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Diameter, girth and cut vertices of the graph of equivalence classes of zero-divisors

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We explore the properties of $\Gamma_E(R)$, the graph of equivalence classes of zero-divisors of a commutative Noetherian ring R. We determine the possible combinations of diameter and girth for the zero-divisor graph $\Gamma(R)$ and the equivalence class graph $\Gamma_E(R)$, and examine properties of cut-vertices of $\Gamma_E(R)$.

Introduction

The zero-divisor graph of a commutative ring R, was first introduced in [Beck 1988] and has since been investigated in various forms. It was shown in [Anderson and Livingston 1999] that the zero-divisor graph of any ring is connected with diameter less than or equal to 3. Mulay [2002] proved many interesting results about cycles in the zero-divisor graph.

In 2009, Spiroff and Wickham [2011] introduced $\Gamma_E(R)$, the graph of equivalence classes of zero-divisors, which is a simplification of the zero-divisor graph $\Gamma(R)$. The vertices of $\Gamma_E(R)$ are, instead of individual zero-divisors of R, equivalence classes of zero-divisors determined by annihilator ideals. The graph $\Gamma_E(R)$ provides a more succinct view of the zero-divisor activity of the ring. In many cases, the equivalence class graph is finite even though the zero-divisor graph is infinite. For example, for $S=\mathbb{Z}[X,Y]/(X^4,XY)$, the graph $\Gamma(S)$ is infinite, while the graph $\Gamma_E(S)$ has only 6 vertices. Specifically, the vertices corresponding to $X^3, 2X^3, 3X^3, \ldots$ are all distinct in $\Gamma(S)$. However, since they all have the same annihilator, they all belong to the same equivalence class, and so are represented by a single vertex $[X^3]$ in $\Gamma_E(S)$.

The equivalence class graph also lets us view the interplay between the annihilator ideals of R and helps to easily identify the associated primes of the ring. The vertices of $\Gamma_E(R)$ which correspond to associated primes have special properties which will help us to prove several interesting results related to $\Gamma_E(R)$. In

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Section 1, we provide basic definitions and background. In Section 2, we determine all possible diameter combinations of $\Gamma(R)$ and $\Gamma_E(R)$, and do the same for the girth of the two graphs in Section 3. In Section 4, we look at properties of the cut-vertices of $\Gamma_E(R)$. Throughout, R will denote a commutative Noetherian ring.

1. Background and basic results

Graph theory. We briefly review basic graph theory terms that we will use throughout the paper. All graphs we deal with will be *simple* graphs in the sense that they contain no loops or double edges. We will denote the set of vertices of a graph Γ by $V(\Gamma)$. If two vertices x and y are joined by an edge, we say x and y are *adjacent*, and write $x - y$. A *path* is defined as an alternating sequence of distinct vertices and edges, and the *length of a path* is the number of edges in the path. If x and y are two vertices, then the *distance between x and y*, denoted $d(x, y)$, is the length of the shortest path from x to y. If there is no path connecting x to y, we say that $d(x, y) = \infty$, and we define $d(x, x) = 0$. The *diameter of a graph* is the maximum distance between any two vertices of the graph. We will denote the diameter of a graph Γ by $\operatorname{diam}\Gamma$. A *cycle* is a closed path, or a path that starts and ends on the same vertex. The *girth* of a graph is the length of its smallest cycle. We denote the girth of a graph Γ by $\mathrm{g}(\Gamma)$ and say that $\mathrm{g}(\Gamma) = \infty$ if the graph Γ contains no cycle. Note that the smallest possible cycle length is 3, so if Γ contains a cycle, $\mathrm{g}(\Gamma) \geq 3$.

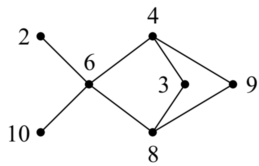
A graph is said to be *connected* if every pair of vertices is joined by a path and *complete* if every pair of vertices is joined by an edge. A *connected component* of a graph Γ is a maximal connected subgraph of Γ. If removing a vertex v from a graph along with all its incident edges increases the number of connected components in the graph, then v is called a *cut vertex*. A graph is *complete bipartite* if its vertices can be partitioned into two subsets, V_1 and V_2, such that every vertex of V_1 is adjacent to every vertex of V_2, but no two vertices of V_1 are adjacent and no two vertices of V_2 are adjacent. Such a graph will be denoted $K_{n,m}$, where $n = |V_1|$ and $m = |V_2|$. If the vertices of a graph can be partitioned into r subsets in a similar fashion, then the graph is said to be *r-partite*.

Zero-divisor graphs. Let $Z(R)$ denote the set of zero-divisors of R and $Z^*(R)$ denote the set $Z(R) \setminus \{0\}$. We define the *zero-divisor graph* of R as the simple graph $\Gamma(R)$ where the vertices of $\Gamma(R)$ are the elements of $Z^*(R)$, and there is an edge between $x, y \in \Gamma(R)$ whenever $xy = 0$.

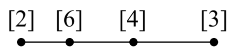
Recall that the annihilator ideal associated to an element $x \in R$ is the set $\operatorname{ann} x = \{r \in R : xr = 0\}$. We define an equivalence relation $\sim$ on R such that for all $x, y \in R$, we say $x \sim y$ if $\operatorname{ann} x = \operatorname{ann} y$. Let $[x]$ denote the equivalence class of x. Notice that $[0] = \{0\}$, $[1] = R \setminus Z(R)$ and the relation $\sim$ partitions the remaining zero-divisors into distinct classes. Furthermore, it follows that the multiplication of these equivalence classes $[x] \cdot [y] = [xy]$ is well-defined.

The *graph of equivalence classes of zero-divisors of* R, $\Gamma_E(R)$, is the graph whose vertices are the classes of nonzero zero-divisors of R determined by the relation $\sim$, where there is an edge between two vertices $[x]$ and $[y]$ if $[x]\cdot[y] = [0]$.

Here, as an example, are the zero-divisor graph of $\mathbb{Z}_{12}$ and the graph of its equivalence classes:



$\Gamma(\mathbb{Z}_{12})$



$\Gamma_E(\mathbb{Z}_{12})$

We see that since $\operatorname{ann} 2 = \operatorname{ann} 10$, the elements 2 and 10 are in the same equivalence class, and therefore collapse to the single vertex [2] in $\Gamma_E(R)$.

Previous results. Spiroff and Wickham [2011] have several interesting results linking the associated primes of R with the structure of $\Gamma_E(R)$. These will be useful in furthering our investigation of $\Gamma_E(R)$. Remember that a prime ideal $\mathfrak{p}$ of R is an *associated prime* if it is the annihilator of some element of R. The set of associated primes is denoted $\operatorname{ass} R$. It is well known that if R is a Noetherian ring, then $\operatorname{ass} R$ is nonempty and finite and that any maximal element of the family of annihilator ideals $\mathfrak{F} = \{\operatorname{ann} x : 0 \neq x \in R\}$ is an associated prime. Note also that since every zero divisor is contained in an annihilator ideal and maximal annihilators are associated primes, the set of zero-divisors of R equals the union of all associated primes of R. Since there is exactly one vertex of $\Gamma_E(R)$ for each distinct annihilator ideal of R, we have a natural injection of $\operatorname{ass} R$ into the vertex set of $\Gamma_E(R)$ given by $\mathfrak{p} \mapsto [y]$ where $\mathfrak{p} = \operatorname{ann} y$. We adopt the conventions of Spiroff and Wickham and by a slight abuse of terminology will refer to the vertex $[y]$ as an associated prime if $\operatorname{ann} y \in \operatorname{ass} R$. It will be clear from context whether $[y]$ refers to an equivalence class, a vertex, or a specific annihilator.

Lemma 1.1 [Spiroff and Wickham 2011, Lemma 1.2]. *Any two distinct elements of* $\operatorname{ass} R$ *are connected by an edge. Furthermore, every vertex* $[v]$ *of* $\Gamma_E(R)$ *is either an associated prime or adjacent to an associated prime maximal in* $\mathfrak{F}$.

Lemma 1.2 [Spiroff and Wickham 2011, Proposition 1.7]. *Let R be a ring such that $\Gamma_E(R)$ is complete r-partite. Then $r = 2$ and $\Gamma_E(R) = K_{n,1}$ for some $n \geq 1$.*

2. Diameter

In this section, we explore the relationship between the diameters of the graphs $\Gamma(R)$ and $\Gamma_E(R)$. It is shown in [Anderson and Livingston 1999] that $\Gamma(R)$ has diameter at most 3 for any commutative ring R. In [Spiroff and Wickham 2011] it is shown that $\operatorname{diam}\Gamma_E(R) \le 3$ for R commutative and Noetherian. The following results further demonstrate the relationship between the diameters of the two graphs.

Proposition 2.1. *If R is a commutative ring, then* $\operatorname{diam}\Gamma_E(R) \le \operatorname{diam}\Gamma(R)$.

Proof. Let $[a], [b] \in \Gamma_E(R)$ with $d([a], [b]) = n$, and let $[a] = [x_1] - [x_2] - \cdots - [x_{n+1}] = [b]$ be a path of minimal length from $[a]$ to $[b]$. From each $[x_i]$, choose one $y_i \in [x_i]$. Then $y_1 - y_2 - \cdots - y_{n+1}$ is a path in $\Gamma(R)$ of length n. We claim that this path is minimal, and thus $d(y_1, y_{n+1}) = n$. If this path is not minimal, there is some shorter path $y_1 = z_1 - z_2 - \cdots - z_{m+1} = y_{n+1}$, with $m < n$. Since either $[z_i] = [z_{i+1}]$ or $[z_i] - [z_{i+1}]$, the path $[y_1] = [z_1] - [z_2] - \cdots - [z_{m+1}] = [y_{n+1}]$ has length less than or equal to m, a contradiction. □

Theorem 2.2. *If* $\operatorname{diam}\Gamma_E(R) = 0$, *then* $\operatorname{diam}\Gamma(R) = 0$ *or* 1.

Proof. Let $\Gamma_E(R)$ have diameter 0. Since $\Gamma_E(R)$ has only one vertex, $[x] = [y]$ for every $x, y \in Z^*(R)$. Since the graph $\Gamma(R)$ is connected and every element in $\Gamma(R)$ has the same annihilator, $xy = 0$ for every $x, y \in Z^*(R)$. Thus the graph $\Gamma(R)$ is complete and $\operatorname{diam}\Gamma(R) = 0$ or 1. □

Theorem 2.3. *If* $\operatorname{diam}\Gamma(R) = 3$, *then* $\operatorname{diam}\Gamma_E(R) = 3$.

Proof. Let $\Gamma(R)$ have diameter 3. Then for some elements $x, w \in \Gamma(R)$, $d(x, w) = 3$ in $\Gamma(R)$. Let $x - y - z - w$ be a path from x to w of minimal length. Since this path is minimal, $xz \ne 0$, but $zw = 0$, so $\operatorname{ann} x \ne \operatorname{ann} w$. By similar reasoning we see that each of $\operatorname{ann} x$, $\operatorname{ann} y$, $\operatorname{ann} z$, and $\operatorname{ann} w$ are distinct. Hence $[x]$, $[y]$, $[z]$, and $[w]$ are distinct equivalence classes in $\Gamma_E(R)$. Thus $[x]$ is not adjacent to $[w]$ and there exist no paths $[x]-[y]-[w]$ or $[x]-[z]-[w]$ in $\Gamma_E(R)$. Now suppose there is some other $[v]$ such that $[x]-[v]-[w]$. This is impossible because it implies that there is a path $x-v-w$ in $\Gamma(R)$, contradicting the supposition that $x-y-z-w$ is a minimal path. Therefore $d([x], [w]) = 3$ and since $\operatorname{diam}\Gamma_E(R) \le 3$, $\operatorname{diam}\Gamma_E(R) = 3$. □

We summarize with Table 1, which shows all possible combinations of diameter for $\Gamma(R)$ and $\Gamma_E(R)$.

We see from our examples that it is possible for the diameter of the zero-divisor graph to shrink under the equivalence relation. We consider the situations where this happens.

If $\operatorname{diam}\Gamma(R) = 1$ and $\operatorname{diam}\Gamma_E(R) = 0$, then R has a unique annihilator ideal $\operatorname{ann} x$. This annihilator is maximal in $\mathfrak{F}$ and an associated prime of the ring. Since $Z(R) = \bigcup_{\mathfrak{p} \in \operatorname{ass} R} \mathfrak{p} = \operatorname{ann} x$, $Z(R)$ forms an ideal of R.

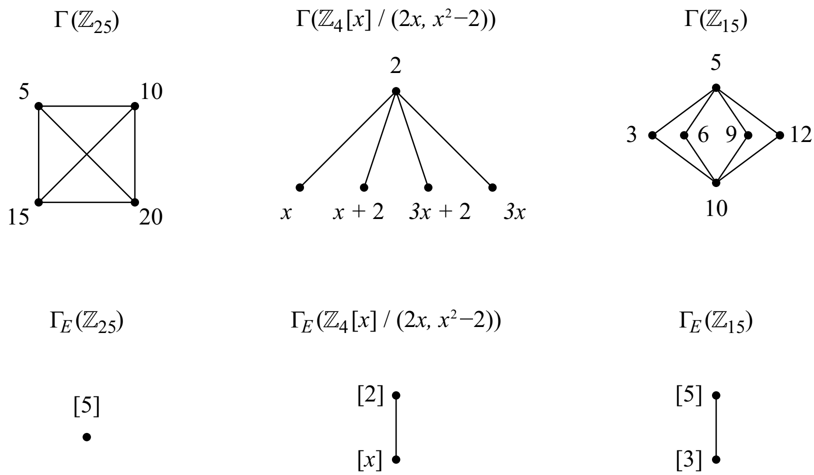
	diam $\Gamma_E(R) =$			
diam $\Gamma(R)$	0	1	2	3
0	$\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$	–	–	–
1	$\mathbb{Z}_9$, $\mathbb{Z}_{25}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	–	–
2	impossible (Theorem 2.2)	$\mathbb{Z}_6$, $\mathbb{Z}_{21}$, $\mathbb{Z}_2[x]/(x^3)$	$\mathbb{Z}_{16}$	–
3	impossible (Theorem 2.3)			$\mathbb{Z}_{12}$

Table 1. Possibilities for diam $\Gamma(R)$ and diam $\Gamma_E(R)$, with examples.

Next we consider the situation in which the diameter reduces from 2 to 1. Since there are no complete equivalence class graphs on 3 or more vertices, by [Spiroff and Wickham 2011, Proposition 1.5], $\Gamma_E(R)$ must have exactly two vertices, and R must have exactly 2 distinct annihilator ideals, $\operatorname{ann} x$ and $\operatorname{ann} y$. Let $\operatorname{ann} x$ be maximal in $\mathfrak{F}$. If $\operatorname{ann} y \subseteq \operatorname{ann} x$, then $Z(R) = \bigcup_{\mathfrak{p} \in \operatorname{ass} R} \mathfrak{p} = \operatorname{ann} x$ forms an ideal of R. Otherwise, both $\operatorname{ann} x$ and $\operatorname{ann} y$ are maximal in $\mathfrak{F}$ and $\operatorname{ann} x \cap \operatorname{ann} y = \{0\}$. If we have nonzero a, b with $a \in \operatorname{ann} x$ and $b \in \operatorname{ann} y$ such that $a + b \in \operatorname{ann} x$, then $b \in \operatorname{ann} x$, a contradiction. So in this case $Z(R) = \bigcup_{\mathfrak{p} \in \operatorname{ass} R} \mathfrak{p} = \operatorname{ann} x \cup \operatorname{ann} y$ does not form an ideal of R.

Therefore we see that if the diameter shrinks in the equivalence class graph, R has 1 or 2 associated primes. If R is a finite ring, this corresponds to R being the direct product of 1 or 2 local rings, since every finite ring R is expressible as the product of finite local rings, with the number of factors equal to the number of associated primes of R.

We show below examples of graphs of rings with shrinking diameter, one from each of the situations considered above. Note that $\mathbb{Z}_{25}$ has a unique annihilator,



$\operatorname{ann} 5 = (5)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$ has two annihilators, $\operatorname{ann} x = (2) \subseteq (2, x) = \operatorname{ann} 2$, and $\mathbb{Z}_{15}$ has two annihilators, $\operatorname{ann} 3 = (5)$ and $\operatorname{ann} 5 = (3)$, which intersect trivially.

3. Girth

Mulay [2002] proved that if the zero-divisor graph, $\Gamma(R)$, contains a cycle then $\mathrm{g}(\Gamma(R)) \leq 4$. In this section we will demonstrate an even stronger restriction on the girth of the equivalence class graph, and find all possible combinations of girth for $\Gamma(R)$ and $\Gamma_E(R)$. The following result gives a girth restriction for $\Gamma_E(R)$ similar to that shown by Mulay for $\Gamma(R)$.

Theorem 3.1. *If R is a commutative Noetherian ring, and if $\Gamma_E(R)$ contains a cycle, then* $\mathrm{g}(\Gamma_E(R)) \leq 4$.

Proof.

Case 1: If R has at least 3 distinct associated primes, say $\operatorname{ann} x$, $\operatorname{ann} y$, and $\operatorname{ann} z$, then the vertices $[x]$, $[y]$, and $[z]$ in $\Gamma_E(R)$ are all adjacent to each other by Lemma 1.1, and therefore span a complete subgraph of $\Gamma_E(R)$. Hence $\Gamma_E(R)$ contains a 3-cycle, so $\mathrm{g}(\Gamma_E(R)) = 3$.

Case 2: If R has exactly one associated prime, $\operatorname{ann} y$, then every other vertex in $\Gamma_E(R)$ is adjacent to $[y]$ by Lemma 1.1. If there is any cycle in $\Gamma_E(R)$, then there are some vertices $[x_1]$, $[x_2]$ distinct from $[y]$ with $[x_1] - [x_2]$. But these are both adjacent to $[y]$, creating the 3-cycle $[y] - [x_1] - [x_2] - [y]$. So $\mathrm{g}(\Gamma_E(R)) = 3$.

Case 3: Now assume that R has exactly 2 associated primes, and let $\operatorname{ass} R = \{\operatorname{ann} x, \operatorname{ann} y\}$. Let $[x_1]$ and $[x_2]$ be two vertices distinct from $[x]$ and $[y]$ such that $[x_1] - [x_2]$. By Lemma 1.1, $[x_1]$ is adjacent to an associated prime. Without loss of generality, let $[x_1] - [x]$. Also, $[x_2]$ is adjacent to either $[x]$ or $[y]$. In the first case, we have a 3-cycle $[x] - [x_1] - [x_2] - [x]$ and in the second case, we have a 4-cycle $[x] - [x_1] - [x_2] - [y]$. Now assume that given any two vertices of $\Gamma_E(R)$, at least one is an associated prime. Let $[x_1] - [x_2] - \cdots - [x_n] - [x_1]$ be a cycle in $\Gamma_E(R)$ of minimal length, and let $n \geq 4$. Since at least one of $[x_1]$ and $[x_2]$ is an associated prime, without loss of generality let $[x_1]$ be an associated prime. Also, at least one of $[x_3]$ and $[x_4]$ is an associated prime. If $[x_3]$ is an associated prime, we have the 3-cycle $[x_1] - [x_2] - [x_3] - [x_1]$, and if $[x_4]$ is an associated prime, we have the 4-cycle $[x_1] - [x_2] - [x_3] - [x_4] - [x_1]$. □

The following corollary is a direct result of the proof of Theorem 3.1.

Corollary 3.2. *If $\Gamma_E(R)$ has girth 4, then R must have exactly 2 associated primes.*

The following proposition gives a relationship between the girths of the two graphs. Note that the inequality is opposite that of the diameter relationship stated in the previous section.

Proposition 3.3. *If $\Gamma_E(R)$ contains a cycle, then* $\mathrm{g}(\Gamma_E(R)) \geq \mathrm{g}(\Gamma(R))$.

Proof. Let $[x_1]-[x_2]-\cdots-[x_n]-[x_1]$ be a cycle in $\Gamma_E(R)$. For each $[x_i]$, choose one $y_i \in [x_i]$. Then by the definition of multiplication of our equivalence classes, $y_1-y_2-\cdots-y_n-y_1$ is a cycle in $\Gamma(R)$ of equal length. So $\mathrm{g}(\Gamma_E(R)) \geq \mathrm{g}(\Gamma(R))$. □

Corollary 3.4. *If* $\mathrm{g}(\Gamma_E) = 3$, *then* $\mathrm{g}(\Gamma) = 3$.

We now examine the situation in which $\Gamma_E(R)$ has girth 4 and conclude that it is impossible.

Theorem 3.5. *For R a commutative Noetherian ring*, $\mathrm{g}(\Gamma_E(R)) \neq 4$.

Proof.

Suppose that $\Gamma_E(R)$ has girth 4. By Corollary 3.2, R has exactly two associated primes, so let $\operatorname{ass} R = \{\operatorname{ann} x, \operatorname{ann} y\}$.

Since $\operatorname{ann} x$ and $\operatorname{ann} y$ are associated primes, $[x]-[y]$ by Lemma 1.1. Let $[z]$ be some other vertex of $\Gamma_E(R)$. Then $[z]$ must be adjacent to at least one of $[x]$ or $[y]$. But if it is adjacent to both $[x]$ and $[y]$ we have a 3-cycle, so $[z]$ is adjacent to exactly one of $[x]$ or $[y]$. Thus the vertex set of $\Gamma_E(R)$ minus $\{[x],[y]\}$ can be partitioned into two disjoint subsets, one adjacent to $[x]$ and one adjacent to $[y]$. We refer to these subsets as X and Y, respectively.

As mentioned earlier, since R is Noetherian, there is at least one maximal element of $\mathfrak{F}$, and this annihilator is an associated prime. Without loss of generality, let $\operatorname{ann} x$ be maximal in $\mathfrak{F}$. We claim that $\operatorname{ann} y$ is also maximal in $\mathfrak{F}$. Now if $\operatorname{ann} y \subseteq \operatorname{ann} w$ for some w, then $\operatorname{ann} w \subseteq \operatorname{ann} m$ for some maximal element $\operatorname{ann} m \in \mathfrak{F}$, but since $\operatorname{ann} m$ is an associated prime, $\operatorname{ann} m = \operatorname{ann} y$ or $\operatorname{ann} m = \operatorname{ann} x$. In the latter case, $\operatorname{ann} y \subseteq \operatorname{ann} x$, so $[x]$ and $[y]$ are both adjacent to a common vertex. This creates a 3-cycle, contradicting that $\mathrm{g}(\Gamma_E(R)) = 4$. So both $\operatorname{ann} y$ and $\operatorname{ann} x$ are maximal in $\mathfrak{F}$.

Suppose that $[x]^2 = [0]$ and $[y]^2 = [0]$, and consider the class $[x+y]$. This class is annihilated by both $[x]$ and $[y]$, so either $[x+y] = [0]$ or $[x+y]$ is in the vertex set of $\Gamma_E(R)$. If $[x+y] = [0]$, then $[x] = [y]$, contrary to our assumption. So $[x+y]$ is in the vertex set of $\Gamma_E(R)$. Since $[y]$ is adjacent to no vertex of X, $[x+y] \neq [x]$. Similarly, since $[x]$ is adjacent to no vertex of Y, $[x+y] \neq [y]$. So $\Gamma_E(R)$ contains the 3-cycle $[x+y]-[x]-[y]-[x+y]$, a contradiction.

Now suppose that $[x]^2 \neq [0]$ and $[y]^2 \neq [0]$. Then $\operatorname{ann} x \cap \operatorname{ann} y = \{0\}$. Now multiplying any $[x_j] \in X$ and $[y_i] \in Y$, we see that since $[x_j] \in \operatorname{ann} x$ and $[y_i] \in \operatorname{ann} y$, $[x_j y_i] \in \operatorname{ann} x \cap \operatorname{ann} y = \{0\}$. If we break up the vertex set of $\Gamma_E(R)$ into $X \cup \{[y]\}$ and $Y \cup \{[x]\}$, we see that $\Gamma_E(R)$ is complete bipartite, and $\Gamma_E(R) = K_{n,m}$ with $n, m \neq 1$, which contradicts Lemma 1.2.

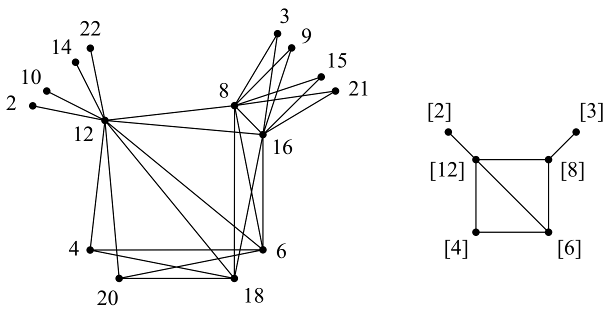
Without loss of generality, let $[x]^2 = [0]$ and $[y]^2 \neq [0]$. Let $[x]-[y]-[z]-[w]-[x]$ be a 4-cycle in $\Gamma_E(R)$, with $[w] \in X$, $[z] \in Y$. Then there is a 4-cycle $x-y-z-w-x$ in $\Gamma(R)$. By the previous discussion, $x^2 = 0$ and $y^2 \neq 0$.

	diam $\gamma_E(R) =$		
diam $\gamma(R)$	∞	3	4
∞	$\mathbb{Z}_4$	impossible (Proposition 3.3)	
3	$\mathbb{Z}_{12}$	$\mathbb{Z}_{24}$	impossible (Theorem 3.5)
4	$\mathbb{Z}_{15}$	impossible (Corollary 3.2)	impossible (Theorem 3.5)

Table 2. Possibilities for $\mathrm{g}(\Gamma(R))$ and $\mathrm{g}(\Gamma_E(R))$, with examples.

Since $\operatorname{ann} y$ is maximal in the set of annihilators of R, there is some m in $\operatorname{ann} y$ but not in $\operatorname{ann} w$. Note that $mw \neq 0$, but $\operatorname{ann} mw \supseteq \{x, z, y\}$. Since $mw - y$ but $y^2 \neq 0$, $\operatorname{ann} mw \neq \operatorname{ann} y$. Also since mw is adjacent to both x and z, and x and z are not adjacent, $\operatorname{ann} mw \neq \operatorname{ann} x$ and $\operatorname{ann} mw \neq \operatorname{ann} z$. So we have the 3-cycles $x - y - mw - x$ and $z - y - mw - z$ that do not reduce under the equivalence relation. So $\Gamma_E(R)$ contains a 3-cycle and $\mathrm{g}(\Gamma_E(R)) \neq 4$. □

We summarize with Table 2, which shows all possible combinations of girths for $\Gamma(R)$ and $\Gamma_E(R)$. We illustrate the case (3, 3) with the graphs of the ring $\mathbb{Z}_{24}$, which does not have shrinking girth:



$\Gamma(\mathbb{Z}_{24})$ $\Gamma_E(\mathbb{Z}_{24})$

4. Cut-vertices

In this section, we examine the properties of cut-vertices of $\Gamma_E(R)$. Since $\Gamma_E(R)$ is connected, the vertex $[a]$ is a cut-vertex of $\Gamma_E(R)$ exactly when removing the vertex $[a]$ and its incident edges causes $\Gamma_E(R)$ to no longer be connected.

We begin with an interesting result concerning cut-vertices and ideals of the ring. The following theorem is very similar to [Axtell et al. 2009, Theorem 4.4], which deals with cut-vertices of the original zero-divisor graph $\Gamma(R)$.

Theorem 4.1. *If $[a]$ is a cut-vertex of $\Gamma_E(R)$, then $[a] \cup \{0\}$ forms an ideal of R.*

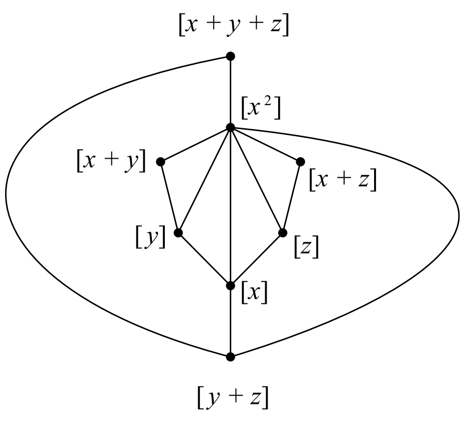
Proof. Let $[a]$ be a cut-vertex of $\Gamma_E(R)$ and let $[a]$ partition $\Gamma_E(R)$ into Γ_b and Γ_c. Let $[b] \in \Gamma_b$ with $[a] - [b]$ and $[c] \in \Gamma_c$ with $[a] - [c]$. Let $a_1, a_2 \in [a] \cup \{0\}$. Since $a_1 + a_2 \in \operatorname{ann} b \cap \operatorname{ann} c$, $a_1 + a_2 \in [a] \cup \{0\}$. If $r \in R$, then $c(ra) = r(ca) = 0$, so $ra \in \operatorname{ann} c$. Similarly, $ra \in \operatorname{ann} b$. So $ra \in \operatorname{ann} b \cap \operatorname{ann} c = [a] \cup \{0\}$. This shows that $[a] \cup \{0\}$ is an ideal of R. $\square$

Theorem 4.2. *If* $[a]$ *is a cut-vertex of* $\Gamma_E(R)$, *then* $\operatorname{ann} a$ *is maximal in* $\mathfrak{F}$.

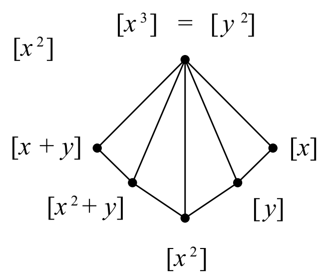
Proof. Let $[a]$ be a cut-vertex of $\Gamma_E(R)$, and let X and Y be mutually separated subgraphs of $\Gamma_E(R)$ with $V(X \cup Y) = V(\Gamma_E(R)) \setminus [a]$. Let $[x] \in X$ and $[y] \in Y$. Then for any $[x_1] \in X$ we have $y \in \operatorname{ann} a \setminus \operatorname{ann} x_1$, and for any $[y_1] \in Y$ we have $x \in \operatorname{ann} a \setminus \operatorname{ann} y_1$. Thus $\operatorname{ann} a \not\subseteq \operatorname{ann} x_1$ and $\operatorname{ann} a \not\subseteq \operatorname{ann} y_1$, and so $\operatorname{ann} a$ is maximal in $\mathfrak{F}$. $\square$

The converse of this theorem does not hold. We may have $\operatorname{ann} x$ maximal in $\mathfrak{F}$, yet not have $[x]$ be a cut-vertex. For example, here are two equivalence graphs, one on 6 vertices and one on 8, each with no cut vertex:

$\Gamma_E(\mathbb{Z}_2[x, y, z] / (x^3, y^2, z^2, xy, xz))$



$\Gamma_E(\mathbb{Z}_2[x, y] / (x^4, xy, x^3 + y^2))$



Both of these rings contains an annihilator ideal which maximal in $\mathfrak{F}$, and therefore an associated prime.

The next corollary follows immediately from Theorem 4.2:

Corollary 4.3. *If* $[a]$ *is a cut-vertex of* $\Gamma_E(R)$, *then* $\operatorname{ann} a$ *is an associated prime.*

Theorem 4.4. *If* $[a]$ *is a cut-vertex of* $\Gamma_E(R)$, *then all other associated primes of* $\Gamma_E(R)$ *are contained in only one connected component of* $\Gamma_E(R) \setminus [a]$.

Proof. Suppose that X and Y are two mutually separated connected components of $\Gamma_E(R) \setminus [a]$, and that each contains an associated prime. By Lemma 1.1, these associated primes are adjacent, and so X and Y are connected, a contradiction. $\square$

Theorem 4.5. *If $\Gamma_E(R)$ has at least 2 cut-vertices, then it has diameter 3.*

Proof. Let $[a]$ and $[b]$ be cut-vertices of $\Gamma_E(R)$. Since $[a]$ is a cut-vertex, there is some $[x_a]$ such that any path connecting $[x_a]$ and $[b]$ must include $[a]$. Similarly, since $[b]$ is a cut-vertex, there is some $[x_b]$ such that any path connecting $[x_b]$ and $[a]$ must include $[b]$. Therefore any path from $[x_a]$ to $[x_b]$ must include both $[a]$ and $[b]$ and so $d([a], [b]) \geq 3$. Since $\Gamma_E(R)$ is connected, $\operatorname{diam} \Gamma_E(R) = 3$. □

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