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The family of ternary cyclotomic polynomials with one free prime

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A cyclotomic polynomial $\Phi_n(x)$ is said to be ternary if $n = pqr$, with p, q and r distinct odd primes. Ternary cyclotomic polynomials are the simplest ones for which the behavior of the coefficients is not completely understood. Here we establish some results and formulate some conjectures regarding the coefficients appearing in the polynomial family $\Phi_{pqr}(x)$ with $p < q < r$, p and q fixed and r a free prime.

1. Introduction

The n-th cyclotomic polynomial $\Phi_n(x)$ is defined by

$$\Phi_n(x) = \prod_{\substack{1\le j\le n\\(j,n)=1}} (x-\zeta_n^j) = \sum_{k=0}^{\infty} a_n(k)x^k,$$

with ζ_n a n-th primitive root of unity (one can take $\zeta_n = e^{2\pi i/n}$). It has degree $\varphi(n)$, with φ Euler's totient function. We write $A(n) = \max\{|a_n(k)| : k \ge 0\}$, and this quantity is called the height of $\Phi_n(x)$. It is easy to see that $A(n) = A(N)$, with $N = \prod_{p|n,\, p>2} p$ the odd squarefree kernel. In deriving this, one uses the observation that if n is odd, then $A(2n) = A(n)$. If n has at most two distinct odd prime factors, then $A(n) = 1$. If $A(n) > 1$, then we necessarily must have that n has at least three distinct odd prime factors. In particular for $n < 105 = 3\cdot 5\cdot 7$ we have $A(n) = 1$. It turns out that $A(105) = 2$ with $a_{105}(7) = -2$. Thus the easiest case where we can expect nontrivial behavior of the coefficients of $\Phi_n(x)$ is the ternary case, where $n = pqr$, with $2 < p < q < r$ odd primes. In this paper we are concerned with the family of ternary cyclotomic polynomials

$$\{\Phi_{pqr}(x) : r > q\}, \tag{1}$$

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where $2 < p < q$ are fixed primes and r is a "free prime". Up to now in the literature the above family was considered, but with also q free. The maximum coefficient (in absolute value) that occurs in that family will be denoted by $M(p)$, thus $M(p) = \max\{A(pqr) : p < q < r\}$, with $p > 2$ fixed. Similarly we define $M(p; q)$ to be the maximum coefficient (in absolute value) that occurs in the family (1), thus $M(p; q) = \max\{A(pqr) : r > q\}$, with $2 < p < q$ fixed primes.

Example. Bang [1895] proved that $M(p) \leq p - 1$. Since $a_{3\cdot5\cdot7}(7) = -2$ we infer that $M(3) = 2$. Using $a_{105}(7) = -2$ and $M(3) = 2$, we infer that $M(3; 5) = 2$.

Let $\mathcal{A}(p; q) = \{a_{pqr}(k) : r > q,\ k \geq 0\}$ be the set of coefficients occurring in the polynomial family (1).

Proposition 1. $\mathcal{A}(p; q) = [-M(p; q), M(p; q)] \cap \mathbb{Z}$.

This shows the relevance of understanding $M(p; q)$. Let us first recall some known results concerning the related function $M(p)$. Here we know thanks to Bachman [2003], who very slightly improved on an earlier result in [Beiter 1971], that $M(p) \leq 3p/4$. It was conjectured by Sister Marion Beiter [1968] (see also [Beiter 1971]) that $M(p) \leq (p+1)/2$. She proved it for $p \leq 5$. Since Möller [1971] proved that $M(p) \geq (p+1)/2$ for $p > 2$, her conjecture actually would imply that $M(p) = (p + 1)/2$ for $p > 2$. The first to show that Beiter's conjecture is false seems to have been Eli Leher (in his PhD thesis), who gave the counterexample $a_{17\cdot29\cdot41}(4801) = -10$, showing that $M(17) \geq 10 > 9 = (17 + 1)/2$. Gallot and Moree [2009b] provided for each $p \geq 11$ infinitely many infinitely many counterexamples $p \cdot q_j \cdot r_j$ with q_j strictly increasing with j. Moreover, they have shown that for every $\epsilon > 0$ and p sufficiently large $M(p) > (\frac{2}{3} - \epsilon)p$. They also proposed the corrected Beiter conjecture: $M(p) \leq 2p/3$. The implications of their work for $M(p; q)$ are described in Section 4.

Proposition 1 together with Möller's result quoted above gives a different proof of the result, due to Bachman [2004], that $\{a_{pqr}(k) : p < q < r\} = \mathbb{Z}$. For references and further results in this direction (begun by I. Schur) see Fintzen [2011].

Jia Zhao and Xianke Zhang [2010] showed that $M(7) = 4$, thus establishing the Beiter conjecture for $p = 7$. In a later paper they established the corrected Beiter conjecture:

Theorem 2 [Zhao and Zhang 2009]. $M(p) \leq 2p/3$.

This result together with some computer computation allows one to extend the list of exactly known values of $M(p)$ (see Table 1).

It is not known whether there is a finite procedure to determine $M(p)$. On the other hand, it is not difficult to see that there is such a procedure for $M(p; q)$.

Proposition 3. *Given primes $2 < p < q$, there is a finite procedure to determine $M(p; q)$.*

p	$M(p)$	smallest n
3	2	$3 \cdot 5 \cdot 7$
5	3	$5 \cdot 7 \cdot 11$
7	4	$7 \cdot 17 \cdot 23$
11	7	$11 \cdot 19 \cdot 601$
13	8	$13 \cdot 73 \cdot 307$
19	12	$19 \cdot 53 \cdot 859$

Table 1. Values of $M(p)$. By "smallest n" we mean the smallest integer n satisfying $A(n) = M(p)$ and with p as its smallest prime divisor.

Recall that a set S of primes is said to have *natural density* δ if

$$\lim_{x\to\infty} \frac{|\{p \le x : p \in S\}|}{\pi(x)} = \delta,$$

where $\pi(x)$ is the number of primes $p \le x$. A further question that arises is how often the maximum value $M(p)$ is assumed. We have:

Theorem 4. *Given primes $2 < p < q$, there exists a prime q_0 with $q_0 \equiv q \pmod{p}$ and an integer d such that $M(p, q) \le M(p, q_0) = M(p, q')$ for every prime $q' \ge q_0$ satisfying $q' \equiv q_0 \pmod{d \cdot p}$. In particular the set of primes q with $M(p; q) = M(p)$ has a subset having a positive natural density.*

A weaker result in this direction, namely that for a fixed prime $p \ge 11$, the set of primes q such that $M(p; q) > (p+1)/2$ has a subset of positive natural density, follows from [Gallot and Moree 2009b] (recall that $M(p) > (p+1)/2$ for $p \ge 11$).

Unfortunately, the proof of Theorem 4 gives a lower bound for the density that seems to be far removed from the true value. In this paper we present some constructions that allow one to obtain much better bounds for the density for small p. These results are subsumed in the following main result of the paper.

Theorem 5. *Let $2 < p \le 19$ be a prime with $p \ne 17$. Then the set of primes q such that $M(p; q) = M(p)$ has a subset having natural density $\delta(p)$ as follows*:

$p =$	3	5	7	11	13	19
$\delta(p) =$	1	1	1	$\frac{2}{5}$	$\frac{1}{12}$	$\frac{1}{9}$

Numerical experimentation suggests that the set of primes q such that $M(p; q) = M(p)$ has a natural density $\delta(p)$ as given in the above table, except when $p = 13$ in which case numerical experimentation suggests $\delta(13) = 1/3$.

In order to prove Theorem 5, we will use the following theorem dealing with $2 < p \le 7$.

Theorem 6. *For $2 < p \le 7$ and $q > p$ we have $M(p; q) = (p+1)/2$, except in the case $p = 7$, $q = 13$, where $M(7; 13) = 3$.*

The fact that $M(7; 13) = 3$ can be explained. It turns out that if $ap + bq = 1$ for integers a and b small in absolute value, then $M(p; q)$ is small. For example:

Theorem 7. *If $p \ge 5$ and $2p - 1$ is a prime, then $M(p; 2p - 1) = 3$.*

This result and similar ones are established in Section 10.

Our main conjecture on $M(p; q)$ is the following one.

Conjecture 8. *Given a prime p, there exists an integer d and a function*

$$g : (\mathbb{Z}/d\mathbb{Z})^* \to \mathbb{Z}_{>0}$$

such that for some $q_0 > d$ we have for every prime $q \ge q_0$ that $M(p; q) = g(\bar{q})$, where $1 \le \bar{q} < d$ satisfies $q \equiv \bar{q} \pmod{d}$. The function g is symmetric, that is we have $g(\alpha) = g(d - \alpha)$.

The smallest integer d with the above properties, if it exists, we call the *ternary conductor* $\mathfrak{f}_p$. The corresponding smallest choice of q_0 (obtained on setting $d = \mathfrak{f}_p$) we call the *ternary minimal prime*. For $p = 7$ we obtain, e.g., $\mathfrak{f}_7 = 1$ and $q_0 = 17$ (by Theorem 6). Note that once we know q_0 it is a finite computation to determine d and the function g. Theorem 6 can be used to obtain the $p \le 7$ part of the following observation concerning the ternary conductor.

Proposition 9. *If $2 < p \le 7$, then the ternary conductor exists and we have $\mathfrak{f}_p = 1$. If $p \ge 11$ and $\mathfrak{f}_p$ exists, then $p | \mathfrak{f}_p$.*

While Theorem 4 only says that the set of primes q with $M(p; q) = M(p)$ has a subset having a positive natural density, Conjecture 8 implies that the set actually has a natural density in $\mathbb{Q}_{>0}$ which can be easily explicitly computed assuming we know q_0. In order to establish this implication one can invoke a quantitative form of Dirichlet's prime number theorem to the effect that, for $(a, d) = 1$, we have, as x tends to infinity,

$$\sum_{\substack{p \le x \\ p \equiv a \pmod{d}}} 1 \sim \frac{x}{\varphi(d) \log x}. \tag{2}$$

This result implies that asymptotically the primes are equidistributed over the primitive congruence classes modulo d. (Recall that Dirichlet's prime number theorem, Dirichlet's theorem for short, says that each primitive residue class contains infinitely many primes.)

The main tool in this paper is Kaplan's lemma, presented in Section 6. The material in that section (except for Lemma 22, which is new) is taken from [Gallot and Moree 2009a]. As a demonstration of working with Kaplan's lemma two

examples (with and without table) are given in Section 6.1. In [Gallot et al. 2010], the full version of this paper, details of further proofs using Kaplan's lemma can be found. In the shorter version we have merely written "Apply Kaplan's lemma".

The above summary of results makes clear how limited presently our knowledge of $M(p;q)$ is. For the benefit of the interested reader we present a list of open problems in Section 11.

2. Proof of two propositions and Theorem 4

Proof of Proposition 1. By the definition of $M(p;q)$ we have

$$\mathcal{A}(p;q) \subseteq [-M(p;q), M(p;q)] \cap \mathbb{Z}.$$

Let $r > q$ be a prime such that $A(pqr) = M(p;q)$ and suppose, without loss of generality, that $a_{pqr}(k) = M(p;q)$. Gallot and Moree [2009a] showed that $|a_n(k) - a_n(k-1)| \leq 1$ for ternary n (see [Bachman 2010; Bzdęga 2010] for alternative proofs). Since $a_{pqr}(k) = 0$ for every k large enough, it then follows that $0, 1, \ldots, M(p;q)$ are in $\mathcal{A}(p;q)$. By a result of Kaplan [2007] (see [Zhao and Zhang 2010] for a different proof), we can find a prime $s \equiv -r \pmod{pq}$ and an integer k_1 such that $a_{pqs}(k_1) = -M(p;q)$. By a similar arguments as above one then infers that $-M(p;q), -M(p;q)+1, \ldots, -1, 0$ are all in $\mathcal{A}(p;q)$. □

Proof of Proposition 3. Let $\mathcal{R}_{pq}$ be a set of primes, all exceeding q such that every primitive residue class modulo pq is represented. By [Kaplan 2007, Theorem 2] we have $A(pqr) = A(pqs)$ if $s \equiv r \pmod{pq}$ with s, r both primes exceeding q and hence

$$M(p;q) = \max\{A(pqr) : r \in \mathcal{R}_{pq}\}.$$

Since the computation of $\mathcal{R}_{pq}$ and $A(pqr)$ is a finite one, the computation of $M(p;q)$ is also finite. □

The remainder of the section is devoted to the proof of Theorem 4.

For coprime positive (not necessary prime) integers p, q, r we define

$$\Phi'_{p,q,r}(x) = \frac{(x^{pqr}-1)(x^p-1)(x^q-1)(x^r-1)}{(x-1)(x^{pq}-1)(x^{pr}-1)(x^{qr}-1)} = \sum_{k=0}^{\infty} a'_{p,q,r}(k)x^k.$$

Here we do not assume $p < q < r$. Hence we have the symmetry $\Phi'_{p,q,r}(x) = \Phi'_{p,r,q}(x)$. A routine application of the inclusion-exclusion principle to the roots of the factors shows that $\Phi'_{p,q,r}(x)$ is a polynomial. It is referred to as a ternary inclusion-exclusion polynomial. Inclusion-exclusion polynomials can be defined in great generality, and the reader is referred to [Bachman 2010] for an introductory discussion. He shows that such polynomials and thus $\Phi'_{p,q,r}(x)$ in particular, can be written as products of cyclotomic polynomials (see Theorem 2 in that reference).

Analogously to $A(pqr)$ and $M(p;q)$ we define

$$A'(p,q,r) = \max\{|a'_{p,q,r}(k)| : k \geq 0\},$$
$$M'(p;q) = \max\{A'(p,q,r) : r \geq 1\},$$
$$M'(p) = \max\{M'(p;q) : q \geq 1\}.$$

We have $\Phi_{pqr}(x) = \Phi'_{p,q,r}(x)$ if p, q, r are distinct primes, so $A(pqr) = A'(p,q,r)$ in this case.

Lemma 10. *For coprime positive (not necessary prime) integers p, q, r we have $A'(p,q,r_1) \leq A'(p,q,r_2) \leq A'(p,q,r_1) + 1$ if $r_2 \equiv r_1 \pmod{pq}$ and $r_2 > r_1$.*

Proof. Note that $r_2 > \max\{p,q\}$. If $r_1 > \max\{p,q\}$, then Kaplan [2007, proof of Theorem 2] showed that $A'(p,q,r_1) = A'(p,q,r_2)$. In the remaining case $r_1 < \max\{p,q\}$, we have $A'(p,q,r_1) \leq A'(p,q,r_2) \leq A'(p,q,r_1) + 1$ by the Theorem in [Bachman and Moree 2011]. □

In [Bachman and Moree 2011] it is remarked that $A'(p,q,r_2) = A'(p,q,r_1)+1$ can occur.

Lemma 11. *If p is a prime, then $M'(p) = M(p)$. If q is also a prime with $q > p$ then $M'(p;q) = M(p;q)$.*

Proof. Let $p < q$ be primes. Assume $M'(p;q) = A'(p,q,r)$, where r is not necessary a prime. By Dirichlet's theorem we can find a prime r' satisfying

$$r' \equiv r \pmod{pq} \quad \text{and} \quad r' > \max(q,r).$$

Therefore we have, by Lemma 10,

$$M'(p;q) = A'(p,q,r) \leq A'(p,q,r') = A(p,q,r') \leq M(p;q).$$

Since obviously $M(p;q) \leq M'(p;q)$, we have $M'(p;q) = M(p;q)$.

Now let only p be a prime. Assume $M'(p) = A'(p,q,r)$, where q and r are not necessary primes. Again by Dirichlet's theorem we find a prime q' with $q' \equiv q \pmod{pr}$ and $q' > \max(p,q)$. Using Lemma 10 we have

$$M'(p) = A'(p,q,r) \leq A'(p,q',r) \leq M'(p,q') = M(p,q') \leq M(p).$$

Since obviously $M(p) \leq M'(p)$, we have $M'(p) = M(p)$. □

Proof of Theorem 4. We set $q_1 := q$. Let r_i be a positive integer satisfying $M'(p;q_i) = A'(p,q_i,r_i)$. Using Lemma 10 (note that $A'(p,q,r)$ is invariant under permutations of p, q and r) we deduce

$$M'(p;q_1) = A'(p,q_1,r_1) \leq A'(p,q_2,r_1) \leq A'(p,q_2,r_2) = M'(p,q_2),$$

where $q_2 = q_1 + pr_1$. By the same argument the sequence $q_1, q_2, q_3, \ldots$ with $q_{i+1} = q_i + pr_i$ satisfies

$$M'(p; q_1) \le M'(p; q_2) \le M'(p; q_3) \le \cdots$$

Since $M'(p; q) \le M'(p) = M(p)$ and by, e.g., Lemma 18, $M(p)$ is finite, there are only finitely many different values for $M'(p; q)$. Hence there is an index k such that $M'(p; q_k) = M'(p; q_{k+i})$ for all $i \ge 0$. That means

$$M'(p; q_k) = A'(p, q_k, r_k) = A'(p, q_{k+1}, r_k) = A'(p, q_{k+1}, r_{k+1}) = M'(p, q_{k+1}),$$

and by induction $A'(p, q_{k+i}, r_k) = A'(p, q_{k+i}, r_{k+i})$. Therefore we can assume $r_{k+i} = r_k$ for $i \ge 0$. Then we have $q_{k+i} = q_k + i \cdot pr_k$. We set $q_0 := q_k$ and $d := r_k$. Certainly we have $q_0 \equiv q \pmod p$. Let $q' \ge q_0$ be a prime with $q' \equiv q_0 \pmod{d \cdot p}$. There must be an integer m such that $q' = q_{k+m}$. Since $M'(p; q) = M(p; q)$ by Lemma 11, we have

$$M(p; q_1) \le M(p; q_0) = M(p; q').$$

Applying this to $M(p; q_1)$ with $M(p; q_1) = M(p)$, where we have chosen q_1 such that $M(p; q_1) = M(p)$, we get infinitely many primes of the form $q_i = q_1 + i \cdot pr_1$ satisfying $M(p; q_i) = M(p)$. On invoking (2) with $a = q_1$ and $d = pr_1$ the proof is then completed. □

3. The bounds of Bachman and Bzdęga

Let q^* and r^*, $0 < q^*, r^* < p$ be the inverses of q and r modulo p respectively. Set $a = \min(q^*, r^*, p - q^*, p - r^*)$. Put $b = \max(\min(q^*, p - q^*), \min(r^*, p - r^*))$. In the sequel we will use repeatedly that $b \ge a$. Bachman [2003] showed that

$$A(pqr) \le \min\left(\frac{p-1}{2} + a,\, p - b\right). \tag{3}$$

This was more recently improved by Bzdęga [Bzdęga 2010] who showed that

$$A(pqr) \le \min(2a + b,\, p - b). \tag{4}$$

It is not difficult to show that $\min(2a + b, p - b) \le \min(\frac{p-1}{2} + a, p - b)$ and thus Bzdęga's bound is never worse than Bachman's and in practice often strict inequality holds.

Note that if $q \equiv \pm 1 \pmod p$, then (3) implies that $A(pqr) \le (p+1)/2$, a result due to Beiter [1968] and, independently, Bloom [1968].

We remark that Bachman and Bzdęga define b as follows:

$$b = \min(b_1, p - b_1), \quad ab_1qr \equiv 1 \pmod p, \quad 0 < b_1 < p.$$

It is an easy exercise to see that our definition is equivalent to this one.

We will show that both (3) and (4) give rise to the same upper bound $f(q^*)$ for $M(p;q)$. Write $q^* \equiv j \pmod p$, $r^* \equiv k \pmod p$ with $1 \le j,k \le p-1$. Thus the right-hand sides of both (3) and (4) are functions of j and k, which we denote respectively by $\mathrm{GB}(j,k)$ and $\mathrm{BB}(j,k)$. We have

$$\mathrm{BB}(j,k) = \min(2a+b,\, p-b) \le \min\left(\frac{p-1}{2}+a,\, p-b\right) = \mathrm{GB}(j,k),$$

with $a = \min(j,k,\, p-j,\, p-k)$ and $b = \max(\min(j,\, p-j),\, \min(k,\, p-k))$.

Lemma 12. *Let* $1 \le j \le p-1$. *Denote* $\mathrm{GB}(j,j)$ *by* $f(j)$. *We have*

$$\max_{1\le k\le p-1} \mathrm{BB}(j,k) = \max_{1\le k\le p-1} \mathrm{GB}(j,k) = f(j),$$

with

$$f(j) = \begin{cases} \frac{1}{2}(p-1)+j & \text{if } j < p/4, \\ p-j & \text{if } p/4 < j \le \frac{1}{2}(p-1), \end{cases}$$

and $f(p-j) = f(j)$ *if* $j > \frac{1}{2}(p-1)$.

Proof. Since the problem is symmetric under replacing j by $p-j$, without loss of generality we may assume that $j \le \frac{1}{2}(p-1)$. If $j < p/4$, then

$$\mathrm{GB}(j,k) \le \frac{p-1}{2} + a \le \frac{p-1}{2} + j = \mathrm{GB}(j,j).$$

If $j > p/4$, then

$$\mathrm{GB}(j,k) \le p - b \le p - j = \mathrm{GB}(j,j).$$

Note that

$$\mathrm{GB}(j,j) = \begin{cases} \mathrm{BB}\left(j, \frac{1}{2}(p+1)-j\right) & \text{if } j < p/4, \\ \mathrm{BB}(j,j) & \text{if } j > p/4. \end{cases}$$

For example, if $j < p/4$, then the choice $q^* = j$, $r^* = \frac{1}{2}(p+1)-j$ leads to $a = j$ and $b = \frac{1}{2}(p+1)-j$ and hence

$$\mathrm{BB}\left(j, \tfrac{1}{2}(p+1)-j\right) = \min\left(\tfrac{1}{2}(p+1)+j,\, \tfrac{1}{2}(p-1)+j\right) = \mathrm{GB}(j,j).$$

Since $\mathrm{BB}(j,k) \le \mathrm{GB}(j,k) \le \mathrm{GB}(j,j)$ we are done. □

Theorem 13. *Let* $2 < p < q$. *Then* $M(p;q) \le f(q^*)$.

Proof. By (4) and the definition of $\mathrm{BB}(j,k)$ we have

$$M(p;q) \le \max_{1\le k\le p-1} \mathrm{BB}(q^*,k) = f(q^*),$$

completing the proof. □

Lemma 12 shows that using either (3) or (4), we cannot improve on the upper bound given in Theorem 13. Since

$$\max_{1\le j\le p-1} f(j) = p-1-\left[\frac{p}{4}\right] = \begin{cases} \frac{3}{4}(p-1) & \text{if } p\equiv 1 \pmod 4,\\ \frac{1}{4}(3p-1) & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

we infer that

$$M(p) \le \max_{1\le j\le p-1}\ \max_{1\le k\le p-1} \mathrm{GB}(j,k) = \max_{1\le j\le p-1} f(j) < \tfrac{3}{4}p.$$

4. Earlier work on $M(p;q)$

Implicit in the literature are various results on $M(p;q)$ (although we are the first to explicitly study $M(p;q)$). Most of these are mentioned in the rest of this paper. Here we rewrite the main result of [Gallot and Moree 2009b] in terms of $M(p;q)$ and use it for $p=11$, to deal with $q\equiv 4 \pmod{11}$, and $p=13$, to deal with $q\equiv 5 \pmod{13}$.

Theorem 14. *Let $p\ge 11$ be a prime. Given any $1\le\beta\le p-1$ we let β^* be the unique integer $1\le\beta^*\le p-1$ with $\beta\beta^*\equiv 1 \pmod p$. Let $\mathcal{B}_-(p)$ be the set of integers satisfying*

$$1\le\beta\le\frac{p-3}{2},\quad p\le\beta+2\beta^*+1,\quad \beta>\beta^*.$$

Let $\mathcal{B}_+(p)$ be the set of integers satisfying

$$1\le\beta\le\frac{p-3}{2},\quad p\le\beta+\beta^*,\quad \beta\ge\beta^*/2.$$

Let $\mathcal{B}(p)$ be the union of these (disjoint) sets. As $(p-3)/2\in\mathcal{B}(p)$, it is nonempty. Let $q\equiv\beta \pmod p$ be a prime satisfying $q>p$. Suppose that the inequality $q>q_-(p):=p(p-\beta^*)(p-\beta^*-2)/(2\beta)$ holds if $\beta\in\mathcal{B}_-(p)$ and

$$q>q_+(p):=\frac{p(p-1-\beta)}{\gamma(p-1-\beta)-p+1+2\beta},$$

with $\gamma=\min((p-\beta^*)/(p-\beta),(\beta^*-\beta)/\beta^*)$ if $\beta\in\mathcal{B}_+(p)$. Then

$$M(p;q)\ge p-\beta>\frac{p+1}{2}$$

and hence $M(p)\ge p-\min\{\mathcal{B}(p)\}$.

We have $\mathcal{B}(11)=\{4\}$, $\mathcal{B}(13)=\{5\}$, $\mathcal{B}(17)=\{7\}$ and $\mathcal{B}(19)=\{8\}$. In general one can show [Cobeli et al. $\ge$ 2011] using Kloosterman sum techniques that

$$\left||\mathcal{B}(p)|-\frac{p}{16}\right|\le 24p^{3/4}\log p.$$

The lower bound for $M(p)$ resulting from this theorem, $p - \min\{\mathcal{B}(p)\}$, never exceeds $2p/3$ and this together with extensive numerical experimentation led in [Gallot and Moree 2009b] to the proposal of a corrected Beiter conjecture, now proved by Zhao and Zhang (Theorem 2).

Under the appropriate conditions on p and q, Theorem 14 says that $M(p;q) \geq p - \beta$, whereas Theorem 13 yields $M(p;q) \leq f(\beta^*)$. Thus studying the case $p - \beta = f(\beta^*)$ with $\beta \in \mathcal{B}(p)$, leads to a small subset of cases where $M(p;q)$ can be exactly computed using Theorem 14.

Theorem 15. *Let $p \geq 13$ with $p \equiv 1 \pmod 4$ be a prime. Let x_0 be the smallest positive integer such that $x_0^2 + 1 \equiv 0 \pmod p$. If $x_0 > p/3$, $q \equiv x_0 \pmod p$ and $q \geq q_+(p)$ (with $\beta = x_0$), then $M(p;q) = p - x_0$.*

Proof. Some easy computations show that if $p - \beta = f(\beta^*)$ and $\beta \in \mathcal{B}(p)$, we must have $\beta \in \mathcal{B}_+(p)$, $\frac{1}{2}(p-1) < \beta^* < \frac{3}{4}p$ and hence $f(\beta^*) = \beta^*$ and so

$$\beta \in \mathcal{B}_+(p), \quad 1 \leq \beta \leq \frac{p-3}{2}, \quad \beta + \beta^* = p, \quad \beta^* \leq 2\beta, \quad \frac{p-1}{2} < \beta^* < \frac{3}{4}p. \tag{5}$$

Note that $\beta + \beta^* = p$, $p \geq 13$, has a solution with $\beta < p/2$ if and only if $p \equiv 1 \pmod 4$ and $\beta = x_0$ (and hence $\beta^* = p - x_0$) with x_0 the smallest solution of $x_0^2 + 1 \equiv 0 \pmod p$. If $x_0 > p/3$, then $\beta = x_0$ satisfies (5). Since by assumption $q \geq q_+(p)$ and $q \equiv x_0 \pmod p$, we have $M(p;q) \geq p - x_0$ by Theorem 14. On the other hand, by Theorem 13, we have $M(p;q) \leq f(p - x_0) = f(x_0) = p - x_0$. □

Remark. The set of primes p satisfying $p \equiv 1 \pmod 4$ and $x_0 > p/3$ (which starts $\{13, 29, 53, 73, 89, 173, \dots\}$) has natural density $\frac{1}{6}$. This follows on taking $\alpha_2 = \frac{1}{2}$ and $\alpha_1 = \frac{1}{3}$ in the result from [Duke et al. 1995] that if f is a quadratic polynomial with complex roots and $0 \leq \alpha_1 < \alpha_2 \leq 1$ are prescribed real numbers, then as x tends to infinity,

$$\#\{(p, v) : p \leq x,\ f(v) \equiv 0 \pmod p,\ \alpha_1 \leq v/p < \alpha_2\} \sim (\alpha_2 - \alpha_1)\pi(x).$$

5. Computation of $M(3; q)$

Note that for all primes q and r with $1 < q < r$, there exists some unique $h \leq (q-1)/2$ and $k > 0$ such that $r = (kq+1)/h$ or $r = (kq-1)/h$. If $n \equiv 0 \pmod 3$ is ternary, then either $A(n) = 1$ or $A(n) = 2$ as $M(3) = 2$. The following result due to Sister Beiter [Beiter 1978] allows one to compute $A(n)$ in this case.

Theorem 16. *Let $n \equiv 0 \pmod 3$ be ternary.*

- *If $h = 1$, then $A(n) = 1$ if and only if $k \equiv 0 \pmod 3$.*
- *If $h > 1$, then $A(n) = 1$ if and only if one of the following conditions holds*:
  (a) $k \equiv 0 \pmod 3$ *and* $h + q \equiv 0 \pmod 3$.
  (b) $k \equiv 0 \pmod 3$ *and* $h + r \equiv 0 \pmod 3$.

We have seen that $M(3; 5) = 2$. The next result extends this.

Theorem 17. *Let $q > 3$ be a prime. We have $M(3; q) = 2$.*

Proof. In case $q \equiv 1 \pmod 3$, then let r be a prime such that $r \equiv 1 + q \pmod{3q}$. Since $(1 + q, 3q) = 1$, Dirichlet's theorem says there are in fact infinitely many such primes. If $q \equiv 2 \pmod 3$, let r be a prime such that $r \equiv 1 + 2q \pmod{3q}$. Since $(1 + 2q, 3q) = 1$, there are infinitely many such primes. The prime r was chosen so as to ensure that $h = 1$ and $3 \nmid k$. Using Theorem 16 it then follows that $A(3qr) = 2$ and hence $M(3; q) = 2$. □

6. Kaplan's lemma reconsidered

Our main tool will be the following result of Kaplan, the proof of which uses the identity

$$\Phi_{pqr}(x) = (1+x^{pq}+x^{2pq}+\cdots)(1+x+\cdots+x^{p-1}-x^q-\cdots-x^{q+p-1})\Phi_{pq}(x^r).$$

Lemma 18 [Kaplan 2007]. *Let $2 < p < q < r$ be primes and $k \geq 0$ be an integer. Put*

$$b_i = \begin{cases} a_{pq}(i) & \text{if } ri \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$a_{pqr}(k) = \sum_{m=0}^{p-1} (b_{f(m)} - b_{f(m+q)}), \tag{6}$$

where $f(m)$ is the unique integer such that $f(m) \equiv r^{-1}(k - m) \pmod{pq}$ and $0 \leq f(m) < pq$.

(If we need to stress the k-dependence of $f(m)$, we will write $f_k(m)$ instead of $f(m)$, see, e.g., Lemma 22 and its proof.) This lemma reduces the computation of $a_{pqr}(k)$ to that of $a_{pq}(i)$ for various i. These binary cyclotomic polynomial coefficients are computed in the following lemma. For a proof see, e.g., [Lam and Leung 1996; Thangadurai 2000].

Lemma 19. *Let $p < q$ be odd primes. Let ρ and σ be the (unique) nonnegative integers for which $1 + pq = (\rho + 1)p + (\sigma + 1)q$. Let $0 \leq m < pq$. Then either $m = \alpha_1 p + \beta_1 q$ or $m = \alpha_1 p + \beta_1 q - pq$ with $0 \leq \alpha_1 \leq q - 1$ the unique integer such that $\alpha_1 p \equiv m \pmod q$ and $0 \leq \beta_1 \leq p - 1$ the unique integer such that $\beta_1 q \equiv m \pmod p$. The cyclotomic coefficient $a_{pq}(m)$ equals*

$$\begin{cases} 1 & \text{if } m = \alpha_1 p + \beta_1 q \text{ with } 0 \leq \alpha_1 \leq \rho,\ 0 \leq \beta_1 \leq \sigma, \\ -1 & \text{if } m = \alpha_1 p + \beta_1 q - pq \text{ with } \rho + 1 \leq \alpha_1 \leq q - 1,\ \sigma + 1 \leq \beta_1 \leq p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We say that $[m]_p = \alpha_1$ is the *p-part of m* and $[m]_q = \beta_1$ is the *q-part of m*. It is easy to see that

$$m = \begin{cases} [m]_p p + [m]_q q & \text{if } [m]_p \le \rho \text{ and } [m]_q \le \sigma; \\ [m]_p p + [m]_q q - pq & \text{if } [m]_p > \rho \text{ and } [m]_q > \sigma; \\ [m]_p p + [m]_q q - \delta_m pq & \text{otherwise,} \end{cases}$$

with $\delta_m \in \{0, 1\}$. Using this observation we find that, for $i < pq$,

$$b_i = \begin{cases} 1 & \text{if } [i]_p \le \rho, [i]_q \le \sigma \text{ and } [i]_p p + [i]_q q \le k/r; \\ -1 & \text{if } [i]_p > \rho, [i]_q > \sigma \text{ and } [i]_p p + [i]_q q - pq \le k/r; \\ 0 & \text{otherwise.} \end{cases}$$

Thus in order to evaluate $a_{pqr}(n)$ using Kaplan's lemma it suffices to compute $[f(m)]_p$, $[f(m)]_q$, and $[f(m+q)]_q$ (note that $[f(m)]_p = [f(m+q)]_p$).

For future reference we provide a version of Kaplan's lemma in which the computation of b_i has been made explicit, and thus is self-contained.

Lemma 20. *Let $2 < p < q < r$ be primes and let $k \ge 0$ be an integer. We put $\rho = [(p-1)(q-1)]_p$ and $\sigma = [(p-1)(q-1)]_q$. Furthermore, we put*

$$b_i = \begin{cases} 1 & \textit{if } [i]_p \le \rho, [i]_q \le \sigma \textit{ and } [i]_p p + [i]_q q \le k/r; \\ -1 & \textit{if } [i]_p > \rho, [i]_q > \sigma \textit{ and } [i]_p p + [i]_q q - pq \le k/r; \\ 0 & \textit{otherwise.} \end{cases}$$

We have

$$a_{pqr}(k) = \sum_{m=0}^{p-1} (b_{f(m)} - b_{f(m+q)}), \tag{7}$$

where $f(m)$ is the unique integer such that $f(m) \equiv r^{-1}(k-m) \pmod{pq}$ and $0 \le f(m) < pq$.

Note that if i and j have the same p-part, then $b_i b_j \ne -1$, that is b_i and b_j cannot be of opposite sign. From this it follows that $|b_{f(m)} - b_{f(m+q)}| \le 1$, and thus we infer from Kaplan's lemma that $|a_{pqr}(k)| \le p$ and hence $M(p) \le p$.

Using the mutual coprimality of p, q and r we arrive at the following trivial, but useful, lemma.

Lemma 21. *We have $\{[f(m)]_q : 0 \le m \le p-1\} = \{0, 1, 2, \ldots, p-1\}$ and $|\{[f(m)]_p : 0 \le m \le p-1\}| = p$. The same conclusions hold if we replace $[f(m)]_q$ and $[f(m)]_p$ by $[f(m+q)]_q$, respectively $[f(m+q)]_p$.*

Working with Kaplan's lemma one first computes $a_{pq}(f(m))$ and then $b_{f(m)}$. As a check on the correctness of the computations we note that the following identity should be satisfied.

Lemma 22. *We have*

$$\sum_{m=0}^{p-1} a_{pq}(f_k(m)) = \sum_{m=0}^{p-1} a_{pq}(f_k(m+q)).$$

Proof. Choose an integer $k_1 \equiv k \pmod{pq}$ such that $k_1 > pqr$. Then $a_{pqr}(k_1) = 0$. By Lemma 18 we find that

$$0 = a_{pqr}(k_1) = \sum_{m=0}^{p-1} \big(a_{pq}(f_{k_1}(m)) - a_{pq}(f_{k_1}(m+q))\big).$$

Since $f_k(m)$ only depends on the congruence class of k modulo pq, $f_{k_1}(m) = f_k(m)$ and the result follows. □

6.1. ***Working with Kaplan's lemma: examples.*** In this section we carry out some sample computations using Kaplan's lemma. For more involved examples the reader is referred to [Gallot and Moree 2009b].

We remark that the result that $a_n(k) = (p+1)/2$ in Lemma 23 is due to Herbert Möller [1971]. The proof we give here of this is rather different. The foundation for Möller's result is due to Emma Lehmer, who showed [1936] that

$$a_n\big(\tfrac{1}{2}(p-3)(qr+1)\big) = \tfrac{1}{2}(p-1)$$

with p, q, r and n satisfying the conditions of Lemma 23.

Lemma 23. *Let $p < q < r$ be primes satisfying*

$$p > 3, \quad q \equiv 2 \pmod{p}, \quad r \equiv \frac{p-1}{2} \pmod{p}, \quad r \equiv \frac{q-1}{2} \pmod{q}.$$

For $k = (p-1)(qr+1)/2$ we have $a_{pqr}(k) = (p+1)/2$.

Proof (taken from [Gallot and Moree 2009a]). Using that $q \equiv 2 \pmod{p}$, we infer from $1 + pq = (\rho+1)p + (\sigma+1)q$ that $\sigma = \frac{1}{2}(p-1)$ and $(\rho+1)p = 1 + \frac{1}{2}(p-1)q$ (and hence $\rho = (p-1)(q-2)/(2p)$). Invoking the Chinese remainder theorem one checks that

$$-r^{-1} \equiv 2 \equiv -\left(\frac{q-2}{p}\right) p + q \pmod{pq}. \tag{8}$$

Furthermore, writing $f(0)$ as a linear combination of p and q we see that

$$f(0) \equiv \frac{k}{r} \equiv \left(\frac{p-1}{2}\right) q + \frac{p-1}{2r} \equiv \left(\frac{p-1}{2}\right) q + 1 - p \equiv \rho p \pmod{pq}. \tag{9}$$

Since $f(m) \equiv f(0) - \frac{m}{r} \pmod{pq}$ we find using (8), (9) and the observation that $\rho - m(q-2)/p \geq 0$ for $0 \leq m \leq (p-1)/2$, that $[f(m)]_p = \rho - m(q-2)/p \leq \rho$

and $[f(m)]_q = m \le \sigma$ for $0 \le m \le (p-1)/2$. Since $[f(m)]_p p + [f(m)]_q q = \rho p + 2m \le \rho p + p - 1 = [k/r]$, we deduce that $a_{pq}(f(m)) = b_{f(m)} = 1$ in this range; see also the following table:

m	$[f(m)]_p$	$[f(m)]_q$	$f(m)$	$a_{pq}(f(m))$	$b_{f(m)}$
0	ρ	0	ρp	1	1
1	$\rho - (q-2)/p$	1	$\rho p + 2$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	1	1
j	$\rho - j(q-2)/p$	j	$\rho p + 2j$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	1	1
$(p-1)/2$	0	$(p-1)/2$	$(p-1)q/2$	1	1

Note that $f(m) \equiv f(0) - m/r \equiv \rho p + 2m \pmod{pq}$, from which one easily infers that $f(m) = \rho p + 2m$ for $0 \le m \le p-1$ (as $\rho p + 2m \le \rho p + 2(p-1) < pq$). In the range $\frac{1}{2}(p+1) \le m \le p-1$ we have $f(m) \ge \rho p + p + 1 = (p-1)q/2 + 2 > k/r$, and hence $b_{f(m)} = 0$.

On noting that $f(m+q) \equiv f(m) - q/r \equiv f(m) + 2q \equiv \rho p + 2m + 2q \pmod{pq}$, one easily finds, for $0 \le m \le p-1$, that $f(m+q) = \rho p + 2m + 2q > k/r$ and hence $b_{f(m+q)} = 0$.

Invoking Kaplan's lemma one finds

$$a_{pqr}(k) = \sum_{m=0}^{p-1} b_{f(m)} - \sum_{m=0}^{p-1} b_{f(m+q)} = \frac{p+1}{2} - 0 = \frac{p+1}{2}. \qquad \square$$

Lemma 24. *Let $3 < p < q < r$ be primes satisfying*

$$q \equiv 1 \pmod{p}, \quad r^{-1} \equiv \frac{p+q}{2} \pmod{pq}.$$

For $k = (p-1)qr/2 - pr + 2$ we have $a_{pqr}(k) = -\min\left(\frac{q-1}{p} + 1, \frac{p+1}{2}\right)$.

Proof. Let $0 \le m \le p-1$. We have

$$\rho = \frac{(p-1)(q-1)}{p} \text{ and } \sigma = 0,$$

$$k \equiv 1 \pmod{p}, \quad k \equiv 0 \pmod{q}, \quad k \equiv 2 \pmod{r},$$

so that we can compute

$$[f(m)]_q \equiv q^{-1} r^{-1} (k-m) \equiv (1-m)/2 \pmod{p},$$
$$[f(m+q)]_q \equiv q^{-1} r^{-1} (k-m-q) \equiv -m/2 \pmod{p},$$
$$[f(m)]_p = [f(m+q)]_p \equiv p^{-1} r^{-1} (k-m) \equiv -m/2 \pmod{q}.$$

This leads to

$$[f(m)]_q = \begin{cases} (p+1-m)/2 & \text{for } m \text{ even,} \\ (2p+1-m)/2 & \text{for } m \text{ odd and } m \neq 1, \\ 0 & \text{for } m = 1, \end{cases}$$

$$[f(m+q)]_q = \begin{cases} (p-m)/2 & \text{for } m \text{ odd,} \\ (2p-m)/2 & \text{for } m \text{ even and } m \neq 0, \\ 0 & \text{for } m = 0, \end{cases}$$

$$[f(m)]_p = [f(m+q)]_p = \begin{cases} (q-m)/2 & \text{for } m \text{ odd,} \\ (2q-m)/2 & \text{for } m \text{ even and } m \neq 0, \\ 0 & \text{for } m = 0. \end{cases}$$

We consider four cases:

Case 1: $[f(m)]_p \leq \rho$ and $[f(m)]_q \leq \sigma$. In this case $m = 1$. Therefore

$$[f(m)]_p p + [f(m)]_q q = \frac{p(q-1)}{2} > \frac{k}{r}.$$

Case 2: $[f(m)]_p > \rho$ and $[f(m)]_q > \sigma$. This case only arises if m is even and $m \geq 2$. Then we have

$$\begin{aligned} [f(m)]_p p + [f(m)]_q q - pq &= \frac{2q-m}{2} p + \frac{p+1-m}{2} q - pq \\ &= \frac{q(p+1-m) - mp}{2} \leq \frac{q(p-1)}{2} - p + \frac{2}{r} = \frac{k}{r}. \end{aligned}$$

However, not all even $m \geq 2$ satisfy $[f(m)]_p > \rho$. For this it is necessary that

$$\frac{2q-m}{2} > \frac{(p-1)(q-1)}{p}.$$

That means

$$\frac{m}{2} < \frac{q-1}{p} + 1$$

and since $0 < \frac{m}{2} \leq \frac{p-1}{2}$ we have exactly $\min\left(\frac{q-1}{p}, \frac{p-1}{2}\right)$ different values of m.

Case 3: $[f(m+q)]_p \leq \rho$ and $[f(m+q)]_q \leq \sigma$. In this case we have $m = 0$. Therefore

$$[f(m+q)]_p p + [f(m+q)]_q q = 0 \leq \frac{k}{r}.$$

Case 4: $[f(m+q)]_p > \rho$ and $[f(m+q)]_q > \sigma$. We must have $2|m$ and $m \geq 2$. We find

$$[f(m+q)]_p p + [f(m+q)]_q q - pq = \frac{2q-m}{2} p + \frac{2p-m}{2} q - pq > \frac{k}{r}.$$

This case analysis shows that (respectively)

$$\sum_{\substack{m=0\\ b_{f(m)}=1}}^{p-1} 1 = 0, \quad \sum_{\substack{m=0\\ b_{f(m)}=-1}}^{p-1} 1 = \min\Big(\frac{q-1}{p}, \frac{p-1}{2}\Big), \quad \sum_{\substack{m=0\\ b_{f(m+q)}=1}}^{p-1} 1 = 1, \quad \sum_{\substack{m=0\\ b_{f(m+q)}=-1}}^{p-1} 1 = 0.$$

Kaplan's lemma then yields

$$a_{pqr}(k) = \Big(0 - \min\Big(\frac{q-1}{p}, \frac{p-1}{2}\Big)\Big) - (1-0) = -\min\Big(\frac{q-1}{p}+1, \frac{p+1}{2}\Big). \quad \square$$

The next two lemmas are proved by application of Kaplan's lemma; see [Gallot et al. 2010] for details.

Lemma 25. *Let $3 < p < q < r$ be primes satisfying*

$$q \equiv -2 (\mathrm{mod}\ p), \quad r^{-1} \equiv p-2 \ (\mathrm{mod}\ pq) \text{ and } q > p^2/2.$$

For $k = \frac{p+1}{2}(1 + r(2-p+q)) + r + q - rq$ we have $a_{pqr}(k) = -(p+1)/2$.

Remark. Numerical experimentation suggests that with this choice of k, a condition of the form $q > p^2 c_1$, with c_1 some absolute positive constant, is unavoidable.

Lemma 26. *Let $3 < p < q < r$ be primes satisfying*

$$q \equiv -1 \ (\mathrm{mod}\ p), \quad r^{-1} \equiv \frac{p+q}{2} \ (\mathrm{mod}\ pq) \text{ and } q \geq p^2 - 2p.$$

For $k = p(q-1)r/2 - rq + p - 1$ we have $a_{pqr}(k) = -(p+1)/2$.

Proof of Proposition 9. The first assertion follows by Theorem 6, so assume $p \geq 11$. We will argue by contradiction. So suppose that $p \nmid \mathfrak{f}_p$. Put $\beta = (p-3)/2$. By the Chinese remainder theorem and Dirichlet's theorem there are infinitely many primes q_1 such that $q_1 \equiv 2 \ (\mathrm{mod}\ p)$ and $q_1 \equiv 1 \ (\mathrm{mod}\ \mathfrak{f}_p)$. Further, there are infinitely many primes q_2 such that $q_2 \equiv \beta \ (\mathrm{mod}\ p)$ and $q_2 \equiv 1 \ (\mathrm{mod}\ \mathfrak{f}_p)$. By the definition of $\mathfrak{f}_p$ there exists an integer c such that $M(p;q) = c$ for all $q \equiv 1 \ (\mathrm{mod}\ \mathfrak{f}_p)$ that are large enough. However, by Lemma 23 we have $M(p;q_1) = (p+1)/2$ and by Theorem 14 (note that $\beta \in \mathcal{B}(p)$) we have $M(p;q_2) > (p+1)/2$ for all q_2 large enough. This contradiction shows that $p \nmid \mathfrak{f}_p$. $\square$

The results from this section together with those from Section 3 allow one to establish the following theorem. In Section 10 we will discuss the sharpness of the lower bounds for q.

Theorem 27. *Let $2 < p < q$ be primes.*

(a) *If $q \equiv 2 \ (\mathrm{mod}\ p)$, then $M(p;q) = (p+1)/2$.*

(b) *If $q \equiv -2 \ (\mathrm{mod}\ p)$ and $q > p^2/2$, then $M(p;q) = (p+1)/2$.*

(c) *If $q \equiv 1 \ (\mathrm{mod}\ p)$ and $q \geq (p-1)p/2 + 1$, then $M(p;q) = (p+1)/2$.*

(d) *If $q \equiv -1 \ (\mathrm{mod}\ p)$ and $q \geq p^2 - 2p$, then $M(p;q) = (p+1)/2$.*

Proof. By Theorem 17 we have $M(3; q) = 2 = (3+1)/2$, so assume $p > 3$.

(a) We have $M(p; q) \geq (p+1)/2$ by Lemma 23, and $M(p; q) \leq f(2^*) = f((p+1)/2) = (p+1)/2$ by Theorem 13.

(b)+(c)+(d) Similar to that of part (a). Note that $f((-2)^*) = f((p-1)/2) = (p+1)/2$ and $f(1) = f(p-1) = (p+1)/2$. □

Theorem 28. *Let $q > 5$ be a prime. Then $M(5; q) = 3$.*

Proof. The proof is most compactly given in a table:

$\bar{q}$	q_0	$M(5; q)$	result
1	11	3	Theorem 27(c)
2	7	3	Theorem 27(a)
3	13	3	Theorem 27(b)
4	19	3	Theorem 27(d)

Interpretation: the third row, for example, says that for $q \equiv 3 \pmod 5$, $q \geq 13$, we have $M(5; q) = 3$ by Theorem 27(b). □

7. Computation of $M(7; q)$

Theorem 27, together with the next two lemmas (again proved by application of Kaplan's lemma), allows one to compute $M(7; q)$. These lemmas concern the computation of $M(p; q)$ with $q \equiv (p \pm 1)/2 \pmod p$.

Lemma 29. *Let $p \geq 5$ be a prime. Let $q \geq \max(3p, p(p+1)/4)$ be a prime satisfying $q \equiv (p-1)/2 \pmod p$. Let $r > q$ be a prime satisfying*

$$r^{-1} \equiv \frac{p+1}{2} \pmod p, \quad r^{-1} \equiv p \pmod q.$$

For $k = p - 1 + r(1 + q(p-1)/2 - p(p+1)/2)$ we have $a_{pqr}(k) = (p+1)/2$.

Lemma 30. *Let $p \geq 5$ be a prime. Let $q \geq \max(3p, p(p-1)/4+1)$ be a prime satisfying $q \equiv (p+1)/2 \pmod p$. Let $r > q$ be a prime satisfying*

$$r^{-1} \equiv \frac{p-1}{2} \pmod p, \quad r^{-1} \equiv p \pmod q.$$

For $k = q + p - 1 + r(q(p-1)/2 - p(p+1)/2)$ we have $a_{pqr}(k) = (p+1)/2$.

Theorem 31.

(a) *If $q \geq \max(3p, p(p+1)/4)$ is a prime satisfying $q \equiv (p-1)/2 \pmod p$, then $(p+1)/2 \leq M(p; q) \leq (p+3)/2$.*

(b) *If $q \geq \max(3p, p(p-1)/4+1)$ is a prime satisfying $q \equiv (p+1)/2 \pmod p$, then $(p+1)/2 \leq M(p; q) \leq (p+3)/2$.*

Proof. This follows on noting that

$$f\Big(\Big(\frac{p+1}{2}\Big)^*\Big) = f(2) = \frac{p+3}{2} = f(p-2) = f\Big(\Big(\frac{p-1}{2}\Big)^*\Big),$$

and combining Lemmas 29 and 30 with Theorem 13. □

Theorem 32. *We have* $M(7;11) = 4$, $M(7;13) = 3$ *and for* $q \geq 17$ *a prime*, $M(7;q) = 4$.

Proof. Again we encode the proof in a table:

$\bar{q}$	q_0	$M(7;q)$	result
1	29	4	Theorem 27(c)
2	23	4	Theorem 27(a)
3	31	4	Theorem 31(a)*
4	53	4	Theorem 31(b)*
5	47	4	Theorem 27(b)
6	41	4	Theorem 27(d)

For the entries marked with asterisks we also need the fact that $M(7) \leq 4$ (see just before Theorem 2). Since $M(7;11) = M(7;17) = M(7;19) = 4$ and $M(7;13) = 3$ (the only cases not covered in the table), the proof is completed. □

Proof of Theorem 6. Combine Theorems 17, 28 and 32. □

8. Computation of $M(11;q)$

We have $M(11;q) \leq M(11) = 7$ (by Theorem 2 and Table 1). Moreover:

Theorem 33 [Gallot and Moree 2009b]. *Let* $q < r$ *be primes with* $q \equiv 4 \pmod{11}$ *and* $r \equiv -3 \pmod{11}$. *Let* $1 \leq \alpha \leq q-1$ *be the unique integer such that* $11r\alpha \equiv 1 \pmod{q}$. *Suppose that* $q/33 < \alpha \leq (3q-1)/77$. *Then* $a_{11qr}(10+(6q-77\alpha)r) = -7$.

Lemma 34. *Let* q *be a prime such that* $q \equiv 4 \pmod{11}$. *For* $q > 37$, $M(11;q) = 7$, *and* $M(11;37) = 6$.

Proof. By computation one finds that $M(11;37) = 6$. Now assume $q > 37$. Notice that it is enough to show that $M(11;q) \geq 7$. For $q \geq 191$ the interval $I(q) := (q/33, (3q-1)/77]$ has length exceeding 1 and so contains at least one integer α_1. Then by the Chinese remainder theorem and Dirichlet's theorem we can find a prime r_1 such that both $r_1 \equiv -3 \pmod{11}$ and $11r_1\alpha_1 \equiv 1 \pmod{q}$. Then we invoke Theorem 33 with $r = r_1$ and $\alpha = \alpha_1$. It remains to deal with the primes 59 and 103. One checks that both intervals $I(59)$ and $I(103)$ contain an integer and so we can proceed as in the case $q \geq 191$ to conclude the proof. □

Lemma 35. *Let* $p = 11$.

(a) *For* ≥ 133, $q \equiv 3 \pmod{11}$, $r^{-1} \equiv \dfrac{q-19}{2} \pmod{pq}$ *and* $k = q + 7r\dfrac{(q-19)}{2}$ *we have* $a_{pqr}(k) = 7$.

(b) *For* $q \equiv 7 \pmod{11}$, $r^{-1} \equiv \dfrac{q+7}{2} \pmod{pq}$ *and* $k = 6qr+4$ *we have* $a_{pqr}(k) = 7$.

(c) *For* $q \equiv 8 \pmod{11}$, $r^{-1} \equiv \dfrac{q-3}{2} \pmod{pq}$ *and* $k = 6qr+4$ *we have* $a_{pqr}(k) = 7$.

The proof is an application of Kaplan's lemma.

Theorem 36. *For* $q \geq 13$ *we have*

$q \pmod{11}$	1	2	3	4	5	6	7	8	9	10
$M(11; q)$	6	6	7	7	6,7	6,7	7	7	6	6

except when $q \in \{17, 23, 37, 43, 47\}$. *We have* $M(11; 17) = 5$, $M(11; 23) = 3$, $M(11; 37) = 6$, $M(11; 43) = 5$ *and* $M(11; 47) = 6$.

Remarks. (1) If $q \equiv \pm 5 \pmod{11}$ and $q \geq 61$, then $M(p, q) \in \{6, 7\}$. We believe that $M(p; q) = 6$.

(2) By Corollary 41 and 42 following Theorem 40, one infers that $M(11; 17) \leq 5$, $M(11; 23) \leq 3$ and $M(11; 43) \leq 5$.

Proof of 36.

$\bar{q}$	q_0	$M(11; q)$	result
1	67	6	Theorem 27(c)
2	13	6	Theorem 27(a)
3	157	7	Lemma 35(a)*
4	59	7	Lemma 34
5	71	6,7	Theorem 31(a)*
6	61	6,7	Theorem 31(b)*
7	29	7	Lemma 35(b)*
8	19	7	Lemma 35(c)*
9	97	6	Theorem 27(b)
10	109	6	Theorem 27(d)

Here the asterisks indicate that we need the fact that $M(11) = 7$. The proof is completed by directly computing the values of $M(p; q)$ not covered by the table. □

9. Computation for $p = 19$

By Theorem 2 we have $M(19) \leq 2 \cdot 19/3$ and hence $M(19) \leq 12$. By Theorem 14 we find that $M(19; q) \geq 11$ for every $q \equiv 8 \pmod{19}$ and $q \geq 179$ and hence

$M(19) \geq 11$. Since $A(19 \cdot 53 \cdot 859) = 12$, it follows that $M(19) = 12$. The next result even shows that $M(19; q) = M(19)$ for a positive fraction of the primes.

Theorem 37. *We have $M(19) = 12$. Moreover, $M(19, q) = 12$ if $q \equiv \pm 4 \pmod{19}$, with $q > 23$. Furthermore, $M(19; 23) = 11$.*

The proof is an almost direct consequence of the following lemma, itself proved by applying Kaplan's lemma.

Lemma 38. *Put $p = 19$ and let $q \equiv \pm 4 \pmod{19}$ be a prime. Suppose there exists an integer a satysifying*

$$qa \equiv -1 \pmod 3 \text{ and } \frac{q}{6p} < a \leq \frac{5q - 18}{6p}. \tag{10}$$

Let $r > q$ be a prime satisfying $r(q - ap) \equiv 3 \pmod{pq}$. Then $a_{pqr}(7qr + q) = -12$, if $q \equiv -4 \pmod{19}$, and $a_{19qr}(7qr + r) = -12$ if $q \equiv 4 \pmod{19}$.

Proof of Theorem 37. For $q > 90$ the interval in (10) is of length > 3 and so contains an integer a satisfying $qa \equiv -1 \pmod 3$. It remains to deal with $q \in \{23, 53, 61\}$. Computation shows that $M(19; 23) = 11$. For $q = 53$ and $q = 61$ one finds an integer a satisfying condition (10). □

Proof of Theorem 5. By Theorem 14 and Dirichlet's theorem the claim follows for $p = 13$. Using Lemmas 34 and 35 the result follows for $p = 11$. On invoking Theorems 6 and 37, the proof is then completed. □

10. Small values of $M(p; q)$

Typically if $M(p; q)$ is constant for all q large enough with $q \equiv a \pmod d$, then $M(p; q)$ assumes a smaller value for some small q in this progression. A (partial) explanation of this phenomenon is provided in this section. We will show that if $ap + bq = 1$ with a and b small in absolute value, then $M(p; q)$ is small. On the other hand we will show that $M(p; q)$ cannot be truly small.

Proposition 39. *Let $2 < p < q$ be odd primes. Then $M(p; q) \geq 2$.*

Proof. We say $\Phi_n(x)$ is flat if $A(n) = 1$. ChunGang Ji [2010] proved that if $p < q < r$ are odd prime and $2r \equiv \pm 1 \pmod{pq}$, then $\Phi_{pqr}(x)$ is flat if and only if $p = 3$ and $q \equiv 1 \pmod 3$. It follows that $M(p; q) \geq 2$ for $p > 3$. Now invoke Theorem 17 to deal with the case $p = 3$. □

Theorem 40. *Let $2 < p < q$ be odd primes and ρ and σ be the (unique) nonnegative integers for which $1 + pq = (\rho + 1)p + (\sigma + 1)q$. Then*

$$M(p; q) \leq \begin{cases} p + \rho - \sigma & \text{if } \rho \leq \sigma, \\ q + \sigma - \rho & \text{if } \rho > \sigma. \end{cases}$$

Corollary 41. *Let h, k be integers with $k > h$ and $q = (kp-1)/h$ a prime. If $p \geq k+h$, then $M(p;q) \leq k+h$.*

Corollary 42. *Let h, k be integers with $k > h$ and $q = (kp+1)/h$ a prime. If $p > h$ and $q > k+h$, then $M(p;q) \leq k+h$.*

Proof of Theorem 40. Let us assume that $\rho \leq \sigma$, the other case being similar. Using Lemma 21 and Lemma 19 we infer that the number of $0 \leq m \leq p-1$ with $b_{f(m)} = 1$ is at most $\rho+1$. Likewise the number of m with $b_{f(m+q)} = -1$ is at most $p-1-\sigma$. By Kaplan's lemma it then follows that $a_{pqr}(k) \leq \rho+1+(p-1-\sigma) = p+\rho-\sigma$. Since the number of $0 \leq m \leq p-1$ with $b_{f(m)} = -1$ is at most $p-1-\sigma$ and the number of m with $b_{f(m+q)} = 1$ is at most $\rho+1$, we infer that $a_{pqr}(k) \geq -(p+\rho-\sigma)$ and hence the result is proved. □

Theorem 43. *Let $q \equiv 1 \pmod{p}$. Then*

$$M(p;q) = \min\Big(\frac{q-1}{p}+1, \frac{p+1}{2}\Big).$$

Proof. For $p = 3$ the result follows by Theorem 17, so assume $p \geq 5$. Sister Beiter [Beiter 1968], and independently Bloom [Bloom 1968], proved that $M(p;q) \leq (p+1)/2$ if $q \equiv \pm 1 \pmod{p}$ (alternatively we invoke Theorem 13). By Corollary 42 we have $M(p;q) \leq (q-1)/p+1$. By Lemma 24 the proof is then completed. □

Numerical experiments suggest that in Theorem 27(b) the condition $q > p^2/2$ can perhaps be dropped. By Theorem 43 the condition $q \geq (p-1)p/2+1$ in part (c) is optimal. In (d) we need $q \geq (p-1)p/2-1$; otherwise $M(p;q) < (p+1)/2$ by Corollary 41.

Lemma 44. *Let $p \geq 7$ be a prime such that $q = 2p-1$ is also a prime. Let $r > q$ be a prime such that $(p+q)r \equiv -2 \pmod{pq}$. Put $k = rq(p-1)/2+2p-pq$. Then $a_{pqr}(k) = 3$.*

The proof is an application of Kaplan's lemma.

Proof of Theorem 7. On combining Lemma 44 with Corollary 41, one deduces that $M(p;2p-1) = 3$ if $p \geq 5$ and $2p-1$ is a prime. □

11. Conjectures, questions, problems

The open problem that we think is the most interesting is Conjecture 8. If one could prove it and obtain an effective upper bound for the ternary conductor $\mathfrak{f}_p$ (say $16p$) and an effective upper bound for the minimal ternary prime (say p^3), one would have a finite procedure to compute $M(p)$.

Problem 45. Bachman [2010] introduced inclusion-exclusion polynomials. These polynomials generalize the ternary cyclotomic polynomials. Study $M(p;q)$ in this setting (here p and q can be any coprime natural numbers), cf. Section 2 where we denoted this function by $M'(p;q)$. For example, using [Bachman 2010, Theorem 3] by an argument similar to that given in Proposition 3 it is easily seen that there is a finite procedure to compute $M'(p;q)$.

Problem 46. The analogue of $M(p;q)$ for inverse cyclotomic polynomials can be defined [Moree 2009]. Study it.

Question 47. Can one compute the average value of $M(p;q)$, that is does the limit

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\sum_{p<q\le x}M(p;q)$$

exist and if yes, what is its value?

Question 48. Is Theorem 5 still true if we put $\delta(13)=1/3$ and cross out the words "a subset having"?

Question 49. If $q>p$ is prime and $q\equiv -2 \pmod p$, then do we have $M(p;q)=(p+1)/2$?

Question 50. Suppose that $p>11$ is a prime.
If $6p-1$ is prime, then do we have $M(p,6p-1)=7$?
If $(5p-1)/2$ is prime, then do we have $M(p,(5p-1)/2)=7$?
If $(5p+1)/2$ is prime then do we have $M(p,(5p+1)/2)=7$?
Find more similar results.

Question 51. Given an integer $k\ge 1$, does there exist $p_0(k)$ and a function $q_k(p)$ such that if $q\equiv 2/(2k+1) \pmod p$, $q\ge q_k(p)$ and $p\ge p_0(k)$, then $M(p;q)=(p+2k+1)/2$?

Question 52. Is it true that $M(11;q)=6$ for all large enough q satisfying $q\equiv \pm 5 \pmod 6$? If so one can finish the computation of $M(11;q)$.

Question 53. Is it true that for q sufficiently large the values of $M(13;q)$, $M(17;q)$, $M(19;q)$ and $M(23;q)$ are given by Table 2 on the next page?

The next question was raised by the referee of this paper.

Question 54. Suppose that for all sufficiently large primes $q\equiv q_0 \pmod{\mathfrak{f}_p}$ we have $M(p;q)<M(p)$. Is it possible to prove that $M(p;q)<M(p)$ for every prime $q\equiv q_0 \pmod{\mathfrak{f}_p}$?

Question 55. For a given prime p, let $m(p)$ denote $\liminf M(p;q)$, with $q>p$. Determine $m(p)$. Is it true that $\lim_{p\to\infty} m(p)/p=c$ for some constant $c>0$?

$q \pmod{13}$	1	2	3	4	5	6	7	8	9	10	11	12
$M(13;q)$	7	7	7	8	8	7	7	8	8	7	7	7

$q \pmod{17}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$M(17;q)$	9	9	9	10	10	9	10	9	9	10	9	10	10	9	9	9

$q \pmod{19}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$M(19;q)$	10	10	10	12	11	9	11	11	10	10	11	11	9	11	12	10
$q \pmod{19}$	(continued)						17	18								
$M(19;q)$							10	10								

$q \pmod{23}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$M(23;q)$	12	12	12	14	14	11	13	11	14	13	12	12	13	14	11	13
$q \pmod{23}$	(continued)						17	18	19	20	21	22				
$M(23;q)$							11	14	14	12	12	12				

Table 2. Conjectural values of $M(13;q)$, $M(17;q)$, $M(19;q)$ and $M(23;q)$ (for q large). See Question 53.

By Proposition 39 we have $m(p) \geq 2$ for $p > 2$. Note that the results in this paper imply that $m(p) = (p+1)/2$ for $2 < p \leq 11$. If the answer to Question 53 is yes, then $m(p) = (p+1)/2$ for $2 < p \leq 17$ and $m(p) = (p-1)/2$ for $19 \leq p \leq 23$. (The issue of lower bounds for $M(p;q)$ was raised by the referee.)

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