:1 (2008), 1–19. MR 2008m:41028 Zbl 1199.42132

[Aldroubi et al. 2001] A. Aldroubi, Q. Sun, and W.-S. Tang, "p-frames and shift invariant subspaces of L^p", *J. Fourier Anal. Appl.* **7**:1 (2001), 1–21. MR 2002c:42046 Zbl 0983.46027

[Ali et al. 1993] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, "Continuous frames in Hilbert space", *Ann. Physics* **222**:1 (1993), 1–37. MR 94e:81107 Zbl 0782.47019

[Asgari and Khosravi 2005] M. S. Asgari and A. Khosravi, "Frames and bases of subspaces in Hilbert spaces", *J. Math. Anal. Appl.* **308**:2 (2005), 541–553. MR 2006b:42042 Zbl 1091.46006

[Cao et al. 2008] H.-X. Cao, L. Li, Q.-J. Chen, and G.-X. Ji, "(p, Y)-operator frames for a Banach space", *J. Math. Anal. Appl.* **347**:2 (2008), 583–591. MR 2009h:46024 Zbl 05344335

[Carothers 2005] N. L. Carothers, *A short course on Banach space theory*, London Math. Soc. Student Texts **64**, Cambridge University Press, Cambridge, 2005. MR 2005k:46001 Zbl 1072.46001

[Cazassa and Christensen 1997] P. G. Cazassa and O. Christensen, "Perturbation of operators and applications to frame theory", *J. Fourier Anal. Appl.* **3**:5 (1997), 543–557. MR 98j:47028 Zbl 0895.47007

[Christensen 2003] O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston, 2003. MR 2003k:42001 Zbl 1017.42022

[Christensen and Stoeva 2003] O. Christensen and D. T. Stoeva, "p-frames in separable Banach spaces", *Adv. Comput. Math.* **18**:2-4 (2003), 117–126. MR 2004b:42060 Zbl 1012.42024

[Daubechies et al. 1986] I. Daubechies, A. Grossmann, and Y. Meyer, "Painless nonorthogonal expansions", *J. Math. Phys.* **27**:5 (1986), 1271–1283. MR 87e:81089 Zbl 0608.46014

[Douglas 1972] R. G. Douglas, *Banach algebra techniques in operator theory*, Pure and Applied Mathematics **49**, Academic Press, New York, 1972. MR 50 #14335 Zbl 0247.47001

[Dragomir 2004] S. S. Dragomir, *Semi-inner products and applications*, Nova Science, Hauppauge, NY, 2004. MR 2005b:46053 Zbl 1060.46001

[Duffin and Schaeffer 1952] R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series", *Trans. Amer. Math. Soc.* **72** (1952), 341–366. MR 13,839a Zbl 0049.32401

[Dunford and Schwartz 1958] N. Dunford and J. T. Schwartz, *Linear operators, I: General theory*, Pure and Applied Math. **7**, Interscience, New York, 1958. MR 22 #8302

[Fabian et al. 2001] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics **8**, Springer, New York, 2001. MR 2002f:46001 Zbl 0981.46001

[Gabardo and Han 2003] J.-P. Gabardo and D. Han, "Frames associated with measurable spaces", *Adv. Comput. Math.* **18**:2-4 (2003), 127–147. MR 2004b:42062 Zbl 1033.42036

[Han and Larson 2000] D. Han and D. R. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc. **697**, American Mathematical Society, Providence, RI, 2000. MR 2001a:47013 Zbl 0971.42023

[Heil and Walnut 1989] C. E. Heil and D. F. Walnut, "Continuous and discrete wavelet transforms", *SIAM Rev.* **31**:4 (1989), 628–666. MR 91c:42032 Zbl 0683.42031

[Heuser 1982] H. G. Heuser, *Functional analysis*, Wiley, New York, 1982. MR 83m:46001 Zbl 0465.47001

[Joveini and Amini 2009] R. Joveini and M. Amini, "Yet another generalization of frames and Riesz bases", *Involve* **2**:4 (2009), 395–407. MR 2010k:42060 Zbl 1184.42026

[Martin 1976] R. H. Martin, Jr., *Nonlinear operators and differential equations in Banach spaces*, Wiley, New York, 1976. MR 58 #11753 Zbl 0333.47023

[Najati and Faroughi 2007] A. Najati and M. H. Faroughi, "p-frames of subspaces in separable Hilbert spaces", *Southeast Asian Bull. Math.* **31**:4 (2007), 713–726. MR 2009d:46045 Zbl 1150.46011

[Rahimi et al. 2006] A. Rahimi, A. Najati, and Y. N. Dehghan, "Continuous frames in Hilbert spaces", *Methods Funct. Anal. Topology* **12**:2 (2006), 170–182. MR 2007d:42061

[Rudin 1973] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973. MR 51 #1315 Zbl 0253.46001

[Rudin 1974] W. Rudin, *Real and complex analysis*, 2nd ed., McGraw-Hill, New York, 1974. MR 49 #8783 Zbl 0278.26001

[Stampfli 1969] J. G. Stampfli, "Adjoint abelian operators on Banach space", *Canad. J. Math.* **21** (1969), 505–512. MR 39 #807 Zbl 0183.14001

[Stoeva 2008] D. T. Stoeva, "Generalization of the frame operator and the canonical dual frame to Banach spaces", *Asian-Eur. J. Math.* **1**:4 (2008), 631–643. MR 2009m:42058

[Sun 2006] W. Sun, "G-frames and g-Riesz bases", *J. Math. Anal. Appl.* **322**:1 (2006), 437–452. MR 2007b:42047 Zbl 1129.42017

Received: 2011-02-17 Accepted: 2011-02-26

mhfaroughi@yahoo.com
Faculty of Mathematical Science, University of Tabriz, 29 Bahman Boulevard, Tabriz, Iran

Department of Mathematics, Islamic Azad University, Shabestar Branch, Shabestar 0098, Iran

osgooei@tabrizu.ac.ir
Faculty of Mathematical Science, University of Tabriz, 29 Bahman Boulevard, Tabriz 0098, Iran

involve

pjm.math.berkeley.edu/involve

See inside back cover or http://pjm.math.berkeley.edu/involve for submission instructions.

The subscription price for 2011 is US $100/year for the electronic version, and $130/year (+$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

involve

2011 vol. 4 no. 2