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The Gauss–Bonnet formula on surfaces with densities

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The celebrated Gauss–Bonnet formula has a nice generalization to surfaces with densities, in which both arclength and area are weighted by positive functions. Surfaces with densities, especially when arclength and area are weighted by the same factor, appear throughout mathematics, including probability theory and Perelman's recent proof of the Poincaré conjecture.

A classic, if somewhat anthropomorphic, question in mathematics is whether an ant moving on a curve embedded in $\mathbb{R}^3$ or in a surface can measure the curvature κ of the curve or say anything about how the curve is embedded in space. The answer, no, stems from the fact that the ant can only measure distance along the curve and has no way to determine changes in direction. Curvature is extrinsic to a curve and must be measured from outside the curve.

Following this one might then ask whether a person moving in a surface embedded in $\mathbb{R}^3$ has any chance of saying something about the surface's curvature in $\mathbb{R}^3$. Whereas the ant could only measure distance along the curve, a person on a surface has the ability to measure both length and area on the surface. Does this change things?

The answer is yes. Gauss's Theorem Egregium declares that a certain measure of surface curvature now known as the Gauss curvature G turns out to be an intrinsic quantity, measurable from within the surface. This is not at all apparent from its definition. G is defined as the product of the principal curvatures κ_1, κ_2, the largest and smallest (or most positive and most negative) curvatures of one-dimensional slices by planes orthogonal to the surface. For a plane, $G = 0$. For a sphere of radius a, we have $G = 1/a^2$. For the hyperbolic paraboloid $\{z = \frac{1}{2}(x^2 - y^2)\}$, at the origin G equals -1: negative because the surface is curving up in one direction and

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down in the other direction; as you move farther out in the surface, G approaches 0 as the surface flattens out.

The fact that the Gauss curvature is actually intrinsic is a consequence of the celebrated Gauss–Bonnet formula (for a general reference see [do Carmo 1976; Morgan 1998]). Gauss–Bonnet relates the integral of the Gauss curvature over a smooth topological disc D in a surface to the integral over the boundary ∂D of the curvature κ of the boundary:

$$\int_{\partial D} \kappa + \int_D G = 2\pi.$$

For example, for a smooth closed curve C in the plane, where $G = 0$,

$$\int_C \kappa = 2\pi,$$

that is, the total curvature of an embedded planar curve is 2π. For a smooth closed curve C enclosing area A on the unit sphere, where $G = 1$,

$$\int_C \kappa + A = 2\pi.$$

For example, the equator, with curvature $\kappa = 0$, encloses area 2π. Note that we are using the intrinsic or "geodesic" curvature κ, not the curvature of the curve in $\mathbb{R}^3$ if the surface is embedded in $\mathbb{R}^3$.

Gauss–Bonnet has extensive applications throughout geometry and topology. It can be used to classify two-dimensional surfaces by genus and to solve isoperimetric problems [Howards et al. 1999; Morgan 1998, Section 9.12]. The Gauss–Bonnet formula provides an intrinsic definition of the Gauss curvature G of a surface at a point p by considering ϵ-balls B_ϵ of area A about p and taking a limit as ϵ approaches 0:

$$G(p) = \frac{1}{A}\int_{B_\epsilon} G = \lim \frac{1}{A}\Big(2\pi - \int_{\partial B_\epsilon} \kappa\Big).$$

This article considers what happens to the Gauss–Bonnet formula under some simple intrinsic alterations of the surface. The most common alteration, called a conformal change of metric, scales distance by a variable factor λ, so that $ds = \lambda\, ds_0$ and $dA = \lambda^2\, dA_0$; that is, arc length is weighted by λ and area is weighted by λ^2. More generally, one can weight arc length and area by unrelated densities:

$$ds = \delta_1 ds_0, \qquad dA = \delta_2\, dA_0.$$

If the two densities are equal, $\delta_1 = \delta_2 = \Psi$, the result is simply called a surface with density Ψ. Surfaces with density appear throughout mathematics, including probability theory and Perelman's recent proof of the Poincaré conjecture [Morgan

2009, Chapter 18]. Important examples include quotients of Riemannian manifolds by symmetries and Gauss space, defined as $\mathbb{R}^n$ with Gaussian density $c\exp(-r^2)$.

Perelman's paper and many other applications require generalizations of curvature to general dimensional surfaces with densities. In higher dimensions, the important intrinsic curvature is the so-called Ricci curvature, for which many generalizations have been proposed, each for its own purpose, one particular choice employed by Perelman (see [Morgan 2009, Section 18.3] and references therein). Corwin et al. [2006, Section 5] proposed a generalization of Gauss curvature and the Gauss–Bonnet formula to surfaces with density Ψ. In principle, their definition generalizes to surfaces with length density δ_1 and area density δ_2 by a conformal change of metric. The following proposition gives a simple, direct presentation of that generalization. The generalized Gauss curvature G' is given by

$$G' = G - \Delta \log \delta_1.$$

An intriguing feature is that G' depends only on the length density δ_1, not on the area density δ_2. For a conformal change of metric ($\delta_1 = \lambda, \delta_2 = \lambda^2$), (1) below agrees with the standard Gauss–Bonnet formula (and gives an easy proof): the first integrand becomes $\kappa\lambda ds_0 = \kappa ds$ and the second integrand becomes the new Gauss curvature $G'\lambda^2 dA_0 = G' dA$ because $G' = (G - \Delta \log \lambda)/\lambda^2$ [Dubrovin et al. 1992, Theorem 13.1.3].

For a disc with density (the case $\delta_2 = \delta_1$), (1) agrees with the formula in [Corwin et al. 2006, Proposition 5.2]. For a disc with area density (the case $\delta_1 = 1$), (1) agrees with the formula in [Carroll et al. 2008, Proposition 3.3].

There are other possible generalizations of Gauss curvature to surfaces with density, for example, coming from the power series expansions for the area and perimeter of geodesic balls [Corwin et al. 2006, Propositions 5.8 and 5.9].

Proposition. *Consider a smooth Riemannian disc D with Gauss curvature G, length density δ_1, area density δ_2, classical boundary curvature κ_0* (*inward normal*), *and hence generalized boundary curvature*

$$\kappa = (\delta_1/\delta_2)\kappa_0 - (1/\delta_2)\partial\delta_1/\partial n.$$

Then

$$\int_{\delta_D} (\delta_2/\delta_1)\kappa \, ds_0 + \int_D (G - \Delta \log \delta_1)\, dA_0 = 2\pi. \tag{1}$$

Proof. We begin by explaining the formula for κ. The geometric interpretation of curvature is minus the rate of change of length per change in enclosed area as you deform the curve normal to itself [Corwin et al. 2006, Proposition 3.2]. First of all, the densities weight this effect by δ_1/δ_2. There is a second effect due to the rate of change $\partial\delta_1/\partial n$ of the length density in the normal direction, divided again by the area density δ_2.

To prove (1), first consider the conformal metric $ds = \delta_1 ds_0$, with area density δ_1^2 and curvature

$$\kappa' = (1/\delta_1)\kappa_0 - (1/\delta_1^2)\partial\delta_1/\partial n.$$

Multiplying the area density by $\mu = \delta_2/\delta_1^2$ multiplies the curvature by $1/\mu = \delta_1^2/\delta_2$:

$$\kappa = (\delta_1/\delta_2)\kappa_0 - (1/\delta_2)\partial\delta_1/\partial n.$$

Hence by substitution, by the classical Gauss–Bonnet Theorem and the divergence theorem, and by trivial algebra,

$$\begin{aligned}\int_{\partial D}(\delta_2/\delta_1)\kappa\, ds_0 &= \int_{\partial D}\kappa_0 ds_0 - \int_{\partial D}\partial\log\delta_1/\partial n\, ds_0\\ &= 2\pi - \int_D G\, dA_0 + \int_D \Delta\log\delta_1\, dA_0\\ &= 2\pi - \int_D (G - \Delta\log\delta_1)\, dA_0,\end{aligned}$$

as desired. □

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