involve

a journal of mathematics

# A BMO theorem for $\epsilon$-distorted diffeomorphisms on $\mathbb{R}^D$ and an application to comparing manifolds of speech and sound

Charles Fefferman, Steven B. Damelin and William Glover

msp

2012 vol. 5, no. 2

INVOLVE 5:2 (2012)
dx.doi.org/10.2140/involve.2012.5.159

# A BMO theorem for $\epsilon$-distorted diffeomorphisms on $\mathbb{R}^D$ and an application to comparing manifolds of speech and sound

Charles Fefferman, Steven B. Damelin and William Glover

(Communicated by Kenneth S. Berenhaut)

This paper deals with a BMO theorem for $\epsilon$-distorted diffeomorphisms on $\mathbb{R}^D$ and an application comparing manifolds of speech and sound.

## 1. Introduction

From the very beginning of time, mathematicians have been intrigued by the fascinating connections which exist between music, speech and mathematics. Indeed, these connections were already in some subtle form in the writings of Gauss. The aim of this paper is to study estimates in measure for diffeomorphisms $\mathbb{R}^D$ to $\mathbb{R}^D$, $D \geq 2$ of small distortion and provide an application to comparing music and speech manifolds.

This paper originated from discussions where Glover, an undergraduate student of Damelin and a passionate practitioner of music (particularly the piano), introduced Damelin to the beautiful world of beats, movements, scales, measures and time signatures. A fruitful and inspiring collaboration ensued, enriched by wonderful contributions from Fefferman.

## 2. Preliminaries

Fix a dimension $D \geq 2$. We work in $\mathbb{R}^D$. We write $B(x, r)$ to denote the open ball in $\mathbb{R}^D$ with centre $x$ and radius $r$. We write $A$ to denote Euclidean motions on $\mathbb{R}^D$. A Euclidean motion may be orientation-preserving or orientation reversing. We write $c$, $C$, $C'$, etc. to denote constants depending on the dimension $D$. These expressions

*MSC2010:* 00A65, 65D19, 97M80.
*Keywords:* measure, BMO, diffeomorphism, small distortion, manifold, music, sound, noise, speech, alignment.
Damelin thanks Princeton University and Wayne Country Day School for financial support. He also thanks Prof Charles Fefferman for wonderfully inspiring collaborations and his generosity.

need not denote the same constant in different occurrences. For a $D \times D$ matrix, $M = (M_{ij})$, we write $|M|$ to denote the Hilbert–Schmidt norm

$$|M| = \Bigl(\sum_{ij} |M_{ij}|^2\Bigr)^{1/2}.$$

Note that if $M$ is real and symmetric and if

$$(1-\lambda)I \le M \le (1+\lambda)I$$

as matrices, where $0 < \lambda < 1$, then

$$|M - I| \le C\lambda. \tag{2-1}$$

This follows from working in an orthonormal basis for which $M$ is diagonal. One way to understand the formulas above is to think of $\lambda$ as being close to zero. See also (2-6) below.

A function $f : \mathbb{R}^D \to \mathbb{R}$ is said to be BMO (Bounded mean oscillation )if there is a constant $K \ge 0$ such that, for every ball $B \subset \mathbb{R}^D$, there exists a real number $H_B$ such that

$$\frac{1}{\operatorname{vol} B} \int_B |f(x) - H_B|\, dx \le K. \tag{2-2}$$

The least such $K$ is denoted by $\|f\|_{\mathrm{BMO}}$.

In harmonic analysis, a function of bounded mean oscillation, also known as a BMO function, is a real-valued function whose mean oscillation is bounded (finite). The space of functions of bounded mean oscillation (BMO), is a function space that, in some precise sense, plays the same role in the theory of Hardy spaces, that the space of essentially bounded functions plays in the theory of $Lp$-spaces: it is also called a John–Nirenberg space, after Fritz John and Louis Nirenberg who introduced and studied it for the first time [John 1961; John and Nirenberg 1961].

The John–Nirenberg inequality asserts the following: Let $f \in \mathrm{BMO}$ and let $B \subset \mathbb{R}^D$ be a ball. Then there exists a real number $H_B$ such that

$$\operatorname{vol}\{x \in B : |f(x) - H_B| > C\lambda\|f\|_{\mathrm{BMO}}\} \le \exp(-\lambda)\operatorname{vol} B, \quad \lambda \ge 1. \tag{2-3}$$

As a corollary of the John–Nirenberg inequality, we have

$$\left(\frac{1}{\operatorname{vol} B} \int_B |f(x) - H_B|^4 dx\right)^{1/4} \le C\lambda\|f\|_{\mathrm{BMO}}. \tag{2-4}$$

There is nothing special about the 4th power in the above; it will be needed later.

The definition of BMO, the notion of the BMO norm, the John–Nirenburg inequality (2-3) and its corollary (2-4) carry through to the case of functions $f$ on $\mathbb{R}^D$ which take their values in the space of $D \times D$ matrices. Indeed, we take

$H_B$ in (2-2)–(2-4) to be a $D \times D$ matrix for such $f$. The matrix valued norms of (2-3)–(2-4) follow easily from the scalar case.

We will need some potential theory. If $f$ is a smooth function of compact support in $\mathbb{R}^D$, then we can write $\Delta^{-1} f$ to denote the convolution of $f$ with the Newtonian potential. Thus, $\Delta^{-1} f$ is smooth and $\Delta(\Delta^{-1} f) = f$ on $\mathbb{R}^D$.

We will use the estimate:

$$\left\| \frac{\partial}{\partial x_i} \Delta^{-1} \frac{\partial}{\partial x_j} f \right\|_{L^2(\mathbb{R}^D)} \leq C \| f \|_{L^2(\mathbb{R}^D)}, \quad i, j = 1, \ldots, D, \tag{2-5}$$

valid for any smooth function $f$ with compact support. Estimate (2-5) follows by applying the Fourier transform.

We will work with a positive number $\varepsilon$. We always assume that $\varepsilon \leq \min(1, C)$. An $\varepsilon$-distorted diffeomorphism of $\mathbb{R}^D$ is a one to one and onto diffeomorphism $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ such as

$$(1-\varepsilon) I \leq (\Phi'(x))^T (\Phi'(x)) \leq (1+\varepsilon) I$$

as matrices. Thanks to (2-1), such $\Phi$ satisfy

$$\left| (\Phi'(x))^T (\Phi'(x)) - I \right| \leq C\varepsilon. \tag{2-6}$$

We end this section with the following inequality from [Fefferman and Damelin $\geq$ 2012]:

**Approximation Lemma.** *Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$-distorted diffeomorphism. Then, there exists an Euclidean motion $A$ such that*

$$|\Phi(x) - A(x)| \leq C\varepsilon \tag{2-7}$$

*for all $x \in B(0, 10)$.*

## 3. An overdetermined system

We will need to study the following elemetary overdetermined system of partial differential equations.

$$\frac{\partial \Omega_i}{\partial x_j} + \frac{\partial \Omega_j}{\partial x_i} = f_{ij}, \quad i, j = 1, \ldots, D, \tag{3-1}$$

on $\mathbb{R}^D$. Here, $\Omega_i$ and $f_{ij}$ are $C^\infty$ functions on $\mathbb{R}^D$. A result concerning (3-1) we need is:

**PDE Theorem.** *Let $\Omega_1, \ldots, \Omega_D$ and $f_{ij}$, for $i, j = 1, \ldots, D$, be smooth functions on $\mathbb{R}^D$. Assume that (3-1) holds and suppose that*

$$\| f_{ij} \|_{L^2(B(0,4))} \leq 1. \tag{3-2}$$

*Then, there exist real numbers* $\Delta_{ij}$, *for* $i, j = 1, \ldots, D$, *such that*

$$\Delta_{ij} + \Delta_{ji} = 0 \quad \textit{for all } i, j \tag{3-3}$$

*and*

$$\left\| \frac{\partial \Omega_i}{\partial x_j} - \Delta_{ij} \right\|_{L^2(B(0,1))} \le C. \tag{3-4}$$

*Proof.* From (3-1), we see at once that

$$\frac{\partial \Omega_i}{\partial x_i} = \tfrac{1}{2} f_{ii}$$

for each $i$. Now, by differentiating (3-1) with respect to $x_j$ and then summing on $j$, we see that

$$\Delta \Omega_i + \frac{1}{2} \frac{\partial}{\partial x_i} \left( \sum_j f_{jj} \right) = \sum_j \frac{\partial f_{ij}}{\partial x_j}$$

for each $i$. Therefore, we may write

$$\Delta \Omega_i = \sum_j \frac{\partial}{\partial x_j} g_{ij}$$

for smooth functions $g_{ij}$ with

$$\|g_{ij}\|_{L^2(B(0,4)} \le C.$$

This holds for each $i$. Let $\chi$ be a $C^\infty$ cutoff function on $\mathbb{R}^D$ equal to 1 on $B(0, 2)$ vanishing outside $B(0, 4)$ and satisfying $0 \le \chi \le 1$ everywhere. Now let

$$\Omega_i^{\mathrm{err}} = \Delta^{-1} \sum_j \frac{\partial}{\partial x_j} \left( \chi g_{ji} \right)$$

and let

$$\Omega_i^* = \Omega_i - \Omega_i^{\mathrm{err}}.$$

Then,

$$\Omega_i = \Omega_i^* + \Omega_i^{\mathrm{err}} \tag{3-5}$$

each $i$. The function

$$\Omega_i^* \tag{3-6}$$

is harmonic on $B(0, 2)$ and

$$\left\| \nabla \Omega_i^{\mathrm{err}} \right\|_{L^2(B(0,2))} \le C \tag{3-7}$$

thanks to (2-5). By (3-1), (3-2), (3-5), (3-7), we can write

$$\frac{\partial \Omega_i^*}{\partial x_j} + \frac{\partial \Omega_j^*}{\partial x_i} = f_{ij}^*, \quad i, j = 1, \dots, D, \tag{3-8}$$

on $B(0, 2)$ and with

$$\| f_{ij}^* \|_{L^2(B(0,2))} \le C. \tag{3-9}$$

From (3-6) and (3-8), we see that each $f_{ij}^*$ is a harmonic function on $B(0, 2)$. Consequently, (3-9) implies

$$sup_{B(0,1)} |\nabla f_{ij}^*| \le C. \tag{3-10}$$

From (3-8), we have for each $i, j, k$,

$$\frac{\partial^2 \Omega_i^*}{\partial x_j \partial x_k} + \frac{\partial^2 \Omega_k^*}{\partial x_i \partial x_j} = \frac{\partial f_{ik}^*}{\partial x_j}, \quad \frac{\partial^2 \Omega_i^*}{\partial x_j \partial x_k} + \frac{\partial^2 \Omega_j^*}{\partial x_i \partial x_k} = \frac{\partial f_{ij}^*}{\partial x_k}, \tag{3-11}$$

$$\frac{\partial^2 \Omega_j^*}{\partial x_i \partial x_k} + \frac{\partial^2 \Omega_k^*}{\partial x_i \partial x_j} = \frac{\partial f_{jk}^*}{\partial x_i}. \tag{3-12}$$

Now adding the first two equations above and subtracting the last, we obtain:

$$2 \frac{\partial^2 \Omega_i^*}{\partial x_j \partial x_k} = \frac{\partial f_{ik}^*}{\partial x_j} + \frac{\partial f_{ij}^*}{\partial x_k} - \frac{\partial f_{jk}^*}{\partial x_i} \tag{3-13}$$

on $B(0, 1)$. Now from (3-10) and (3-13), we obtain the estimate

$$\left| \frac{\partial^2 \Omega_i^*}{\partial x_j \partial x_k} \right| \le C \tag{3-14}$$

on $B(0, 1)$ for each $i, j, k$. Now for each $i, j$, let

$$\Delta_{ij}^* = \frac{\partial \Omega_i^*}{\partial x_j}(0). \tag{3-15}$$

By (3-14), we have

$$\left| \frac{\partial \Omega_i^*}{\partial x_j} - \Delta_{ij}^* \right| \le C \tag{3-16}$$

on $B(0, 1)$ for each $i, j$. Recalling (3-5) and (3-7), we see that (3-16) implies that

$$\left\| \frac{\partial \Omega_i}{\partial x_j} - \Delta_{ij}^* \right\|_{L^2(B(0,1))} \le C. \tag{3-17}$$

Unfortunately, the $\Delta_{ij}^*$ need not satisfy (3-3). However, (3-1), (3-2) and (3-17) imply the estimate

$$|\Delta_{ij}^* + \Delta_{ji}^*| \le C$$

for each $i, j$. Hence, there exist real numbers $\Delta_{ij}$, $(i, j = 1, \dots, D)$ such that

$$\Delta_{ij} + \Delta_{ji} = 0 \tag{3-18}$$

and

$$|\Delta^*_{ij} - \Delta_{ij}| \le C \tag{3-19}$$

for each $i, j$. From (3-17) and (3-19), we see that

$$\left\| \frac{\partial \Omega_i}{\partial x_j} - \Delta_{ij} \right\|_{L^2(B(0,1))} \le C \tag{3-20}$$

for each $i$ and $j$.

Thus (3-18) and (3-20) are the desired conclusions of the theorem. □

## 4. A BMO theorem

**BMO Theorem 1.** *Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$ diffeomorphism and let $B \subset \mathbb{R}^D$ be a ball. Then, there exists $T \in O(D)$ such that*

$$\frac{1}{\text{vol } B} \int_B |\Phi'(x) - T|\, dx \le C\varepsilon^{1/2}. \tag{4-1}$$

*Proof.* Estimate (4-1) is preserved by translations and dilations. Hence we may assume that

$$B = B(0, 1). \tag{4-2}$$

Now we know that there exists an Euclidean motion $A : \mathbb{R}^D \to \mathbb{R}^D$ such that

$$|\Phi(x) - A(x)| \le C\varepsilon \tag{4-3}$$

for $x \in B_{(0,10)}$. Our desired conclusion (4-1) holds for $\Phi$ if and only if it holds for $A^{-1} o \Phi$ (with a different T). Hence, without loss of generality, we may assume that $A = I$. Thus, (4-3) becomes

$$|\Phi(x) - x| \le C\varepsilon, \quad x \in B(0, 10). \tag{4-4}$$

We set up some notation: We write the diffeomorphism $\Phi$ in coordinates by setting:

$$\Phi(x_1, \dots, x_D) = (y_1, \dots, y_D) \tag{4-5}$$

where for each $i$, $1 \le i \le D$,

$$y_i = \psi_i(x_1, \dots, x_D). \tag{4-6}$$

First claim: For each $i = 1, \dots, D$,

$$\int_{B(0,1)} \left| \frac{\partial \psi_i(x)}{\partial x_i} - 1 \right| \le C\varepsilon. \tag{4-7}$$

For this, for fixed $(x_2, \ldots, x_D) \in B'$, we apply (4-4) to the points $x^+ = (1, \ldots, x_D)$ and $x^- = (1, \ldots, x_D)$. We have

$$\left|\psi_1(x^+) - 1\right| \leq C\varepsilon$$

and

$$\left|\psi_1(x^{-1}) + 1\right| \leq C\varepsilon.$$

Consequently,

$$\int_{-1}^{1} \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) dx_1 \geq 2 - C\varepsilon. \tag{4-8}$$

On the other hand, since,

$$\left(\psi'(x)\right)^T \left(\psi'(x)\right) \leq (1+\varepsilon) I,$$

we have for each $i = 1, \ldots, D$ the inequality

$$\left(\frac{\partial \psi_i}{\partial x_i}\right)^2 \leq 1 + \varepsilon.$$

Therefore,

$$\left|\frac{\partial \psi_i}{\partial x_i}\right| - 1 \leq \sqrt{1+\varepsilon} - 1 \leq \varepsilon. \tag{4-9}$$

Set

$$I^+ = \left\{x_1 \in [-1, 1] : \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1 \leq 0\right\},$$

$$I^{-1} = \left\{x_1 \in [-1, 1] : \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1 \geq 0\right\},$$

$$\Delta^+ = \int_{I^+} \left(\frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1\right) dx_1,$$

$$\Delta^- = \int_{I^-} \left(\frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1\right) dx_1.$$

The inequality (4-8) implies that $-\Delta^{-1} \leq C\varepsilon + \Delta^+$. The inequality (4-9) implies that

$$\frac{\partial \psi_1}{\partial x_1} - 1 \leq C\varepsilon.$$

Integrating the last inequality over $I^+$, we obtain $\Delta^+ \leq C\varepsilon$. Consequently,

$$\int_{-1}^{1} \left|\frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1\right| dx_1 = \Delta^+ - \Delta^- \leq C\varepsilon. \tag{4-10}$$

Integrating this last equation over $(x_2, \dots, x_D) \in B'$ and noting that $B(0,1) \subset [-1,1] \times B'$, we conclude that

$$\int_{B(0,1)} \left| \frac{\partial \psi_1}{\partial x_1}(x_1, \dots, x_D) - 1 \right| dx \le C\varepsilon.$$

Similarly, for each $i = 1, \dots, D$, we obtain (4-7).

Second claim: For each $i, j = 1, \dots, D,\ i \ne j$, we have

$$\int_{B(0,1)} \left| \frac{\partial \psi_i(x)}{\partial x_j} \right| dx \le C\sqrt{\varepsilon}. \tag{4-11}$$

Since

$$(1-\varepsilon)I \le (\Phi'(x))^T(\Phi'(x)) \le (1+\varepsilon)I,$$

we have

$$\sum_{i,j=1}^{D} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \le (1 + C\varepsilon)D. \tag{4-12}$$

Therefore,

$$\sum_{i \ne j} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \le C\varepsilon + \sum_{i=1}^{D} \left( 1 - \frac{\partial \psi_i}{\partial x_i} \right) \left( 1 + \frac{\partial \psi_i}{\partial x_i} \right).$$

Using (4-9) for $i$, we have $|\partial \psi_i / \partial x_i| + 1 \le C$. Therefore,

$$\sum_{i \ne j} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \le C\varepsilon + C \left| \frac{\partial \psi_i}{\partial x_i} - 1 \right|.$$

Now integrating the last inequality over the unit ball and using (4-7), we find that

$$\int_{B(0,1)} \sum_{i \ne j} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 dx \le C\varepsilon + \int_{B(0,1)} \left| \frac{\partial \psi_i}{\partial x_i} - 1 \right| dx \le C\varepsilon. \tag{4-13}$$

Consequently, by the Cauchy–Schwarz inequality, we have

$$\int_{B(0,1)} \sum_{i \ne j} \left| \frac{\partial \psi_i}{\partial x_j} \right| dx \le C\sqrt{\varepsilon}.$$

Third claim:

$$\int_{B(0,1)} \left| \frac{\partial \psi_i}{\partial x_i} \right| dx \le C\sqrt{\varepsilon} \tag{4-14}$$

Since,

$$\int_{B(0,1)} \left(\frac{\partial \psi_i}{\partial x_i} - 1\right)^2 dx \le \int_{B(0,1)} \left|\frac{\partial \psi_i}{\partial x_i} - 1\right| \left|\frac{\partial \psi_i}{\partial x_i} + 1\right| dx,$$

using (4-7) and $|\partial\psi_i/\partial x_i| \le 1 + C\varepsilon$, we obtain

$$\int_{B(0,1)} \left(\frac{\partial \psi_i}{\partial x_i}\right)^2 dx \le C\varepsilon.$$

Thus, an application of Cauchy–Schwarz, yields (4-14).

Final claim: By the Hilbert–Schmidt definition, we have

$$\begin{aligned}\int_{B(0,1)} |\Psi'(x) - I|\, dx &= \int_{B(0,1)} \left(\sum_{i,j=1}^{D} \left(\frac{\partial \psi_i}{\partial x_j} - \delta_{ij}\right)^2\right)^{1/2} \\ &\le \int_{B(0,1)} \sum_{i,j=1}^{D} \left|\frac{\partial \psi_i}{\partial x_j} - \delta_{ij}\right| dx.\end{aligned}$$

The estimate (4-11) combined with (4-14) yields:

$$\int_{B(0,1)} \left|\Phi'(x) - I\right| dx \le C\varepsilon^{1/2}.$$

Thus we have proved (4-1) with $T = I$. The proof of the BMO Theorem 1 is complete. □

**Corollary.** *Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$-distorted diffeomorphism. For each, ball $B \subset \mathbb{R}^D$, there exists $T_B \in O(D)$, such that*

$$\left(\frac{1}{\operatorname{vol} B} \int_B \left|\Phi'(x) - T\right|^4 dx\right)^{1/4} \le C\varepsilon^{1/2}.$$

*Proof.* The proof follows from that of BMO Theorem 1 just proved and the John Nirenberg inequality. (See (2-4).) □

## 5. A refined BMO theorem

**BMO Theorem 2.** *Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$ diffeomorphism and let $B \in \mathbb{R}^D$ be a ball. Then, there exists $T \in O(D)$ such that*

$$\frac{1}{\operatorname{vol} B} \int_B \left|\Phi'(x) - T\right| dx \le C\varepsilon. \tag{5-1}$$

*Proof.* We may assume without loss of generality that

$$B = B(0,1). \tag{5-2}$$

We know that there exists $T_B^* \in O(D)$ such that

$$\left(\int_B |\Phi'(x) - T_B^*|^4 dx\right)^{1/4} \le C\varepsilon^{1/2}.$$

Our desired conclusion holds for $\Phi$ if and only if it holds for $(T_B^*)^{-1} o \Phi$. Hence without loss of generality, we may assume that $T_B^* = I$. Thus we have

$$\left(\int_B |\Phi'(x) - I)|^4 dx\right)^{1/4} \le C\varepsilon^{1/2}. \tag{5-3}$$

Let

$$\Omega(x) = \big(\Omega_1(x), \Omega_2(x), \dots, \Omega_D(x)\big) = \Phi(x) - x, \quad x \in \mathbb{R}^D. \tag{5-4}$$

Thus (5-3) asserts that

$$\left(\int_{B(0,1)} |\nabla\Omega(x)|^4 \, dx\right)^{1/4} \le C\varepsilon^{1/2}. \tag{5-5}$$

We know that

$$\left|(\Phi'(x))^T \Phi'(x) - I\right| \le C\varepsilon, \quad x \in \mathbb{R}^D. \tag{5-6}$$

In coordinates, $\Phi'(x)$ is the matrix $\left(\delta_{ij} + \dfrac{\partial\Omega_i(x)}{\partial x_j}\right)$; hence $\Phi'(x)^T \Phi'(x)$ is the matrix whose $ij$-th entry is

$$\delta_{ij} + \frac{\partial\Omega_j(x)}{\partial x_i} + \frac{\partial\Omega_i(x)}{\partial x_j} + \sum_l \frac{\partial\Omega_l(x)}{\partial x_i}\frac{\partial\Omega_l(x)}{\partial x_j}.$$

Thus (5-6) says that

$$\left|\frac{\partial\Omega_j}{\partial x_i} + \frac{\partial\Omega_i}{\partial x_j} + \sum_l \frac{\partial\Omega_l}{\partial x_i}\frac{\partial\Omega_l}{\partial x_j}\right| \le C\varepsilon \tag{5-7}$$

on $\mathbb{R}^D$, $i, j = 1, \dots, D$. Thus, we have from (5-5), (5-7) and the Cauchy–Schwarz inequality the estimate

$$\left\|\frac{\partial\Omega_i}{\partial x_j} + \frac{\partial\Omega_j}{\partial x_i}\right\|_{L^2(B(0,10))} \le C\varepsilon.$$

By the PDE Theorem, there exists, for each $i, j$, an antisymmetric matrix $S = (S)_{ij}$, such that

$$\left\|\frac{\partial\Omega_i}{\partial x_j} - S\right\|_{L^2(B(0,1))} \le C\varepsilon. \tag{5-8}$$

Recalling (5-4), this is equivalent to

$$\left\|\Phi' - (I + S)\right\|_{L^2(B(0,1))} \le C\varepsilon. \tag{5-9}$$

Note that (5-5) and (5-8) show that

$$|S| \leq C\varepsilon^{1/2}$$

and thus,

$$|\exp(S) - (I + S)| \leq C\varepsilon.$$

Hence, (5-9) implies via Cauchy–Schwarz.

$$\int_{B(0,1)} \left|\Phi'(x) - \exp(S)(x)\right| dx \leq C\varepsilon^{1/2}. \tag{5-10}$$

This implies the result because $S$ is antisymmetric, which means that $\exp S \in O(D)$. □

## 6. A BMO theorem for diffeomorphisms of small distortion

**Theorem.** *Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$ distorted diffeomorphism. Let $B \subset \mathbb{R}^D$ be a ball. Then, there exists $T_B \in O(D)$ such that for every $\lambda \geq 1$,*

$$\mathrm{vol}\left\{x \in B : |\Phi'(x) - T_B| > C\lambda\varepsilon\right\} \leq \exp(-\lambda)\mathrm{vol}\,(B). \tag{6-1}$$

*Moreover, the result (6-1) is sharp in the sense of small volume if one takes a slow twist defined as follows: For $x \in \mathbb{R}^D$, let $S_x$ be the block-diagonal matrix*

$$\begin{pmatrix} D_1(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & D_2(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & D_r(x) \end{pmatrix}$$

*where, for each $i$, either $D_i(x)$ is the $1 \times 1$ identity matrix or else*

$$D_i(x) = \begin{pmatrix} \cos f_i(|x|) & \sin f_i(|x|) \\ -\sin f_i(|x|) & \cos f_i(|x|) \end{pmatrix}$$

*for a function $f_i$ of one variable.*

*Now define for each $x \in \mathbb{R}^D$, $\Phi(x) = \Theta^T S_x(\Theta x)$ where $\Theta$ is any fixed matrix in $SO(D)$. One checks that $\Phi$ is $\varepsilon$-distorted, provided for each $i$, $t|f_i'(t)| < c\varepsilon$ for all $t \in [0, \infty)$.*

*Proof.* The theorem follows from BMO Theorem 2 and the Nirenberg inequality. The sharpness can be easily checked. □

## 7. On the approximate and exact alignment of data in Euclidean space, speech and music manifolds

***Approximate and exact alignment of data.*** A classical problem in geometry goes as follows. Suppose we are given two sets of $D$-dimensional data, that is, sets of points in Euclidean $D$-space, where $D \geq 1$. The data sets are indexed by the same set, and we know that pairwise distances between corresponding points are equal in the two data sets. In other words, the sets are isometric. Can this correspondence be extended to an isometry of the ambient Euclidean space?

In this form the question is not terribly interesting; the answer has long known to be yes (see [Wells and Williams 1975], for example). But a related question is actually fundamental in data analysis: here the known points are samples from larger, unknown sets — say, manifolds in $\mathbb{R}^D$ — and we seek to know what can be said about the manifolds themselves. A typical example might be a face recognition problem, where all we have is multiple finite images of people's faces from various views.

An added complication is that in general we are not given exact distances. We have noise and so we need to demand that instead of the pairwise distances being equal, they should be close in some reasonable metric. Some results on almost isometries in Euclidean spaces can be found in [John 1961; Alestalo et al. 2003].

In [Fefferman and Damelin $\geq$ 2012], the following two theorems are established which tell us about how to handle manifold identification when the point set function values given are not exactly equal but are close.

**Theorem.** *Given $\varepsilon > 0$ and $k \geq 1$, there exists $\delta > 0$ such that the following holds. Let $y_1, \dots, y_k$ and $z_1, \dots, z_k$ be points in $\mathbb{R}^D$. Suppose*

$$(1+\delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq 1+\delta, \quad i \neq j.$$

*Then, there exists a Euclidean motion $\Phi_0 : x \to Tx + x_0$ such that*

$$|z_i - \Phi_0(y_i)| \leq \varepsilon \operatorname{diam}\{y_1, \dots, y_k\}$$

*for each $i$. If $k \leq D$, then we can take $\Phi_0$ to be a proper Euclidean motion on $\mathbb{R}^D$.*

**Theorem.** *Let $\varepsilon > 0$, $D \geq 1$ and $1 \leq k \leq D$. Then there exists $\delta > 0$ such that the following holds: Let $E := y_1, \dots, y_k$ and $E' := z_1, \dots z_k$ be distinct points in $\mathbb{R}^D$. Suppose that*

$$(1+\delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq (1+\delta), \quad 1 \leq i, j \leq k,\ i \neq j.$$

*Then there exists a diffeomorphism $\Psi : \mathbb{R}^D \to \mathbb{R}^D$ with*

$$(1+\varepsilon)^{-1} \leq \frac{|\Psi(x)-\Psi(y)|}{|x-y|} \leq (1+\varepsilon), \quad x, y \in \mathbb{R}^D,\ x \neq y$$

*satisfying*

$$\Psi(y_i) = z_i, \quad 1 \leq i \leq k.$$

The theorem above shows that any $1+\delta$ bilipchitz mapping $\Phi$ of $1 \leq k \leq D$ points from $\mathbb{R}^D$ to $\mathbb{R}^D$ may be extended to a $1+\varepsilon$ bilipchitz diffeomorphism of $\mathbb{R}^D$ to $\mathbb{R}^D$.

Given the two theorems above, we now need to ask ourselves. Can we take, in any particular data application, an $\varepsilon$-distorted map and replace it by a Euclidean motion or visa versa. Clearly this is very important since the theorems themselves provide in the once case a Euclidean motion and in the other a diffeomorphism of small distortion. We understand that our main BMO theorems tell us that at least in measure, diffeomorphisms of small distortion are very close to Euclidean motions motions.

***Speech and music manifolds.*** Recently (see [Damelin and Miller 2012] and the references cited therein) there has been much interest in geometrically motivated dimensionality reduction algorithms. The reason for this is that these algorithms exploit low dimensional manifold structure in certain natural datasets to reduce dimensionality while preserving categorical content. In [Jansen and Niyogi 2006], the authors motivated the existence of low dimensional manifold structure to voice and speech sounds. As an immediate application of our results from this paper and from [Fefferman and Damelin ≥ 2012], we are now able to answer the following question related to speech and music manifolds. Suppose that we are given two collections of data functions in time which arise from vocal tract functions used in speech and music production. These manfolds exist; see the results of [Jansen and Niyogi 2006]. Suppose that all we know is that the functions are the same within a small $\delta$ distrortion. Then what can one say about the manifolds themselves. For example, can one identify different musical instruments or people/animals via speech using Euclidean motions or diffeomorphisms of $\varepsilon$ distortion? What can one say about the differences in measure between the Euclidean motions or diffeomorphisms themselves? The theorems proved in this paper and in [Fefferman and Damelin ≥ 2012] provide a fascinating insight into these very interesting questions.

## References

[Alestalo et al. 2003] P. Alestalo, D. A. Trotsenko, and J. Väisälä, "The linear extension property of bi-Lipschitz mappings", *Sibirsk. Mat. Zh.* **44**:6 (2003), 1226–1238. In Russian; translated as Linear bilipschitz extension property in *Sib. Math. J.* **44**:6 (2003), 959–968. MR 2004k:30042 Zbl 1063.30019

[Damelin and Miller 2012] S. B. Damelin and W. Miller, Jr., *The mathematics of signal processing*, Cambridge University Press, Cambridge, 2012. MR 2883645 Zbl 06002972

[Fefferman and Damelin $\geq$ 2012] C. Fefferman and S. B. Damelin, "On the approximate and exact alignment of $\delta$ distorted data in Euclidean space I", manuscript.

[Jansen and Niyogi 2006] A. Jansen and P. Niyogi, "Intrinsic Fourier analysis on the manifold of speech sounds", in *Acoustics, speech, and signal processing: proceedings of ICASSP* (Toulouse, 2006), IEEE, Piscataway, NJ, 2006.

[John 1961] F. John, "Rotation and strain", *Comm. Pure Appl. Math.* **14** (1961), 391–413. MR 25 #1672 Zbl 0102.17404

[John and Nirenberg 1961] F. John and L. Nirenberg, "On functions of bounded mean oscillation", *Comm. Pure Appl. Math.* **14** (1961), 415–426. MR 24 #A1348 Zbl 0102.04302

[Wells and Williams 1975] J. H. Wells and L. R. Williams, *Embeddings and extensions in analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete **84**, Springer, New York, 1975. MR 57 #1092 Zbl 0324.46034

Received: 2011-04-01 Revised: 2011-10-06 Accepted: 2011-10-08

cf@math.princeton.edu *Department of Mathematics, Princeton University, Princeton, NJ 08544, United States*

steve.damelin@gmail.com *Department of Mathematics, Wayne Country Day School, Goldsboro, NC 27530, United States*

*Department of Mathematics and Physics, Wake Technical Community College, Raleigh, NC 27603, United States*

will25655@yahoo.com *Department of Music, Albany State University, Albany, GA 31701, United States*

# involve

msp.berkeley.edu/involve





See inside back cover or http://msp.berkeley.edu/involve for submission instructions.

The subscription price for 2012 is US $105/year for the electronic version, and $145/year (+$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

# involve

2012 vol. 5 no. 2