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A bifurcus semigroup is a semigroup in which every nonunit nonatom can be written as the product of exactly two atoms. We generalize this notion to k-furcus semigroups: every element that can be factored as the product of at least k nonunits can be factored as the product of exactly k atoms. We compute some factorization-theoretic invariants of k-furcus semigroups that generalize the bifurcus results. We then define two variations on the k-furcus property, one stronger (presumabaly strictly) and the other strictly weaker than the k-furcus property.

1. Introduction

Vadim Ponomarenko and coworkers [Adams et al. 2009] introduced and studied the notion of *bifurcus semigroups*, a class of semigroups with "bad" factorization properties: a semigroup S is bifurcus if every nonunit nonatom can be bifurcated, that is, expressed as the product of exactly two atoms in S. They gave examples, showed that certain important families of semigroups cannot be bifurcus, and calculated several factorization-theoretic invariants of bifurcus semigroups. Further examples of bifurcus semigroups can be found in [Baeth et al. 2011]. Our goal is to generalize the concept of bifurcus and to provide analogous results for what we call *k-furcus semigroups*. We also give in Section 3 two modified definitions, one which appears to strengthen and one which weakens the original definition. Finally, in Section 4, we consider irreducible divisor graphs — a graphical interpretation of the factorization of an element in a semigroup — of elements in k-furcus semigroups.

First, some basic background. The reader is referred to [Geroldinger and Halter-Koch 2006] for a thorough treatment of factorization theory.

A *semigroup* is a set together with an associative operation. A nonunit a of a semigroup S is an *atom* if it is impossible to write $a = b \cdot c$ with b and c nonunits. The set $\mathcal{A}(S)$ denotes the set of all atoms of S. We will restrict our attention to *atomic* semigroups, those in which every element can be expressed as a (finite)

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product of atoms. We now define several important invariants which describe how unique or nonunique factorization is in a given semigroup.

An element $a \in S$ is a *strong atom* if whenever $a^m = bc$ for some $b, c \in S$ with $b \neq 1$, then $b = \epsilon a^n$ for some unit ϵ and some integer $n \leq m$. If $x \in S$, then $\mathfrak{L}(x) = \{n : x = a_1 a_2 \cdots a_n$ with each a_i an atom of $S\}$ is called the *set of factorization lengths* of x. The *elasticity* of an element x, defined by

$$\rho(x) = \frac{\sup \mathfrak{L}(x)}{\inf \mathfrak{L}(x)},$$

gives a coarse measure of how far away x is from having unique factorization. The *elasticity* of the semigroup S is then $\rho(S) = \sup\{\rho(x) : x \text{ is a nonunit of S}\}$. If $\mathfrak{L}(x) = \{t_1, t_2, \ldots\}$ is the set of factorization lengths of x with $t_i < t_{i+1}$ for each i, the *delta set* of x is defined to be $\Delta(x) = \{t_{i+1} - t_i : t_i, t_{i+1} \in \mathfrak{L}(x)\}$ and $\Delta(S) = \bigcup \Delta(x)$. If $\Lambda = \{\min \mathfrak{L}(x) : x$ is a nonunit of $S\}$, then the *critical length* of S is $cr(S) = \max \Lambda + 1$. The *catenary degree* of S, denoted $C(S)$, is the smallest integer N such that for all $a \in S$, and for any two factorizations z and z' of a, there exists factorizations $z_0, \ldots, z_k$ of a such that for all $i \in [1, k]$, z_i arises from z_{i-1} by replacing at most N atoms from z_{i-1} by at most N new atoms; that is, $d(z_i, z_{i-1}) \leq N$. Finally, we define the *tame degree* $t(S)$ of the semigroup S to be the smallest natural number N such that whenever $a \in S$ and x is an atom of S occurring in some factorization of a, given a factorization z of a, there exists a factorization z' of a containing x such that $d(z, z') \leq N$.

2. *k*-furcus semigroups

Let S be an atomic semigroup and let $k \geq 2$ be an integer. We say S is *k-furcus* if whenever an element can be factored as a product of at least k nonunits, then it can be factored as the product of exactly k atoms of S. Note that when $k = 2$, a semigroup is k-furcus if and only if it is bifurcus.

The following result generalizes (2)–(9) of [Adams et al. 2009, Theorem 1.1] for k-furcus semigroups, and can be proved by straightforward modifications of the arguments in that paper.

Theorem 2.1. *Let S be a nontrivial k-furcus semigroup, and let $0 \neq x \in S$ be a nonunit nonatom. Then*:

(1) *S contains no strong atoms.*

(2) *If $k \geq \sup \mathfrak{L}(x)$, then $[k, \sup \mathfrak{L}(x)] \subseteq \mathfrak{L}(x) \subseteq [2, \sup \mathfrak{L}(x)]$.*

(3) $k\rho(x) \in \mathbb{N} \cup \{\infty\}$.

(4) $\rho(S) = \infty$.

(5) (a) *If* $k \in \{2, 3\}$, *then* $\Delta(S) = \{1\}$.
(b) *If* $k > 3$, *then* $\{1\} \subseteq \Delta(S) \subseteq \{1, 2, \ldots, k-2\}$.

(6) $3 \le C(S) \le k+1$.

(7) $t(S) = \infty$.

(8) $3 \le cr(S) \le k+1$.

We note that the statements (2), (5), and (8) of Theorem 2.1 are strictly weaker than their analogs (3), (6), and (9) from [Adams et al. 2009, Theorem 1.1]. Any improvements on these statements would require knowledge of how elements with no factorizations of length k or greater can be written as products of atoms.

Suppose that S is a k-furcus semigroup and that m is an integer larger than k. Further suppose that $x \in S$ can be written as the product of at least m nonunits of S. Then $k \le m \le \sup \mathfrak{L}(x)$ and thus, by Theorem 2.1(2), $m \in \mathfrak{L}(x)$ and so x can be factored into exactly m atoms. Therefore:

Corollary 2.2. *If a semigroup S is k-furcus then S is m-furcus for every $m \ge k$.*

Example 2.3. As shown in [Adams et al. 2009, Section 2], the following semigroups are bifurcus, and hence (by Corollary 2.2) k-furcus for all $k \ge 2$:

(1) $n\mathbb{Z}$, where n is not a prime power;

(2) $m\mathbb{Z} \times n\mathbb{Z}$ for natural numbers $m, n > 1$;

(3) the multiplicative subsemigroup of $n \times n$ matrices with all entries identical integers, for n not a prime power.

To be more concrete, consider the semigroup $n\mathbb{Z}$, where $n = pqr$ with p and q distinct primes, and suppose that $x \in n\mathbb{Z}$ can be written as $x = (nx_1)(nx_2)\cdots(nx_k)$ for some $k \ge 2$. Then we can factor x in $\mathbb{Z}$ as $x = p^a q^b r^k s$ where $a, b \ge k$ and $p, q \nmid s$. Then we can factor x as $x = (np^{a-k}s)(nq^{b-k})(n)^{k-2}$ as a product of exactly k atoms of $n\mathbb{Z}$. Therefore, $n\mathbb{Z}$ is k-furcus for all $k \ge 2$.

We now provide an example of a k-furcus semigroup that is not m-furcus for any $m < k$, showing that the term k-furcus properly extends the term bifurcus. We thank Vadim Ponomarenko (private communication) for providing this example.

Example 2.4. Consider k distinct primes $p_1, p_2, \ldots, p_k$ and let $N = p_1 p_2 \cdots p_k$. Define S to be the multiplicative semigroup with the infinitely many generators $Np_1^{a_1}, Np_2^{a_2} \ldots, Np_k^{a_k}$, where each a_i is a nonnegative integer. If x is an element of S that can be written as the product of at least k elements of S, then $x = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for some $a_1, a_2, \ldots, a_k$. Note that $a_i \ge k$ for each i because $N = p_1 p_2 \cdots p_k$ divides (in $\mathbb{Z}$) every element in S. Now we can write $x = (Np_1^{(a_1-k)})(Np_2^{(a_2-k)})\cdots(Np_k^{(a_k-k)})$ as a product of exactly k atoms and hence S is k-furcus. However, we shall see that it is impossible to write x as the product of less than k atoms.

Suppose $x = b_1 b_2 \cdots b_{k-1}$, where each b_i is an atom of S. Since p_1 occurs at least k times in the factorization of x, it must occur at least twice in a factorization of some b_i and hence each factorization of x must contain, for each i, an atom of the form $Np_i^{c_i}$ with $c_i \geq 1$. Thus every factorization of x must have length at least k. Since S is k-furcus we know that N will occur at least k times in any factorization of x. By factoring x into $k-1$ elements we can see that N will still have to occur k times, so x can not be written as a product of atoms of length less than k.

In [Adams et al. 2009], it is shown that several important families of semigroups are not bifurcus. Straightforward modifictions show that these same families of semigroups cannot be k-furcus for any $k \geq 2$.

Proposition 2.5. *These classes of semigroups are not k-furcus for any $k \geq 2$:*

(1) *block monoids $B(G_0)$ for any subset G_0 of an abelian group G*;

(2) *Krull monoids*;

(3) *diophantine monoids.*

3. Variations of k-furcus semigroups

In this section we consider variations of k-furcus semigroups, one weaker and one stronger.

We call a semigroup S *quasi k-furcus* if every nonunit has a factorization of length at most k. This definition is motivated by the following example, which follows from [Banister et al. 2007, Theorem 2.3]:

Example 3.1. If $M(a, b) = \{a + kb : k \in \mathbb{N}_0\} \cup \{1\}$ is an arithmetical congruence monoid with $\gcd(a, b)$ not a prime power, then $M(a, b)$ is quasi k-furcus for some k.

If S is k-furcus, then whenever an element can be factored into at least k elements, it can be factored as the product of exactly k atoms of S; thus every nonunit has a factorization of length at most k. This yields:

Proposition 3.2. *A k-furcus semigroup S is also quasi k-furcus.*

The converse to Proposition 3.2 is false, as the following example illustrates.

Example 3.3. Consider the arithmetical congruence monoid $S = M(6, 30)$ (in the notation of Example 3.1); its first few elements are $1, 6, 36, 66, 96, 126, 156, \ldots$. From the proof of Theorem 2.3 in [Banister et al. 2007] we know that S is quasi 15-furcus. We will now consider all factorizations of the element 6^{16} in S. In $\mathbb{N}$, 6^{16} factors as $6^{16} = 2^{16}3^{16}$. Since elements of S are multiples of 6 that are congruent to 1 modulo 5, the only factorizations of 6^{16} in S are

$$6^{16}, \quad 96 \cdot 486 \cdot 6^{10}, \quad 1536 \cdot 39366 \cdot 6^{6}, \quad \text{and} \quad 24576 \cdot 3188646 \cdot 6^{2}.$$

Therefore, $\mathcal{L}(6^{16}) = \{4, 8, 12, 16\}$ and S is quasi 15-furcus but not 15-furcus.

In spite of the nonequivalence of k-furcus and quasi k-furcus semigroups, these classes of semigroups share many properties. We illustrate this in Theorem 3.4, which parallels Theorem 2.1 in both statement and proof.

Theorem 3.4. *Let S be a nontrivial quasi k-furcus semigroup and let $0 \neq x \in S$ be a nonunit nonatom. Then*:

(1) *S contains no strong atoms.*

(2) $\rho(S) = \infty$.

(3) $C(S) \leq k+1$.

(4) $3 \leq cr(S) \leq k+1$.

We now give a definition that appears stronger than that of k-furcus, although we have no examples to justify this claim. From Example 2.4, the semigroup with generators $Np_1^{a_1}, Np_2^{a_2}, \ldots, Np_k^{a_k}$ has the property that every factorization of an element x with $\mathfrak{L}(x) \geq k$ must have length at least k. This motivates the following definition. We call a semigroup S *strongly k-furcus* if S is k-furcus and no element that can be written as the product of k atoms can be written as the product of less than k atoms. The following theorem gives improvements to Theorem 2.1 when S is strongly k-furcus.

Theorem 3.5. *Let S be a nontrivial strongly k-furcus semigroup*, *and let $0 \neq x \in S$ be a nonunit nonatom. Then*:

(1) $\mathfrak{L}(x) = [k, \sup \mathfrak{L}(x)]$ *or* $\mathfrak{L}(x) \subseteq \{2, 3, \ldots, k-1\}$.

(2) $\Delta(x) = \{1\}$ *or* $\Delta(x) \subseteq \{1, 2, \ldots, k-3\}$.

(3) $\{1\} \subseteq \Delta(S) \subseteq \{1, 2, \ldots, k-3\}$.

We now give an analog to [Adams et al. 2009, Theorem 1.1(1)] when S is strongly k-furcus. This analog was omitted from our Theorem 2.1 since the hypothesis of S being k-furcus but not strongly k-furcus seems not to be enough information to guarantee these results. Again, we point out that we have no example of a k-furcus semigroup that is not also strongly k-furcus.

Proposition 3.6. *If S is a nontrivial strongly k-furcus semigroup and is either left or right cancellative*, *then S contains infinitely many atoms.*

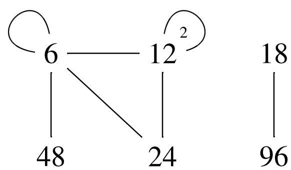
4. Irreducible divisor graphs

In this section we give a means of visually representing the factorization of an element in a k-furcus semigroup. The concept of the irreducible divisor graph of an element in a commutative integral domain was introduced in [Coykendall and Maney 2007] and further studied in [Axtell et al. 2011]. We now give a similar definition for the irreducible divisor graph of an element in a multiplicative semigroup. Given a semigroup S and an element $x \in S$, the *irreducible divisor graph of* x, denoted $G(x)$, is defined as follows. The vertices of $G(x)$ are the atoms $a \in S$ such that $a \mid x$. Two vertices a and b of $G(x)$ are connected by an edge provided $ab \mid x$. Moreover, we place n loops (for $n > 1$ this is denoted by a single loop labeled with an n) on vertex a if $a^n \mid x$ but yet $a^{n+1} \nmid x$. We now provide two examples to illustrate this definition.

Example 4.1. Consider the element $y = 1728$ in the multiplicative bifurcus semigroup $S = 6\mathbb{Z}$. The only factorizations of y in S are

$$\begin{aligned} 1728 &= 6 \cdot 6 \cdot 48 \\ &= 12 \cdot 12 \cdot 12 \\ &= 6 \cdot 12 \cdot 24 \\ &= 18 \cdot 96. \end{aligned}$$

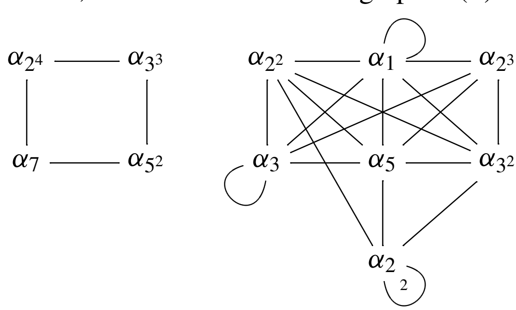
Therefore, $G(y)$, the irreducible divisor graph of y in S is



Example 4.2. Let S be the strongly 4-furcus multiplicative semigroup with generators $210 \cdot 2^{a_1}, 210 \cdot 3^{a_2}, 210 \cdot 5^{a_3}, 210 \cdot 7^{a_4}$, where each a_i is a nonnegative integer given in Example 2.4. Let $x = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7^5 \in S$ and note that x factors only as

$$\begin{aligned} x &= (210 \cdot 2^4)(210 \cdot 3^3)(210 \cdot 5^2)(210 \cdot 7) \\ &= (210 \cdot 2^3)(210 \cdot 3^2)(210 \cdot 5)(210)(210) \\ &= (210 \cdot 2^3)(210 \cdot 3)(210 \cdot 5)(210 \cdot 3)(210) \\ &= (210 \cdot 2^2)(210 \cdot 3^2)(210 \cdot 5)(210 \cdot 2)(210) \\ &= (210 \cdot 2^2)(210 \cdot 3)(210 \cdot 5)(210 \cdot 2)(210 \cdot 3) \\ &= (210 \cdot 2)(210 \cdot 3^2)(210 \cdot 5)(210 \cdot 2)(210 \cdot 2) \end{aligned}$$

Setting $\alpha_{b^j} := 210 \cdot b^j$, the irreducible divisor graph $G(x)$ is



The fundamental result in the theory of irreducible divisor graphs, proved in [Coykendall and Maney 2007; Axtell et al. 2011] tells us that an atomic integral domain R is a UFD if and only if $G(x)$ is connected (equivalently, complete) for all nonunits $x \in R$. In fact, the proof of this result goes through for any commutative, cancellative semigroup. As should be no surprise, the examples above give disconnected graphs. In fact, the following theorem illustrates that this is nearly always the case for strongly k-furcus semigroups, thus giving another demonstration of how "bad" factorization is in k-furcus semigroups.

Theorem 4.3. *Let S be a commutative, cancellative strongly k-furcus semigroup. Then $G(x)$ is disconnected for every nonatom, nonunit x of S with* $\sup \mathfrak{L}(x) > k$.

Proof. Divide the set of vertices of $G(x)$ into two subsets:

$$A = \{a \in \mathcal{A}(S) : x = aa_1a_2 \cdots a_{k-1};\, a_i \in \mathcal{A}(S)\},$$

containing the vertices involved in factorizations of length k, and

$$B = \{b \in \mathcal{A}(S) : x = bb_1b_2 \cdots b_m;\, b_i \in \mathcal{A}(S), m \geq k\},$$

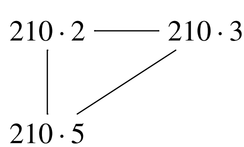
containing those involved in factorizations of length greater than k. Assume $b \in \mathcal{A}(S)$ with $b \in A \cap B$.

Since $b \in A$, $x = bc_1c_2 \cdots c_t$, where $t = k-1$. Since $b \in B$, $x = bd_1 \cdots d_s$, $s \geq k$. Thus $\frac{x}{b} = c_1c_2 \cdots c_t = d_1d_2 \cdots d_s$ has a factorization of length greater than or equal to k and a factorization of length less than k, which is impossible since S is strongly k-furcus. Therefore $A \cap B = \varnothing$.

Now assume $a \in A$ and $b \in B$ with an edge connecting a and b in $G(x)$. Then $x = abc_1 \cdots c_t$ with c_i atoms of S. If $t = k-2$, then $b \in A$, a contradiction since $b \in B$. If $t > k-2$, then $a \in B$, a contradiction since $a \in A$. Therefore no edges connect vertices in A with vertices in B, and hence $G(x)$ is disconnected. □

The requirement that $\sup \mathfrak{L}(x) > k$ is necessary as the following example illustrates.

Example 4.4. Consider the element $x = 2^4 \cdot 3^4 \cdot 5^4 \cdot 7^3$ in the strongly 4-furcus semigroup from Example 4.2 which factors only as $x = (210\cdot 2)(210\cdot 3)(210\cdot 5)$ with $\alpha_{b^j} = 210 \cdot b^j$. Since x has no factorization of length greater than 3, its irreducible divisor graph, shown below is connected.



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References

[Adams et al. 2009] D. Adams, R. Ardila, D. Hannasch, A. Kosh, H. McCarthy, V. Ponomarenko, and R. Rosenbaum, "Bifurcus semigroups and rings", *Involve* **2**:3 (2009), 351–356. MR 2011b:20161 Zbl 1190.20046

[Axtell et al. 2011] M. Axtell, N. R. Baeth, and J. Stickles, "Irreducible divisor graphs and factorization properties of domains", *Comm. Algebra* **39**:11 (2011), 4148–4162. MR 2855118

[Baeth et al. 2011] N. R. Baeth, V. Ponomarenko, D. Adams, R. Ardila, D. Hannasch, A. Kosh, H. McCarthy, and R. Rosenbaum, "Number theory of matrix semigroups", *Linear Algebra Appl.* **434**:3 (2011), 694–711. MR 2012c:11265 Zbl 05833978

[Banister et al. 2007] M. Banister, J. Chaika, S. T. Chapman, and W. Meyerson, "On the arithmetic of arithmetical congruence monoids", *Colloq. Math.* **108**:1 (2007), 105–118. MR 2007m:20096 Zbl 1142.20038

[Coykendall and Maney 2007] J. Coykendall and J. Maney, "Irreducible divisor graphs", *Comm. Algebra* **35**:3 (2007), 885–895. MR 2008a:13001 Zbl 1114.13001

[Geroldinger and Halter-Koch 2006] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations: algebraic, combinatorial and analytic theory*, Pure and Applied Mathematics **278**, CRC, Boca Raton, FL, 2006. MR 2006k:20001 Zbl 1113.11002

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