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Total positivity of a shuffle matrix

Audra McMillan

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Holte introduced a $n \times n$ matrix P as a transition matrix related to the carries obtained when summing n numbers base b . Since then Diaconis and Fulman have further studied this matrix proving it to also be a transition matrix related to the process of b -riffle shuffling n cards. They also conjectured that the matrix P is totally nonnegative. In this paper, the matrix P is written as a product of a totally nonnegative matrix and an upper triangular matrix. The positivity of the leading principal minors for general n and b is proven as well as the nonnegativity of minors composed from initial columns and arbitrary rows.

1. Introduction

Holte [1997] introduced an $n \times n$ matrix P , with entries

$$P(i, j) = \frac{1}{b^n} \sum_{r=0}^{j - \lfloor i/b \rfloor} (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n}$$

where the $P(i, j)$ entry gives the probability that when adding n random numbers base b , the next carry will be j , given that the previous carry was i . This matrix was then further studied in [2009a; Diaconis and Fulman 2009b], where it is noted that this is also a transition matrix related to card shuffling, where the $P(i, j)$ entry records the probability that a b -riffle shuffle of a permutation with i descents will lead to a permutation with j descents. Note that the rows and columns of this matrix are indexed by $0, \dots, n-1$.

Holte proved a number of properties of the matrix P , including that P has eigenvalues given by the geometric sequence $1, b^{-1}, \dots, b^{-(n-1)}$, implying that the determinant is positive for positive b .

A matrix will be referred to as *totally nonnegative* if every minor is nonnegative and *totally positive* if every minor is positive. Note that in some texts, such as [Pinkus 2010] and [Karlin 1968] these terms are replaced by *totally positive* and

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strictly totally positive respectively. Totally nonnegative matrices figure prominently in a wide range of mathematical disciplines including, but not limited to, combinatorics, stochastic processes and probability theory. Many properties of totally nonnegative matrices are known including eigenvalue/eigenvector properties and factorisation of such matrices. A good reference for the theory and applications of total positivity is [Pinkus 2010] and some further results on stochastic totally nonnegative matrices are included in [Gasca and Micchelli 1996].

Diaconis and Fulman [2009a, Remark after Lemma 4.2] conjectured that the matrix P is totally nonnegative for all positive integers n and b . Their paper included a proof that for all n and b , P is totally nonnegative of order 2, that is all the 2×2 minors are nonnegative, and that when b is a power of 2, P is totally nonnegative. Unfortunately, their method of proof does not generalise to other b . The aim of this paper is to make progress on the general conjecture.

Recall the following result:

Theorem 1. *Let $A = (a_{ij})$ be an $n \times n$ nonsingular matrix whose rows and columns are indexed by $0, \dots, n-1$. Then A is totally nonnegative if and only if A satisfies*

- (i) $A \begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} > 0$ for $k = 1, \dots, n$,
- (ii) $A \begin{pmatrix} i_1, \dots, i_k \\ 0, \dots, k-1 \end{pmatrix}$ for $0 \leq i_1 < \dots < i_k \leq n-1$ and $k = 1, \dots, n$,
- (iii) $A \begin{pmatrix} 0, \dots, k-1 \\ j_1, \dots, j_k \end{pmatrix}$ for $0 \leq j_1 < \dots < j_k \leq n-1$ and $k = 1, \dots, n$,

where $A \begin{pmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{pmatrix}$ denotes the minor composed of rows i_1, \dots, i_k and columns j_1, \dots, j_k .

A proof of this can be found in [Pinkus 2010, Proposition 2.15].

In this paper, we will prove (i) and (ii) for the matrix P , hence reducing the conjecture to condition (iii). Proving these conditions hold for P is equivalent to proving that they hold for $P' = b^n P$, so this matrix will be dealt with instead.

2. Proof of total nonnegativity claims

Firstly, note that

$$\begin{aligned} P'(i, j) &= \sum_{r=0}^{j-\lfloor i/b \rfloor} (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n} \\ &= \sum_{r=0}^j (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n}, \end{aligned}$$

which implies

$$P' = \left(\binom{n-1-i+(j+1)b}{n} \right)_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq n-1}} \left((-1)^{j-i} \binom{n+1}{j-i} \right)_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq n-1}},$$

where $\binom{n}{k} = 0$ if $k < 0$. Let's call the first matrix A , and the second B . Note that B is upper unitriangular.

Using the Vandermonde convolution note that

$$\sum_{k=0}^{n-i} \binom{n-i}{n-i-k} \binom{jb}{i+k} = \binom{n-i+jb}{n},$$

so A can be further factored as

$$A = \left[\binom{n-i-1}{j-i} \right]_{i,j} \left[\binom{(j+1)b}{i+1} \right]_{i,j}.$$

Let's call these matrices C and D , respectively. Note that C is upper unitriangular, so this factorisation of P' implies that $\det P' = \det D$.

Lemma 2. C is totally nonnegative.

Proof. Obviously all the leading principal minors of C are 1, and all other minors composed of k initial columns and k arbitrary rows are 0 since C is upper unitriangular.

Now let $k \in \mathbb{Z}$, $1 \leq k \leq n$ and $0 \leq j_1 < \dots < j_k \leq n-1$. Again using the Vandermonde convolution we observe that

$$\sum_{p=0}^{k-i-1} \binom{k-i-1}{p} \binom{n-k}{j_{l+1}-i-p} = \binom{n-i-1}{j_{l+1}-i},$$

so

$$\begin{aligned} C \begin{pmatrix} 0, \dots, k-1 \\ j_1, \dots, j_k \end{pmatrix} &= \left| \left[\binom{n-i-1}{j_{l+1}-i} \right]_{i,l} \right| = \left| \left[\binom{k-i-1}{l-i} \right]_{i,l} \right| \left| \left[\binom{n-k}{j_{l+1}-i} \right]_{i,l} \right| \\ &= \left| \left[\binom{n-k}{j_{l+1}-i} \right]_{i,l} \right|. \end{aligned} \quad (*)$$

A sequence $(a_i)_{0 \leq i < \infty}$ is called a Pólya frequency sequence of infinite order if the corresponding infinite kernel matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative. The matrix (*) is nonnegative since it is a submatrix of the infinite kernel matrix of the sequence

$$\left(\binom{n-k}{0}, \binom{n-k}{1}, \dots, \binom{n-k}{n-k} \right),$$

which is a Pólya frequency sequence of infinite order according to the classification of Pólya frequency sequences in [Karlin 1968, Theorem 5.3, Chapter 8].

Therefore, by Theorem 1, matrix C is totally nonnegative for all n . \square

Lemma 3. D is totally nonnegative.

Proof. D is a submatrix of the upper triangular Pascal matrix

$$\left[\binom{j}{i} \right]_{i,j}$$

which is simply the reflection of C about the antidiagonal where the dimension is $nb + 1$, and hence is totally nonnegative [Pinkus 2010, Propositions 1.2 and 1.3]. Therefore D is totally nonnegative. \square

Corollary 4. A is totally nonnegative.

Proof. Since the product of totally nonnegative matrices is totally nonnegative, A is totally nonnegative. \square

Proposition 5. Conditions (i) and (ii) of Theorem 1 hold for matrix P' for general n and b .

Proof. Let $k \in \mathbb{Z}$, $1 \leq k \leq n$ and $0 \leq i_1 < \dots < i_k \leq n - 1$. From the Cauchy–Binet formula and the fact that B is upper unitriangular,

$$P' \begin{pmatrix} i_1, \dots, i_k \\ 0, \dots, k-1 \end{pmatrix} = A \begin{pmatrix} i_1, \dots, i_k \\ 0, \dots, k-1 \end{pmatrix} \geq 0$$

and

$$\begin{aligned} P' \begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} &= A \begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} \\ &= \sum_{0 \leq m_1 < \dots < m_k \leq n-1} C \begin{pmatrix} 0, \dots, k-1 \\ m_1, \dots, m_k \end{pmatrix} D \begin{pmatrix} m_1, \dots, m_k \\ 0, \dots, k-1 \end{pmatrix} \\ &\geq D \begin{pmatrix} 0, \dots, k-1 \\ 0, \dots, k-1 \end{pmatrix} = \left| \left[\binom{(j+1)b}{i+1} \right]_{i,j} \right|. \end{aligned}$$

Here the inequality follows from the fact that C and D are totally nonnegative and C is upper unitriangular.

However this is simply the determinant of a smaller version of D , with n replaced by k and therefore by the previous factorisation of P' , this is equal to the

determinant of the P' matrix of dimension k , which is positive (as stated earlier) so we are done. \square

One might hope that condition (iii) could be proved similarly by noting that

$$P' \begin{pmatrix} 0, \dots, k-1 \\ j_1, \dots, j_k \end{pmatrix} = \sum_{0 \leq m_1 < \dots < m_k \leq n-1} A \begin{pmatrix} 0, \dots, k-1 \\ m_1, \dots, m_k \end{pmatrix} B \begin{pmatrix} m_1, \dots, m_k \\ j_1, \dots, j_k \end{pmatrix}.$$

However the proof of condition (ii) relied on the fact that the minors of B involved were clearly seen to be 0 or 1 so this equation easily simplified. This is not the case for the above equation since little has been established about general minors of B . Progress might still be made if all minors of size k were nonnegative for some k however small examples show this to be unlikely, for example this is not true for minors of size 2 for any n . If the conjecture is true, it seems likely that a new approach is required to prove condition (iii).

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amcm7623@uni.sydney.edu.au *School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia*

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