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Maximality of the Bernstein polynomials

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For fixed a and b, let Q_n be the family of polynomials $q(x)$ all of whose roots are real numbers in $[a, b]$ (possibly repeated), and such that $q(a) = q(b) = 0$. Since an element of Q_n is completely determined by it roots (with multiplicity), we may ask how the polynomial is sensitive to changes in the location of its roots. It has been shown that one of the Bernstein polynomials $b_i(x) = (x-a)^{n-i}(x-b)^i$, $i = 1, \ldots, n-1$, is the member of Q_n with largest supremum norm in $[a, b]$. Here we show that for $p \geq 1$, $b_1(x)$ and $b_{n-1}(x)$ are the members of Q_n that maximize the L^p norm in $[a, b]$. We then find the associated maximum values.

1. Introduction

A monic polynomial $q(x)$ is completely determined by its roots (with multiplicity), since it can be written as the product

$$q(x) = \prod_{i=1}^{n}(x - r_i),$$

where the r_i are the roots. So it is a fair question to ask how the polynomial q is sensitive to changes in the location of its roots. Boelkins, Miller and Vugteveen [Boelkins et al. 2006] have shown that, among degree-n monic polynomials $q(x)$ all of whose roots are real, belong to $[a, b]$, and include a and b, the value of the supremum norm, $\max\limits_{a \leq x \leq b} q(x)$, is maximized by the polynomials

$$(x-a)^{n-1}(x-b) \quad \text{and} \quad (x-a)(x-b)^{n-1}.$$

So these are in some sense the "largest" polynomials in the class just described.

We will show that these are also the largest polynomials with respect to another measure of size, namely, the L^p norm for $p \geq 1$. (For $p = 1$ this is simply the area enclosed by the graph between a and b.)

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Throughout this paper we let $q(x)$ be a monic polynomial of degree n all of whose roots are real and lie in $[a, b]$; we assume further that $q(a) = q(b) = 0$. We denote the family of all such polynomials by Q_n. We show that given any $q \in Q_n$,

$$\int_a^b |q(x)|\ dx \le (b-a)^{n+1} \frac{1}{n(n+1)},$$

and for any $p \in \mathbb{N}$

$$\int_a^b |q(x)|^p\ dx \le (b-a)^{pn+1} \frac{1}{pn+1} \left(\frac{(pn-p)!\, p!}{(pn)!} \right).$$

We then use these bounds to verify the results of [Boelkins et al. 2006]. That is, for $a < x < b$,

$$|q(x)| \le \frac{(b-a)^n}{n} \left(\frac{n-1}{n} \right)^{n-1}.$$

2. Preliminary information

We are interested in how "large" a polynomial in Q_n can be and therefore need a way to tell when one polynomial is larger than another. We will use the L^p norms to measure the size of a polynomial. Given a polynomial q we use the notation $\|q\|_{L^p_{[a,b]}}$ to denote the L^p norm of q:

$$\|q\|_{L^p_{[a,b]}} = \left(\int_a^b |q(x)|^p\, dx \right)^{1/p}$$

and

$$\|q\|_{L^\infty_{[a,b]}} = \max_{x \in [a,b]} |q(x)|.$$

In particular, the L^1 norm of q,

$$\|q\|_{L^1_{[a,b]}} = \int_a^b |q(x)|\, dx,$$

measures the area enclosed by q.

Our goal is to understand how the L^p norm of $q \in Q_n$ is a function of the location of its roots. Specifically, we would like to understand how the smallest root of q which is greater than a will affect the L^p norm of q. We let $r_0 = a$ and r_1 represent the smallest root greater than r_0. With this in mind, we study how r_1 affects the L^p norm of polynomials of the form

$$q(x) = (x - r_1)^k s(x)$$

where $s(x) = (x - r_0)^l (x - r_2)(x - r_3) \cdots (x - r_{m-1})$ and $n = l + k + m - 2$. That is, q is a degree n polynomial with roots

$$r_0 = a < r_1 < r_2 \le r_3 \cdots \le r_{m-1} = b,$$

which takes into account having possibly repeated roots at r_0 and r_1. To understand how r_1 affects the L^p norm of q we study the function

$$A_p(q)(r_1) = \|q\|^p_{L^p_{[a,b]}} = \int_a^{r_1} (r_1 - x)^{kp} |s(x)|^p \, dx + \int_{r_1}^b (x - r_1)^{kp} |s(x)|^p \, dx,$$

where we allow $r_1 \in [r_0, r_2]$.

The following two basic results of calculus will be used later, when we optimize the L^p norm.

Lemma 2.1. *If $f(x)$ is twice differentiable and concave up on $[a, b]$, then*

$$\max\{f(a), f(b)\} > f(x)$$

for all $x \in (a, b)$.

Lemma 2.2 (Leibniz's formula). *If $F(x, y)$ and $F_x(x, y)$ are continuous in both x and y in some region of the xy-plane including $a \le y \le x$ and $u(x)$ is a continuous function of x, then*

$$\frac{d}{dx} \int_a^{u(x)} F(x, y) \, dy = F(x, u(x)) \frac{d}{dx} u(x) + \int_a^{u(x)} F_x(x, y) \, dy.$$

3. Maximizing the enclosed area

We are now ready to find the member of Q_n that encloses the largest area. In order to do so we show that $A_1(q)(r_1)$ is concave up on $[r_0, r_2]$.

Theorem 3.1. *If $q(x) = (x - r_1)^k s(x)$, where $s(x) = (x - r_0)^l (x - r_2) \cdots (x - r_{m-1})$ and $r_0 < r_1 < r_2 \le r_3 \le \cdots \le r_{m-1}$, then*

$$\frac{d^2}{dr_1^2} A_1(q)(r_1) > 0 \quad \text{on } [r_0, r_2].$$

Proof. Let $F(r_1, x) = (x - r_1)^k s(x)$, and observe that $F(r_1, r_1) = 0$. Applying Leibniz's formula to each term in $dA_1(q)(r_1)/dr_1$, we have

$$\frac{d}{dr_1} \int_a^{r_1} (r_1 - x)^k |s(x)| \, dx = k \int_a^{r_1} (r_1 - x)^{k-1} |s(x)| \, dx$$

and

$$\frac{d}{dr_1} \int_{r_1}^b (x - r_1)^k |s(x)| \, dx = -k \int_{r_1}^b (x - r_1)^{k-1} |s(x)| \, dx.$$

If $k = 1$, the fundamental theorem of Calculus implies that

$$\frac{d^2}{dr_1^2}\int_a^{r_1}(r_1 - x)|s(x)|\,dx = |s(r_1)| \quad \text{and} \quad \frac{d^2}{dr_1^2}\int_{r_1}^{b}(x - r_1)|s(x)|\,dx = |s(r_1)|.$$

Since r_1 is not a root of $s(x)$, it follows that

$$\frac{d^2}{dr_1^2}A_1(q)(r_1) = 2|s(r_1)| > 0.$$

If $k \geq 2$, then

$$\frac{d^2}{dr_1^2}\int_a^{r_1}(r_1 - x)^k|s(x)|\,dx = k(k-1)\int_a^{r_1}(r_1 - x)^{k-2}|s(x)|\,dx$$

and

$$\frac{d^2}{dr_1^2}\int_{r_1}^{b}(x - r_1)^k|s(x)|\,dx = k(k-1)\int_{r_1}^{b}(x - r_1)^{k-2}|s(x)|\,dx.$$

Therefore,

$$\frac{d^2}{dr_1^2}A_1(q)(r_1) = k(k-1)\int_a^b |(x - r_1)^{k-2}s(x)|\,dx > 0$$

and $A_1(q)(r_1)$ is concave up on $[r_0, r_2]$. □

Corollary 3.2. *One of the Bernstein polynomials*

$$b_i(x) = (x-a)^{n-i}(x-b)^i, \quad i = 1, \ldots, n-1,$$

is the member of Q_n that encloses the largest area on $[a, b]$.

Theorem 3.1, along with Lemma 2.1, tells us that we can always find a polynomial in Q_n with a larger L^1 norm by "dragging" r_1 to either r_0 or r_2. Playing this game a finite number of times leaves us a polynomial with roots only at a and b. So, one of the Bernstein polynomials,

$$b_i(x) = (x-a)^{n-i}(x-b)^i, \quad i = 1, \ldots, n-1,$$

will be the member of Q_n that encloses the largest area.

4. Other values of p

We now extend the method of the previous section to values of $p > 1$. Let

$$q(x) = (x - r_1)^k s(x),$$

where

$$s(x) = (x - r_0)^l (x - r_2) \cdots (x - r_{m-1})$$

with $r_0 < r_1 < r_2 \le r_3 \le \cdots \le r_{m-1}$, and consider

$$A_p(q)(r_1) = \int_a^{r_1} (r_1 - x)^{kp} |s(x)|^p \, dx + \int_{r_1}^b (x - r_1)^{kp} |s(x)|^p \, dx. \tag{1}$$

If we can show that $A_p(q)(r_1)$ is concave up on $[r_0, r_2]$, then one of the Bernstein polynomials will be the member of Q_n with the largest L^p norm. Using the same argument as the $p = 1$ case, two applications of Leibniz's formula yields

$$\frac{d^2}{dr_1^2} A_p(q)(r_1) = kp(kp-1) \int_a^b |(x - r_1)^{kp-2}| |s(x)|^p \, dx > 0,$$

and $A_p(q)(r_1)$ is concave up on the interval $[r_0, r_2]$ when $p > 1$.

In the above calculation, we have to be careful when $kp - 2 < 0$. Since $kp - 1 > 0$ ($k \ge 1$ and $p > 1$) the hypothesis of Leibniz's formula are satisfied for the first application with

$$\frac{d}{dr_1} A_p(q)(r_1) = kp \int_a^{r_1} (r_1 - x)^{kp-1} |s(x)|^p \, dx - kp \int_{r_1}^b (x - r_1)^{kp-1} |s(x)|^p \, dx. \tag{2}$$

When applying Leibniz's formula to the first term on the right-hand side, we need

$$\frac{\partial}{\partial r_1} (r_1 - x)^{kp-1} |s(x)|^p$$

to be continuous in both x and r_1 in some region including $a \le x \le r_1$. Although this may not be true at $x = r_1$, we can still justify the application of Leibniz's formula by considering the interval $[a, r_1 - \epsilon]$ and letting $\epsilon \to 0^+$. That is,

$$\frac{d^2}{dr_1^2} \int_a^{r_1} (r_1 - x)^{kp} |s(x)|^p \, dx = \lim_{\epsilon \to 0^+} \left(\frac{d}{dr_1} kp \int_a^{r_1 - \epsilon} (r_1 - x)^{kp-1} |s(x)|^p \, dx \right).$$

Because the integrand is positive, the result will follow if the limit exists.

The polynomial $s(x)$ does not change sign on the interval (a, r_2), so we may assume without loss of generality that $s(x) \ge 0$ on $[a, r_1 - \epsilon]$, with $s(x) = 0$ only at $x = a$. Applying Leibniz's formula on $[a, r_1 - \epsilon]$ yields

$$\begin{aligned}
&\lim_{\epsilon \to 0^+} \left(\frac{d}{dr_1} kp \int_a^{r_1 - \epsilon} (r_1 - x)^{kp-1} s(x)^p \, dx \right) \\
&\qquad = \lim_{\epsilon \to 0^+} kp(kp-1) \int_a^{r_1 - \epsilon} (r_1 - x)^{kp-2} s(x)^p \, dx + \lim_{\epsilon \to 0^+} (\epsilon)^{kp-1} s(r_1 - \epsilon)^p \\
&\qquad = \lim_{\epsilon \to 0^+} kp(kp-1) \int_a^{r_1 - \epsilon} (r_1 - x)^{kp-2} s(x)^p \, dx.
\end{aligned}$$

In order to see that this limit exists, we integrate by parts to get

$$kp(kp-1)\lim_{\epsilon\to 0^+}\left(-s(r_1-\epsilon)^p\frac{(\epsilon)^{kp-1}}{kp-1}+\frac{p}{kp-1}\int_a^{r_1-\epsilon}(r_1-x)^{kp-1}s(x)^{p-1}s'(x)\,dx\right)$$
$$=kp^2\int_a^{r_1}(r_1-x)^{kp-1}s(x)^{p-1}s'(x)\,dx,$$

where equality follows as $kp-1>0$ and the integrand is a continuous function of x on $[a,r_1]$. Hence the limit exists and is positive from an earlier observation. A similar argument applied to the second term on the right in (2) shows that

$$\frac{d^2}{dr_1^2}\int_{r_1}^b (x-r_1)^{kp}|s(x)|^p\,dx=\lim_{\epsilon\to 0^+}\frac{d}{dr_1}\left(-kp\int_{r_1+\epsilon}^b (x-r_1)^{kp-1}|s(x)|^p\,dx\right)$$

exists and is positive. Therefore, $\dfrac{d^2}{dr_1^2}A_p(q)(r_1)>0$.

From an argument similar to Theorem 3.1, we have the following result:

Theorem 4.1. *If $p\geq 1$, one of the Bernstein polynomials is the member of Q_n that has the largest L^p norm on $[a,b]$.*

Finally, we consider the case $p=\infty$. Since $[a,b]$ has finite measure,

$$\lim_{p\to\infty}\|f(x)\|_{L^p_{[a,b]}}=\|f(x)\|_{L^\infty_{[a,b]}};\tag{3}$$

see [Wheeden and Zygmund 1977, p. 126].

Corollary 4.2. *One of the Bernstein polynomials is the member of Q_n that has the largest L^∞ norm on $[a,b]$.*

Proof. Let $m(x)\in Q_n$ with $m(x)\neq b_i(x)$ for $i=1,\dots,n-1$. If we restrict p to the positive integers, it follows from (3) that the sequences

$$\left\{\|m(x)\|_{L^p_{[a,b]}}\right\}_p\to\|m(x)\|_{L^\infty_{[a,b]}}\quad\text{and}\quad\left\{\|b_i(x)\|_{L^p_{[a,b]}}\right\}_p\to\|b_i(x)\|_{L^\infty_{[a,b]}}$$

as $p\to\infty$. Theorem 4.1 implies that for each $p\in\mathbb{N}$

$$\|m(x)\|_{L^p_{[a,b]}}\leq\|b_i(x)\|_{L^p_{[a,b]}},$$

so that

$$\lim_{p\to\infty}\|m(x)\|_{L^p_{[a,b]}}\leq\lim_{p\to\infty}\|b_i(x)\|_{L^p_{[a,b]}}.$$

Therefore $\|m(x)\|_{L^\infty_{[a,b]}}\leq\|b_i(x)\|_{L^\infty_{[a,b]}}$ and we have the desired result. □

5. Evaluating the maximum

The process of increasing the L^p norm lead us to a finite class of polynomials that must contain the "largest" polynomial in Q_n. Specifically, we arrived at the class of Bernstein polynomials

$$b_i(x) = (x-a)^{n-i}(x-b)^i, \quad i = 1, \dots, n-1.$$

We would like to determine which of these polynomials will maximize the L^p norm. To do so, we recall (from [Dennery and Krzywicki 1996, pp. 94–98], for example) the beta function, defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} dt$ satisfies the property $\Gamma(n+1) = n!$.

Initially, we answer the question when $a = 0$ and $b = 1$, and then translate the result back to general a and b by the appropriate substitution. We observe that

$$\int_0^1 x^{n-i}(x-1)^i \, dx = (-1)^i B(n-i+1, i+1) = (-1)^i \frac{\Gamma(n-i+1)\Gamma(i+1)}{\Gamma(n+2)}.$$

Since the polynomials $b_i(x)$ are either entirely positive or entirely negative on [0, 1], we have

$$\|b_i(x)\|_{L^1_{[0,1]}} = \left| \int_0^1 x^{n-i}(x-1)^i \, dx \right| = \frac{\Gamma(n-i+1)\Gamma(i+1)}{\Gamma(n+2)} = \frac{1}{n+1} \frac{i!\,(n-i)!}{n!}.$$

Note that $\dfrac{i!\,(n-i)!}{n!}$ is the reciprocal of the binomial coefficient $\dbinom{n}{i}$. Since n is fixed, we need to pick the value of i that minimizes this binomial coefficient. Clearly this happens when $i = 1$ or $i = n-1$. Therefore, the maximum value of the norm is obtained for $b_1(x)$ and $b_{n-1}(x)$:

$$\|b_1(x)\|_{L^1_{[0,1]}} = \|b_{n-1}(x)\|_{L^1_{[0,1]}} = \frac{1}{n+1}\binom{n}{1}^{-1} = \frac{1}{n(n+1)}. \tag{4}$$

This can be generalized to the interval $[a, b]$ by using the substitution $u = (x-a)/(b-a)$; for any monic degree-n polynomial $q(x)$ with all real zeros in $[a, b]$ such that $q(x)$ has roots at a and b, we have

$$\|q(x)\|_{L^1_{[a,b]}} \le (b-a)^{n+1} \frac{1}{n(n+1)}.$$

If $p \in \mathbb{N}$, the same method can be used to evaluate the L^p norm of the Bernstein polynomials. We have

$$\|b_i(x)\|^p_{L^p_{[0,1]}} = \left[\frac{\Gamma(pn-pi+1)\Gamma(pi+1)}{\Gamma(pn+2)}\right]^{1/p} = \left[\frac{1}{pn+1}\,\frac{(pn-pi)!\,(pi)!}{(pn)!}\right]^{1/p}. \quad (5)$$

The maximum value is still achieved by $b_1(x)$ and $b_{n-1}(x)$. Inequality (5) can be generalized to the interval $[a, b]$ by using the substitution $u = (x-a)/(b-a)$; for any monic degree-n polynomial $q(x)$ with all real zeros in $[a, b]$ such that $q(x)$ has roots at a and b,

$$\|q(x)\|_{L^p_{[a,b]}} \leq \left[(b-a)^{pn+1}\frac{1}{pn+1}\,\frac{(pn-pi)!\,(pi)!}{(pn)!}\right]^{1/p}.$$

If p is not a natural number, the first equality in (5) is still valid, though we can no longer express the result in terms of factorials. Therefore (again passing to the case of $[a, b]$) we can write

$$\|b_i(x)\|_{L^p_{[a,b]}} = \left[(b-a)^{pn+1}\frac{\Gamma(pn-pi+1)\Gamma(pi+1)}{\Gamma(pn+2)}\right]^{1/p}. \quad (6)$$

To find the values of i that maximize this expression, we can differentiate it with respect to i. (Although only integer values of i make sense in our context, the quotient in (6) makes sense for all real i in the range of interest, $1 \leq i \leq n-1$. The domain of definition and differentiability of the gamma function includes $(0, \infty)$.) The derivative of the gamma function involves another transcendental function, known as polygamma. The upshot is that the quotient in (6) has only one critical point in the interval $1 \leq i \leq n-1$, and it is a minimum rather than a maximum. It follows that, once more, the local maxima in this interval must be at the endpoints of the interval, that is, $i = 1$ and $i = n-1$.

6. Recovering the supremum norm

As mentioned in the introduction, it was established in [Boelkins et al. 2006] that the Bernstein polynomials $b_1(x)$ and $b_{n-1}(x)$ are the members of Q_n with the largest L^∞ norm on $[a, b]$. In fact, they found that

$$\|b_1(x)\|_{L^\infty_{[a,b]}} = \frac{(b-a)^n}{n}\left(\frac{n-1}{n}\right)^{n-1},$$

a result that we now reproduce as a consequence of the work in the previous section.

We have seen that, for $p \in \mathbb{N}$,

$$\|b_1(x)\|_{L^p_{[a,b]}} = \left[\frac{(b-a)^{pn+1}}{pn+1}\left(\frac{(pn-p)!\,p!}{(pn)!}\right)\right]^{1/p}.$$

Applying Sterling's approximation, $\lim_{n\to\infty}\left(n! - \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right) = 0$, we obtain

$$\begin{aligned}\|b_1(x)\|_{L^\infty_{[a,b]}} &= \lim_{p\to\infty} \|b_1(x)\|_{L^p_{[a,b]}} \\ &= \lim_{p\to\infty}\left[\frac{(b-a)^{pn+1}}{pn+1}\frac{(pn-p)!\,p!}{(pn)!}\right]^{1/p} \\ &= \lim_{p\to\infty}\left[\frac{(b-a)^{pn+1}}{pn+1}\frac{\sqrt{2\pi p(n-1)}\left(\frac{p(n-1)}{e}\right)^{p(n-1)}\sqrt{2\pi p}\left(\frac{p}{e}\right)^p}{\sqrt{2\pi pn}\left(\frac{pn}{e}\right)^{pn}}\right]^{1/p}.\end{aligned}$$

After simplification, this becomes

$$\begin{aligned}\|b_1(x)\|_{L^\infty_{[a,b]}} &= \frac{(b-a)^n}{n}\left(\frac{n-1}{n}\right)^{n-1}\lim_{p\to\infty}\left[\frac{(b-a)}{pn+1}\left(\frac{\sqrt{2\pi p(n-1)}}{\sqrt{n}}\right)\right]^{1/p} \\ &= \frac{(b-a)^n}{n}\left(\frac{n-1}{n}\right)^{n-1}\lim_{p\to\infty}\left(\frac{b-a}{\sqrt{n}}\right)^{1/p}\lim_{p\to\infty}\left(\frac{\sqrt{2\pi p(n-1)}}{pn+1}\right)^{1/p} \\ &= \frac{(b-a)^n}{n}\left(\frac{n-1}{n}\right)^{n-1}\lim_{p\to\infty}\left(\frac{\sqrt{2\pi p(n-1)}}{pn+1}\right)^{1/p}.\end{aligned}$$

L'Hopital's rule implies

$$\lim_{p\to\infty}\left(\frac{\sqrt{2\pi p(n-1)}}{pn+1}\right)^{1/p} = 1$$

and it follows that

$$\|b_1(x)\|_{L^\infty_{[a,b]}} = \frac{(b-a)^n}{n}\left(\frac{n-1}{n}\right)^{n-1}.$$

We can now reasonably claim that the Bernstein polynomials are the largest monic polynomials with all real roots in $[a, b]$ in the full sense of all possible L^p norms.

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