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Theoretical properties of the length-biased inverse Weibull distribution

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We investigate the length-biased inverse Weibull (LBIW) distribution, deriving its density function, hazard and reverse hazard functions, and reliability function. The moments, moment-generating function, Fisher information and Shannon entropy are also given. We discuss parameter estimation via the method of moments and maximum likelihood, and hypothesis testing for the LBIW and parent distributions.

1. Introduction

Weighted distributions occur in many areas, including medicine, ecology, reliability, and branching processes. Results and applications in these and other areas can be seen in [Patil and Rao 1978; Gupta and Kirmani 1990; Gupta and Keating 1986; Oluyede 1999]. In a weighted distribution problem, a realization x of X enters into the investigator's record with probability proportional to a weight function $w(x)$. The recorded x is not an observation of X, but rather an observation of a weighted random variable X_w.

In this article we are interested in the case where $w(x) = x$. This is called *length bias*; it approximates situations common in practice (see [Arratia and Goldstein 2009] for an introductory discussion). We will apply length bias to the inverse Weibull distribution (see Section 2 below), which has a wide range of applications in diverse areas such as medicine, reliability and ecology; for example, Keller et al. [1985] found it to be a good fit in their investigation of failures of mechanical components subject to degradation. As a result, the inverse Weibull distribution is well studied; see [Johnson et al. 1994] or [Rinne 2009] for a tabulation of results.

To proceed, we need some standard terminology. If X is a continuous, nonnegative random variable with distribution function F and probability density function (pdf) f (so that $f(u) = dF(u)/du$), we call $\bar{F}(x) = 1 - F(x)$ the associated *reliability function*, from the situation where $\bar{F}(x)$ describes the probability that

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some piece of equipment, say, will still be working at time x. The *hazard function* $\lambda_F(x)$ and *mean residual life function* $\delta_F(x)$ are defined by

$$\lambda_F(x) = \frac{f(x)}{\bar{F}(x)} \quad \text{and} \quad \delta_F(x) = \int_x^\infty \frac{\bar{F}(u)}{\bar{F}(x)}\, du. \tag{1}$$

The *reverse hazard function* is $\tau_F(x) = f(x)/F(x)$. When λ_F is monotone increasing, we say that F is an *increasing hazard rate* (*IHR*) *distribution*. Likewise, a *decreasing mean residual life* (*DMRL*) *distribution* is one where δ_F is monotone decreasing. It can be shown that IHR implies DMRL. IHR distributions have a number of nice properties, including finiteness of moments of all orders.

Now let $w(x)$, $x \geq 0$, be a positive function, and assume that the expectation of $w(X)$ is positive and finite:

$$0 < E[w(X)] := \int_0^\infty f(x) w(x)\, dx < \infty. \tag{2}$$

We define the weighted random variable X_w by specifying its pdf:

$$f_w(x) = \frac{w(x) f(x)}{E[w(X)]}, \quad x \geq 0. \tag{3}$$

(The denominator ensures that the total mass is 1.)

As mentioned, we will be interested in the case of *length bias*, where $w(x) = x$. In Section 2 we apply this weighting to the inverse Weibull distribution to obtain our main object of study, the LBIW (length-biased inverse Weibull) distribution. We briefly study the shape of the LBIW pdf. In Section 3 we calculate the LBIW moments and moment-generating function, together with the variance, skewness and kurtosis. Section 4 deals with Fisher information and Shannon entropy. In Section 5 we discuss the estimation of the parameters of an LBIW, and describe a test for the detection of length bias. Section 6 showcases a numerical example.

2. The inverse Weibull distribution and its length-biased version

The inverse Weibull distribution function is defined by

$$F(x; x_0, \alpha, \beta) = \exp\bigl(-(\alpha(x - x_0))^{-\beta}\bigr), \quad x \geq 0,\ \alpha > 0,\ \beta > 0, \tag{4}$$

where α, x_0 and β are the scale, location and shape parameters, respectively. We will consider only the case $x_0 = 0$, so our distribution function of departure is

$$F(x; \alpha, \beta) = \exp\bigl(-(\alpha x)^{-\beta}\bigr), \quad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{5}$$

(When $\alpha = 1$, this is known as the Fréchet distribution, and its value at $x = 1$ is independent of β; it equals $e^{-1} = 0.3679$, and is known as the characteristic life of the distribution.) By differentiation we get the corresponding pdf:

$$f(x;\alpha,\beta)=\beta\alpha^{-\beta}x^{-\beta-1}\exp\bigl(-(\alpha x)^{-\beta}\bigr),\quad x\ge 0,\ \alpha>0,\ \beta>0. \tag{6}$$

To introduce the length bias we first multiply this pdf by the weighting function $w(x)=x$, obtaining

$$\begin{aligned} xf(x;\alpha,\beta)&=\beta\alpha^{-\beta}x^{-\beta}\exp\bigl(-(\alpha x)^{-\beta}\bigr)\\ &=\beta F(x;\alpha,\beta)\bigl(-\log F(x;\alpha,\beta)\bigr),\quad x\ge 0,\ \alpha>0,\ \beta>0. \end{aligned}\tag{7}$$

As we saw in (3), we need to divide this function by its integral (2), which is of course the mean of the original distribution, denoted by μ_F. Evaluation yields

$$\mu_F=\frac{\Gamma\bigl(1-\frac{1}{\beta}\bigr)}{\alpha}.$$

Therefore the LBIW (length-biased inverse Weibull) pdf is

$$\begin{aligned} g_w(x;\alpha,\beta)&:=\frac{\alpha}{\Gamma\bigl(1-\frac{1}{\beta}\bigr)}\beta F(x;\alpha,\beta)\bigl(-\log F(x;\alpha,\beta)\bigr)\\ &=\frac{\beta\alpha^{-\beta+1}x^{-\beta}}{\Gamma\bigl(1-\frac{1}{\beta}\bigr)}\exp\bigl(-(\alpha x)^{-\beta}\bigr)\quad x\ge 0,\ \alpha>0,\ \beta>1. \end{aligned}\tag{8}$$

(We use the notation g_w instead of f_w as in (3) to make it more distinctive.) The corresponding distribution function is given by

$$G_w(x;\alpha,\beta)=\int_0^x g_w(u;\alpha,\beta)\,du=\frac{1}{\Gamma\bigl(1-\frac{1}{\beta}\bigr)}\int_0^{(\alpha x)^{-\beta}} y^{-1/\beta}\exp(-y)\,dy, \tag{9}$$

the last equality resulting from rewriting the integral in the variable $y=(\alpha u)^{-\beta}$.

We now turn to the shape of g_w. From (8) we see that $\lim_{x\to 0} g_w(x;\alpha,\beta)=0$ and $\lim_{x\to\infty} g_w(x;\alpha,\beta)=0$. Next we look for extrema. It is easier to work with the logarithmic derivative. Since

$$\eta_w(x):=\frac{\partial\log g_w(x;\alpha,\beta)}{\partial x}=\frac{\beta}{x}\bigl((\alpha x)^{-\beta}-1\bigr), \tag{10}$$

we see that an extremum requires that $(\alpha x)^{-\beta}=1$. Thus the only extremizer is $x=1/\alpha$; the pdf increases to a maximum at $1/\alpha$ and then decreases.

For the study of the hazard function it will be useful to consider the *second* derivative of $\log g_w(x;\alpha,\beta)$, namely

$$\eta'_w(x)=-\beta\frac{(\beta+1)(\alpha x)^{-\beta}-1}{x^2}. \tag{11}$$

The numerator on the right has only one zero, at $x=x^*:=(\beta+1)^{1/\beta}/\alpha$, so the same is true of η'_w. More precisely, we have

$$\begin{aligned} \eta'_w(x) &< 0 \quad \text{if } x < x^*, \\ \eta'_w(x) &= 0 \quad \text{if } x = x^*, \\ \eta'_w(x) &> 0 \quad \text{if } x > x^*. \end{aligned} \tag{12}$$

A criterion of Glaser [1980, Theorem on p. 668, case (d)(i), and Lemma on p. 669, case (iii)] then implies that the hazard function is "upside-down bathtub-shaped"; that is, it is initially increasing, reaches a maximum, and decreases thereafter. The conditions of the criterion are that the pdf is twice differentiable and positive for $x > 0$, that it tends to 0 as $x \to 0+$, and that the second derivative of its log satisfies (12) for some x^*. (Note that our η_w differs from Glaser's η by a sign.)

With the qualitative behavior of the hazard function in hand, there remains to write its formula. Recalling the definition in (1), we write

$$\overline{G}_w(x;\alpha,\beta) = \frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)} \int_x^\infty t^{-\beta} \exp(-(\alpha t)^{-\beta})\,dt \tag{13}$$

and

$$\lambda_{G_w}(x;\alpha,\beta) = \frac{g_w(x;\alpha,\beta)}{\overline{G}_w(x)} = \frac{x^{-\beta}\exp(-(\alpha x)^{-\beta})}{\int_x^\infty t^{-\beta}\exp(-(\alpha t)^{-\beta})\,dt}. \tag{14}$$

3. Moments and moment-generating function

In this section we derive the moments, moment-generating function, mean, variance, coefficients of variation, skewness, and kurtosis for the LBIW distribution.

The moments of a length-biased random variable X_w are related to those of the original or parent random variable X by

$$E_{G_w}[X_w^k] = \frac{E_F[X^{k+1}]}{E_F[X]}, \quad k = 1, 2, \ldots, \tag{15}$$

provided $E_F[X^{k+1}]$ exists. Noting that the moments of F are given by

$$E_F[X^k] = \gamma_k := \frac{\Gamma\left(1-\frac{k}{\beta}\right)}{\alpha^k}, \quad k \geq 1,\ \beta > k, \tag{16}$$

we obtain the moments of X_w as follows:

$$E_{G_w}[X_w^k] = \frac{\Gamma\left(1-\frac{k+1}{\beta}\right)}{\alpha^k\Gamma\left(1-\frac{1}{\beta}\right)} = \frac{\gamma_{k+1}}{\gamma_1}, \quad k \geq 1,\ \beta > k. \tag{17}$$

In particular, the mean of X_w is

$$\mu_{G_w} = E_{G_w}[X_w] = \frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\alpha\Gamma\left(1-\frac{1}{\beta}\right)} = \frac{\gamma_2}{\gamma_1} \tag{18}$$

and the variance is

$$\sigma^2_{G_w} = E_{G_w}[X_w^2] - E_{G_w}[X_w]^2 = \frac{\gamma_1\gamma_3 - \gamma_2^2}{\gamma_1^2}, \tag{19}$$

where $\gamma_k = \Gamma(1 - k/\beta)/\alpha^k$. The coefficient of variation (CV) is

$$\mathrm{CV} = \frac{\sigma_{G_w}}{\mu_{G_w}} = \sqrt{\frac{\gamma_3\gamma_1}{\gamma_2^2} - 1}. \tag{20}$$

The coefficients of skewness (CS) and kurtosis (CK) are given by

$$\mathrm{CS} = \frac{E[(X_w - \mu_{G_w})^3]}{E[(X_w - \mu_{G_w})^2]^{3/2}} = \frac{\gamma_1^2\gamma_4 - 3\gamma_1\gamma_2\gamma_3 + 2\gamma_2^3}{(\gamma_1\gamma_3 - \gamma_2^2)^{3/2}} \tag{21}$$

and

$$\mathrm{CK} = \frac{E[(X_w - \mu_{G_w})^4]}{E[(X_w - \mu_{G_w})^2]^2} = \frac{\gamma_1^3\gamma_5 - 4\gamma_1^2\gamma_2\gamma_4 + 6\gamma_1\gamma_2^2\gamma_3 - 3\gamma_2^4}{\gamma_1^2\gamma_3^2 - 2\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4}. \tag{22}$$

The moment-generating function is given by

$$\begin{aligned} M_{X_w}(t) &= \frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)} \int_0^\infty e^{ty} y^{-\beta} e^{-(\alpha y)^{-\beta}}\, dy \\ &= \frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^\infty y^{j-\beta} e^{-(\alpha y)^{-\beta}}\, dy = \frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \Psi_{j,\alpha,\beta}, \end{aligned} \tag{23}$$

where

$$\Psi_{j,\alpha,\beta} = \int_0^\infty y^{j-\beta} e^{-(\alpha y)^{-\beta}}\, dy.$$

4. Fisher information and Shannon entropy

The information (or Fisher information) that a random variable X contains about the parameter θ is given by

$$I(\theta) = E\left[\left(\frac{\partial}{\partial\theta} \log f(X,\theta)\right)^2\right]. \tag{24}$$

If, in addition, the second derivative with respect to θ of $f(x,\theta)$ exists for all x and θ, and if the second derivative with respect to θ of $\int f(x,\theta)\, dx = 1$ can be obtained by differentiating twice under the integral sign, then

$$I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2} \log f(X,\theta)\right]. \tag{25}$$

The Shannon entropy of a random variable X is a measure of the uncertainty and is given by $E_F[-\log f(X)]$, where $f(x)$ is the pdf of the random variable X.

For the LBIW distribution, the Fisher information that X_w (now renamed X for simplicity) contains about the parameters $\theta = (\alpha, \beta)$ is obtained as follows:

$$\begin{aligned}
&E\Big[\Big(\frac{\partial \log g_w(X;\alpha,\beta)}{\partial\alpha}\Big)^2\Big] \\
&\quad = \int_0^\infty \Big(\frac{1-\beta}{\alpha} + \beta\alpha^{-\beta-1}x^{-\beta}\Big)^2 g_w(x;\alpha,\beta)\,dx \\
&\quad = (1-\beta)^2\alpha^{-2}\int_0^\infty g_w(x;\alpha,\beta)\,dx + \frac{2\beta^2(1-\beta)\alpha^{-2\beta-1}}{\Gamma\big(1-\frac{1}{\beta}\big)}\int_0^\infty x^{-2\beta}e^{-(\alpha x)^{-\beta}}\,dx \\
&\qquad\qquad + \frac{\beta^3\alpha^{-3\beta-1}}{\Gamma\big(1-\frac{1}{\beta}\big)}\int_0^\infty x^{-3\beta}e^{-(\alpha x)^{-\beta}}\,dx \\
&\quad = (1-\beta)^2\alpha^{-2} + \frac{2\beta(1-\beta)\alpha^{-2}}{\Gamma\big(1-\frac{1}{\beta}\big)}\Gamma\big(2-\tfrac{1}{\beta}\big) + \frac{\beta^2\alpha^{-2}}{\Gamma\big(1-\frac{1}{\beta}\big)}\Gamma\big(3-\tfrac{1}{\beta}\big) \\
&\quad = \beta(\beta-1)\alpha^{-2},
\end{aligned} \tag{26}$$

$$\begin{aligned}
&E\Big[\Big(\frac{\partial \log g_w(X;\alpha,\beta)}{\partial\beta}\Big)^2\Big] \\
&\quad = \int_0^\infty \Big(\frac{1}{\beta} - \frac{\Gamma'\big(1-\frac{1}{\beta}\big)}{\beta^2\Gamma\big(1-\frac{1}{\beta}\big)} + \log(\alpha x)\big((\alpha x)^{-\beta}-1\big)\Big)^2 g_w(x;\alpha,\beta)\,dx \\
&\quad = \Big(\frac{1}{\beta} - \frac{\Gamma'\big(1-\frac{1}{\beta}\big)}{\beta^2\Gamma\big(1-\frac{1}{\beta}\big)}\Big)^2 - 2\Big(\frac{1}{\beta} - \frac{\Gamma'\big(1-\frac{1}{\beta}\big)}{\beta^2\Gamma\big(1-\frac{1}{\beta}\big)}\Big)\frac{\Gamma'\big(2-\frac{1}{\beta}\big) - \Gamma'\big(1-\frac{1}{\beta}\big)}{\beta\Gamma\big(1-\frac{1}{\beta}\big)} \\
&\qquad\qquad + \frac{\beta^2\big(\Gamma''\big(3-\frac{1}{\beta}\big) - 2\Gamma''\big(2-\frac{1}{\beta}\big) + \Gamma''\big(1-\frac{1}{\beta}\big)\big)}{\Gamma\big(1-\frac{1}{\beta}\big)},
\end{aligned} \tag{27}$$

$$\begin{aligned}
&E\Big[\frac{\partial^2 \log g_w(X;\alpha,\beta)}{\partial\alpha\,\partial\beta}\Big] = E\Big[\frac{\partial^2 \log g_w(X;\alpha,\beta)}{\partial\beta\,\partial\alpha}\Big] \\
&\quad = \int_0^\infty \Big(\alpha^{-\beta-1}x^{-\beta}(1-\beta\log\alpha-\beta\log x) - \frac{1}{\alpha}\Big) g_w(x;\alpha,\beta)\,dx \\
&\quad = \alpha^{-\beta-1}(1-\beta\log\alpha)\int_0^\infty x^{-\beta}g_w(x;\alpha,\beta)\,dx \\
&\qquad\qquad - \frac{\alpha^{-2\beta}\beta^2}{\Gamma\big(1-\frac{1}{\beta}\big)}\int_0^\infty x^{-2\beta}\log x e^{-(\alpha x)^{-\beta}}\,dx - \frac{1}{\alpha}\int_0^\infty g_w(x;\alpha,\beta)\,dx \\
&\quad = \alpha^{-1}\beta^{-2}(1-\beta) + \alpha^{-1}\beta^{-3}(\beta-1)\frac{\Gamma'\big(1-\frac{1}{\beta}\big)}{\Gamma\big(1-\frac{1}{\beta}\big)} = \frac{\beta-1}{\alpha\beta^3}\Big(\frac{\Gamma'\big(1-\frac{1}{\beta}\big)}{\Gamma\big(1-\frac{1}{\beta}\big)} - \beta\Big).
\end{aligned} \tag{28}$$

Thus the information matrix, namely

$$\boldsymbol{I}(\alpha,\beta)=\begin{pmatrix} E\left[\left(\dfrac{\partial\log g_w(X;\alpha,\beta)}{\partial\alpha}\right)^2\right] & E\left[\dfrac{\partial^2\log g_w(X;\alpha,\beta)}{\partial\alpha\,\partial\beta}\right] \\ E\left[\dfrac{\partial^2\log g_w(X;\alpha,\beta)}{\partial\beta\,\partial\alpha}\right] & E\left[\left(\dfrac{\partial\log g_w(X;\alpha,\beta)}{\partial\beta}\right)^2\right] \end{pmatrix}, \tag{29}$$

is given by

$$\boldsymbol{I}(\alpha,\beta)=\begin{pmatrix} E\left[\left(\dfrac{\partial\log g_w(X;\alpha,\beta)}{\partial\alpha}\right)^2\right] & \dfrac{\beta-1}{\alpha\beta^3}\left(\dfrac{\Gamma'\left(1-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}-\beta\right) \\ \dfrac{\beta-1}{\alpha\beta^3}\left(\dfrac{\Gamma'\left(1-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}-\beta\right) & E\left[\left(\dfrac{\partial\log g_w(X;\alpha,\beta)}{\partial\beta}\right)^2\right] \end{pmatrix}, \tag{30}$$

where the diagonal entries are stated in (26) and (27).

Note that, for fixed β, the top left entry of this matrix is monotonically decreasing in α, since

$$\frac{\beta(\beta-1)}{\alpha_1^2}\geq\frac{\beta(\beta-1)}{\alpha_2^2}\iff\alpha_2^2\geq\alpha_1^2\iff\alpha_2\geq\alpha_1. \tag{31}$$

On the other hand, for fixed α, the same function is monotonically increasing in β, since

$$\begin{aligned}\frac{\beta_1(\beta_1-1)}{\alpha^2}\geq\frac{\beta_2(\beta_2-1)}{\alpha^2}&\iff\beta_1(\beta_1-1)\geq\beta_2(\beta_2-1)\iff\beta_1^2-\beta_2^2-(\beta_1-\beta_2)\geq0\\&\iff(\beta_1-\beta_2)(\beta_1+\beta_2-1)\geq0\iff\beta_1\geq\beta_2,\end{aligned} \tag{32}$$

the last equivalence being a consequence of the inequalities $\beta_1>1,\ \beta_2>1$.

Under the LBIW distribution, the Shannon entropy is given by

$$\begin{aligned}&E_G(-\log g_w(X;\alpha;\beta))\\&=\int_0^\infty\left(-\log\frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)}+\beta\log x+(\alpha x)^{-\beta}\right)g_w(x;\alpha,\beta)\,dx\\&=-\log\frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)}+\beta\int_0^\infty(\log x)g_w(x;\alpha,\beta)\,dx+\int_0^\infty(\alpha x)^{-\beta}g_w(x;\alpha,\beta)\,dx\\&=-\log\frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)}+\beta\left(-\log\alpha-\frac{\Gamma'\left(1-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)}\right)+\frac{\beta-1}{\beta}\\&=\log\frac{\Gamma\left(1-\frac{1}{\beta}\right)}{\alpha\beta}-\frac{\Gamma'\left(1-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}+\frac{\beta-1}{\beta}.\end{aligned} \tag{33}$$

5. Estimation of parameters

In this section we derive formulas to estimate the parameters α and β for an unknown LBIW distribution. We also present a test for the detection of length bias in a sample.

(For the inverse Weibull parent distribution, Calabria and Pulcini [1990; 1994] derived maximum likelihood, least squares and Bayes estimates for the parameters. They also obtained confidence limits for reliability and tolerance limits for the same distribution [Calabria and Pulcini 1989].)

We continue to use X for the LBIW random variable whose parameters α and β we wish to estimate. We use two standard methods to obtain the estimators: the method of moments and maximum likelihood.

Method of moments estimators. The method of moments with two parameters involves setting the first two moments $E[X]$ and $E[X^2]$ equal to the corresponding moments of an independent sample $X_1, X_2, \ldots, X_n$ of the LBIW random variable. In view of (18) and (19), this leads to the equations

$$\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\alpha\Gamma\left(1-\frac{1}{\beta}\right)} = \frac{1}{n}\sum_{j=1}^{n} X_j \quad \text{and} \quad \frac{\Gamma\left(1-\frac{3}{\beta}\right)}{\alpha^2\Gamma\left(1-\frac{1}{\beta}\right)} = \frac{1}{n}\sum_{j=1}^{n} X_j^2. \tag{34}$$

These equations are then solved (numerically, for example) for α and β, leading to the estimators $\hat{\alpha}$ and $\hat{\beta}$.

If β is known, we only need the first equation in (34). In that case (i.e., for fixed $\beta > 1$), the method of moments estimate (MME) of α is given by

$$\hat{\alpha} = \frac{n}{\sum_{j=1}^{n} X_j} \frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}. \tag{35}$$

Maximum likelihood estimators. In this method we take the log-likelihood function of the distribution, take its partial derivatives with respect to the parameters, and equate their expectations to 0. The log-likelihood function for a single observation x of X is

$$\begin{aligned} l(\alpha,\beta) &= \log\left(\frac{\beta\alpha^{-\beta+1}}{\Gamma\left(1-\frac{1}{\beta}\right)} x^{-\beta}\exp(-(\alpha x)^{-\beta})\right) \\ &= \log\beta - (\beta-1)\log\alpha - \beta\log x - (\alpha x)^{-\beta} - \log\Gamma\left(1-\tfrac{1}{\beta}\right), \end{aligned} \tag{36}$$

which leads to

$$\frac{\partial l}{\partial\alpha} = -\frac{\beta-1}{\alpha} + \frac{\beta(\alpha x)^{-\beta}}{\alpha}, \tag{37}$$

$$\frac{\partial l}{\partial\beta} = \frac{1}{\beta} - \log\alpha - \log x + (\alpha x)^{-\beta}\log(\alpha x) + \frac{\Gamma'\left(1-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)}. \tag{38}$$

From $E[\partial l/\partial \alpha] = 0$, we obtain

$$E[X^{-\beta}] = \frac{\alpha^{\beta}(\beta - 1)}{\beta}, \tag{39}$$

and from $E[\partial l/\partial \beta] = 0$, we have

$$E[-\log X + (\alpha X)^{-\beta} \log(\alpha X)] = \log \alpha - \frac{1}{\beta} - \frac{\Gamma'\left(1-\frac{1}{\beta}\right)}{\beta^2 \Gamma\left(1-\frac{1}{\beta}\right)}. \tag{40}$$

The full log-likelihood function is given by

$$L(\alpha, \beta) = n \log \beta - n(\beta - 1) \log \alpha - \beta \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} (\alpha x_j)^{-\beta} - n \log \Gamma\left(1-\tfrac{1}{\beta}\right).$$

The normal equations are

$$\frac{\partial L(\alpha, \beta)}{\partial \alpha} = \frac{-n(\hat{\beta} - 1)}{\hat{\alpha}} + \hat{\beta}\hat{\alpha}^{-\hat{\beta}-1} \sum_{j=1}^{n} x_j^{-\hat{\beta}} = 0, \tag{41}$$

$$\frac{\partial L(\alpha, \beta)}{\partial \beta} = \frac{n}{\hat{\beta}} - n \log \hat{\alpha} - \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \frac{\log(\hat{\alpha} x_j)}{(\hat{\alpha} x_j)^{\hat{\beta}}} - \frac{n}{\hat{\beta}^2} \Psi(1-1/\hat{\beta}) = 0. \tag{42}$$

From (41), the MLE of α is

$$\hat{\alpha} = \left(\frac{n(\hat{\beta} - 1)}{\hat{\beta} \sum_{j=1}^{n} x_j^{-\hat{\beta}}} \right)^{-1/\hat{\beta}}. \tag{43}$$

Now replace $\hat{\alpha}$ in (42) to obtain

$$\begin{aligned}
\left.\frac{\partial L(\alpha, \beta)}{\partial \beta}\right|_{\hat{\alpha},\hat{\beta}} &= \frac{n}{\hat{\beta}} - n \log \left(\frac{n(\hat{\beta}-1)}{\hat{\beta} \sum_{j=1}^{n} x_j^{-\hat{\beta}}} \right)^{-1/\hat{\beta}} - \sum_{j=1}^{n} \log x_j \\
&\quad - \sum_{j=1}^{n} \left(\left(\frac{n(\hat{\beta}-1)}{\hat{\beta} \sum_{j=1}^{n} x_j^{-\hat{\beta}}} \right)^{-1/\hat{\beta}} x_j \right)^{-\hat{\beta}} \log \left(\left(\frac{n(\hat{\beta}-1)}{\hat{\beta} \sum_{j=1}^{n} x_j^{-\hat{\beta}}} \right)^{-1/\hat{\beta}} x_j \right) \\
&\quad - \frac{1}{\hat{\beta}^2} \sum_{j=1}^{n} \frac{\Gamma'(1-1/\hat{\beta})}{\Gamma(1-1/\hat{\beta})} = 0.
\end{aligned} \tag{44}$$

This equation does not have a closed form solution and must be solved iteratively to obtain the MLE of the scale parameter β. When α is unknown and β is known, the MLE of α is obtained from (41) with the value of β in place of $\hat{\beta}$. When both α and β are unknown the MLEs of α and β are obtained by solving the normal equations in (41) and (42). The MLEs of the reliability and hazard functions can be obtained by replacing α and β by their MLEs $\hat{\alpha}$ and $\hat{\beta}$.

The expectations in the Fisher information matrix (FIM) can be obtained numerically. Under the conditions that the parameters are in the interior of the parameter space, but not on the boundary, we have

$$\sqrt{n}\begin{pmatrix}\hat{\alpha}-\alpha\\ \hat{\beta}-\beta\end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, I^{-1}(\alpha,\beta)\right) \quad \text{as } n\to\infty,$$

where $I(\alpha,\beta)=\lim_{n\to\infty} n^{-1}I_n(\alpha,\beta)$ and

$$I_n(\alpha,\beta)=n\begin{pmatrix}I(1,1) & I(1,2)\\ I(2,1) & I(2,2)\end{pmatrix}.$$

The entries $I(i,j)$, $i=1,2$ and $j=1,2$, are given in (30). The multivariate normal distribution with mean vector $(0,0)^T$ and covariance matrix $I_n(\alpha,\beta)$ can be used to construct confidence intervals for the model parameters.

Test for generalized length bias. We now seek to discriminate whether a random variable, represented by a random sample of size n, is likely to be the result of length-biased sampling. More precisely, we compare the *null hypothesis* H_0, to the effect that the random variable has the inverse Weibull pdf (6) with given α and β, to the *alternative hypothesis* H_c, which says that the random variable is LBIW ($c=1$) or perhaps inverse Weibull with some other power weighting $w(x)=x^c$. In this context it's natural to allow this extra generality (and in our particular case this doesn't demand much extra effort). A calculation similar to the one leading to (8) shows that the pdf under the alternative hypothesis is

$$g_w(x;\alpha,\beta,c)=\frac{\beta\alpha^{c-\beta}}{\Gamma(1-c/\beta)}x^{c-\beta-1}\exp\bigl(-(\alpha x)^{-\beta}\bigr),$$
$$x\ge 0,\ \alpha>0,\ \beta>0,\ c>0. \quad (45)$$

To decide whether it's plausible that our random sample $x_1,\dots,x_n$ represents the parent inverse Weibull distribution (null hypothesis H_0) relative to the weighted inverse Weibull distribution (alternative hypothesis H_c), we use the following test statistic, where α and β are assumed known and c is also fixed (several values can be tried, including $c=1$ for the LBIW):

$$\Lambda=\prod_{i=1}^{n}\frac{g_w(x_i;\alpha,\beta,c)}{f(x_i;\alpha,\beta)}=\prod_{i=1}^{n}\frac{\dfrac{\beta\alpha^{c-\beta}}{\Gamma(1-c/\beta)}x_i^{c-\beta-1}\exp(-(\alpha x_i)^{-\beta})}{\beta\alpha^{-\beta}x_i^{-\beta-1}\exp(-(\alpha x_i)^{-\beta})}$$
$$=\prod_{i=1}^{n}\frac{\alpha^c x_i^c}{\Gamma\bigl(1-\frac{1}{\beta}\bigr)}=\frac{\alpha^{nc}\prod_{i=1}^{n}x_i^c}{\bigl(\Gamma\bigl(1-\frac{1}{\beta}\bigr)\bigr)^n}. \quad (46)$$

We reject H_0 when

$$\Lambda = \frac{\alpha^{nc} \prod_{i=1}^{n} x_i^c}{\left(\Gamma\left(1-\frac{1}{\beta}\right)\right)^n} > K, \tag{47}$$

where $K > 0$ is some threshold chosen beforehand, indicating the level of confidence we want to have in our prediction. Equivalently, we reject the null hypothesis when

$$\Lambda^* = \prod_{i=1}^{n} x_i^c > K^*, \quad \text{where } K^* = \frac{K\Gamma(1-c/\beta)^n}{\alpha^{nc}} > 0. \tag{48}$$

The choice of K is related to the *p-value*, defined as the probability that, under H_0, the expected value of the test statistic Λ^* is at least as high as the one actually observed. For large n we have $2\log\Lambda^* \sim \chi^2$, and from the χ^2 one obtains the p-value using the χ^2 table (or software). The p-value can also be readily computed via Monte Carlo simulation: simulate N samples from the distribution under H_0, for some large value of N, and compute the test statistic Λ_i^* for each sample. Then take

$$p\text{-value} = \frac{\#\{i : \Lambda_i^* > \Lambda^*\}}{N}.$$

Reject the null hypothesis if the p-value is less than the desired level of significance (typically 5% or 1%).

6. Examples

In this section we apply the formulas obtained in the previous section to two examples from the literature. The first set of data, given in Table 1, represents the waiting times (in minutes) before service of 100 bank customers [Ghitany et al. 2008]. The second data set, shown in Table 2, represents the number of millions of revolutions before failure of each of 23 ball bearings in a life testing experiment [Lawless 2003].

We modeled these data sets using the weighted inverse Weibull distribution with unknown parameters α and β (we keep the assumption made after (4) that $x_0 = 0$). The normal equations were solved by numerical methods to estimate the model parameters. Specifically, the MLEs of the parameters were computed by maximizing the objective function with the trust-region algorithm in the NLPTR subroutine in SAS. We present in Table 3 the estimated values of the parameters α and β and corresponding gradient objective functions (normal equations) under the length-biased inverse Weibull distribution for both sets of data.

We also conducted, for each set of data, a test for the detection of length bias, to compare the hypothesis that the waiting time distribution follows the LBIW distribution is to be preferred to the null hypothesis that the distribution is unweighted inverse Weibull.

0.8	0.8	4.3	5.0	6.7	8.2	9.7	11.9	14.1	19.9
0.8	0.8	4.3	5.3	6.9	8.6	9.8	12.4	15.4	20.6
1.3	1.3	4.4	5.5	7.1	8.6	10.7	12.5	15.4	21.3
1.5	1.5	4.4	5.7	7.1	8.6	10.9	12.9	17.3	21.4
1.8	1.8	4.6	5.7	7.1	8.8	11.0	13.0	17.3	21.9
1.9	1.9	4.7	6.1	7.1	8.8	11.0	13.1	18.1	23.0
1.9	1.9	4.7	6.2	7.4	8.9	11.1	13.3	18.2	27.0
2.1	2.1	4.8	6.2	7.6	8.9	11.2	13.6	18.4	31.6
2.6	2.6	4.9	6.2	7.7	9.5	11.2	13.7	18.9	33.1
2.7	2.7	4.9	6.3	8.0	9.6	11.5	13.9	19.0	38.5

Table 1. Waiting times of 100 bank customers, from [Ghitany et al. 2008].

17.88	28.92	33.00	41.52	42.12	45.60	48.80	51.84	51.96	54.12
55.56	67.80	68.64	68.64	68.88	84.12	93.12	98.64	105.12	105.84
127.92	128.04	173.40	–	–	–	–	–	–	–

Table 2. Lifetimes of 23 ball bearings, from [Lawless 2003].

Data	α	β	$\partial L/\partial\alpha$	$\partial L/\partial\beta$
I ($n = 100$)	0.400	1.819	7.46×10^{-4}	-5.98×10^{-5}
II ($n = 23$)	0.02795	2.4610	1.990×10^{-9}	1.930×10^{-11}

Table 3. Estimated values of the parameters.

For the set of waiting times given in Table 1, where (as shown in Table 3) the estimated values of the parameters α and β are $\hat{\alpha} = 0.3997$ and $\hat{\beta} = 1.81887$, we obtained for the test statistic the value $2 \log \Lambda = 270.927$, and the p-value for the test was less than 0.000001. Therefore, we have strong statistical evidence that the hypothesis that the waiting time distribution follows the LBIW distribution is to be preferred to the null hypothesis.

For the second set of data, the estimated values of the parameters are $\hat{\alpha} = 0.027952$ and $\hat{\beta} = 2.46097$. The value of the test statistic is $2 \log \Lambda = 170.893$, and the p-value is less than 0.00001. Again, the null hypothesis corresponding to the parent distribution is rejected.

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