involve
a journal of mathematics

Induced trees, minimum semidefinite rank, and zero forcing

Rachel Cranfill, Lon H. Mitchell, Sivaram K. Narayan and Taiji Tsutsui

msp

2012 vol. 5, no. 4

INVOLVE 5:4 (2012)
dx.doi.org/10.2140/involve.2012.5.411

Induced trees, minimum semidefinite rank, and zero forcing

Rachel Cranfill, Lon H. Mitchell, Sivaram K. Narayan and Taiji Tsutsui

(Communicated by Chi-Kwong Li)

We prove that the ordered subgraph number of a connected graph that has no duplicate vertices is at most three if and only if the complement does not contain a cycle on four vertices. The duality between zero forcing and ordered subgraphs then provides a complementary characterization for positive semidefinite zero forcing. We also provide some necessary conditions for when the minimum semidefinite rank can be computed using tree size.

1. Introduction

Graph theory provides a natural way to describe patterns in the entries of matrices and a large body of research and terminology to help study those patterns. Conversely, matrices that are associated to graphs can provide structural information about the graph. For example, the second-smallest eigenvalue of the Laplacian matrix of a graph is nonzero if and only if the graph is connected [Merris 1995].

The research described in this paper was inspired by the question of finding the smallest possible rank among matrices with a given zero/nonzero (off-diagonal) entry pattern. Depending on the type of matrices one allows (for example, real or complex, symmetric or not), different answers for the same pattern are possible [Berman et al. 2008; IMA-ISU 2010; Barioli et al. 2009], and a complete solution to this problem for any large class of matrices seems difficult. On the other hand, for certain types of patterns (graphs), there are very satisfying complete answers. For example, for trees and positive semidefinite (psd) real symmetric or complex Hermitian matrices, the minimum rank is equal to one less than the number of vertices [van der Holst 2003; Johnson and Duarte 2006]; for trees and symmetric matrices over any field, the minimum rank plus the zero forcing number gives the number of vertices [Chenette et al. 2007; Johnson and Duarte 1999].

MSC2010: 05C50, 15A18, 15B48.
Keywords: minimum semidefinite rank.
Research supported in part by NSF grant 05-52594.

One part of our work, described in Section 4, seeks to use the detailed knowledge we have for trees in general graphs. In particular, if a graph contains a tree as an induced subgraph, under what conditions will matrices associated to the larger graph behave like those for the tree with respect to minimum rank?

Rather than looking for trees, participants in the 2004 Research Experience for Undergraduates at Central Michigan University sought to find an alternative that would provide just as much rank information. The result, designed specifically for Hermitian psd matrices, was called *ordered subgraphs* [Hackney et al. 2009]. For some time, it was conjectured that ordered subgraphs would in fact determine minimum rank, but a counterexample on eight vertices was found: the Möbius ladder on eight vertices has psd minimum rank (msr) five and an ordered subgraph (OS) number of four [Mitchell et al. 2010].

Results on ordered subgraphs are of additional interest thanks to their connection to "zero forcing." Defined by the AIM Minimum Rank-Special Graphs Work Group [AIM 2008], zero forcing was also the result of looking for approaches to solving a minimum rank problem, but has since been shown to be of interest in quantum physics [Burgarth et al. 2011]. It turns out that the OS number and the positive semidefinite zero forcing number are two sides of the same coin, as for any graph they sum to the number of vertices [Barioli et al. 2010]. Moreover, the complement of an OS set is a zero forcing set and vice versa. This duality means that our OS results have an equivalent formulation in terms of zero forcing.

One of the many open questions concerning ordered subgraphs (and zero forcing) is how large the class of graphs is for which minimum rank and the ordered subgraph number differ. If the msr of a graph is one or two, then so is the OS number. The Möbius ladder example means that msr three is the remaining case[1] in which we might hope that msr and the ordered subgraph number coincide. In Section 3, we study graphs that have msr 3, show that msr 3 implies OS number 3, and give a characterization of those graphs with OS number 3. Whether OS number equal to 3 implies msr 3 remains open, although we are able to use our work on maximum induced trees from Section 4 to present some partial results in Section 5.

2. Preliminaries

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a set of vertices and $E(G)$ is a set of unordered pairs of vertices. In this paper, we assume all graphs are simple (that is, have no multiple edges or loops). Two vertices u and v are said to be *adjacent* if they share an edge. If u and v are adjacent, we write $uv \in E(G)$.

[1] For small rank, that is — some results are known for small nullity as well; see for example [van der Holst 2003].

For any $n \times n$ Hermitian matrix $A = [a_{ij}]$, we associate a simple graph $G(A)$ with vertex set $V(G) = \{v_1, \dots, v_n\}$ and $v_i v_j \in E(G)$ if and only if $a_{ij} \neq 0$ in A. Note that $G(A)$ is independent of the diagonal elements of A. For a given graph G, we define $\mathcal{P}(G)$ to be the set of all positive semidefinite matrices with graph G. The *minimum semidefinite rank* of G is

$$\operatorname{msr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{P}(G)\}.$$

If there is a path between two vertices u and v in G, the *distance* from u to v, $d_G(u, v)$, is the length of the shortest path between u and v. If no such path exists, we say $d_G(u, v) = \infty$.

The *tree size* of a graph G, $\operatorname{ts}(G)$, is the maximum size of a subset of $V(G)$ that induces a tree [Erdős et al. 1986]. Since $\operatorname{msr}(G) = |G| - 1$ if and only if G is a tree, this gives a general lower bound of $\operatorname{msr}(G) \geq \operatorname{ts}(G) - 1$ [Booth et al. 2008].

Let the *neighborhood* of a vertex v in G be $N(v) = \{w \in V(G) : vw \in E(G)\}$, and let the *closed neighborhood* of v be $N[v] = N(v) \cup \{v\}$. We say vertices u and w are *duplicate vertices* if $N[u] = N[w]$.

If $S \subseteq V(G)$ such that all of the vertices in S are pairwise nonadjacent, we say S is an *independent set*. The maximum cardinality of all independent sets of a graph G is called the *independence number* of G and is denoted by $\alpha(G)$ [West 1996, p. 113].

The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the disconnected graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We frequently write the union of k copies of a graph G as kG. The *join* of G_1 and G_2, written $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set consisting of all of the edges in $E(G_1)$ and $E(G_2)$ as well as the edges $\{uv : u \in V(G_1), v \in V(G_2)\}$ [West 1996, p. 118].

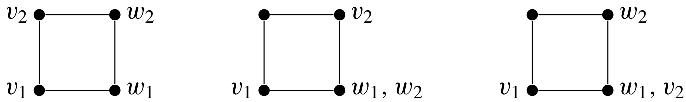
Suppose $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an n-tuple of vectors in $\mathbb{C}^m$ such that, for $i \neq j$, we have $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ if and only if $v_i v_j \notin E(G)$. We call $\vec{V}$ a *vector representation* of G [Parsons and Pisanski 1989]; the rank of $\vec{V}$ is defined as the dimension of the span of the vectors.

Let $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a vector representation of G. If $V = [\vec{v}_1 \cdots \vec{v}_n]$, then $V^*V \in \mathcal{P}(G)$. If $A \in \mathcal{P}(G)$, then $A = B^*B$ for some matrix B with the same rank [Horn and Johnson 1990, p. 407]. Thus, for any $A \in \mathcal{P}(G)$, we can find a vector representation of G that produces A. This implies that finding a vector representation for a graph is equivalent to finding a positive semidefinite matrix of the graph.

Let G be a graph on n vertices and let $S = (v_1, \dots, v_m)$ be an ordered set of vertices of G. Let G_k be the subgraph of G induced by $\{v_1, \dots, v_k\}$ for $k \leq m$, and let H_k be the connected component of G_k containing v_k. If for each k there exists a vertex w_k of G such that $w_k \notin G_k$, $w_k v_k \in E(G)$, and $w_k v_l \notin E(G)$ for all $v_l \in V(H_k)$ with $l \neq k$, we say S is a *vertex set of ordered subgraphs* (OS-*set*) of G [Hackney et al. 2009].

For every v_k in an OS-set, we call its corresponding w_k its OS-*neighbor*. The maximum cardinality of all OS-sets of a graph G is called the OS-*number* of G, denoted by $\mathrm{OS}(G)$.

Example 2.1. In the cycle C_4, $\mathrm{OS}(C_4) = 2$. Here are some examples of OS-sets of C_4:



Proposition 2.2 [Hackney et al. 2009]. *If G is a connected graph then* $\mathrm{msr}(G) \geq \mathrm{OS}(G) \geq \mathrm{ts}(G) - 1$. *In particular, if T is a tree, for every $v \in V(T)$, $V(T)\backslash\{v\}$ is an* OS-*set.*

If H is an induced subgraph of G, then $\mathrm{OS}(H) \leq \mathrm{OS}(G)$. The OS-number is related to the positive semidefinite zero forcing number, $Z_+(G)$, by $\mathrm{OS}(G)+Z_+(G) = |G|$ [Barioli et al. 2010].

3. Graphs with minimum semidefinite rank three

An open question that has been of interest is a complete characterization of all graphs for which $\mathrm{msr}(G) = 3$. Some prior results [Booth et al. 2011; AIM 2008] give sufficient conditions, including if $\overline{G} = P_n$ with $n \geq 4$ or $\overline{G} = C_n$ with $n \geq 5$ then $\mathrm{msr}(G) = 3$, and a sufficient condition for when $\mathrm{msr}(G) \leq 3$:

Proposition 3.1 [Booth et al. 2011]. *If the cycle C_m is not a subgraph of $\overline{G}$ for all $m \geq 4$, then* $\mathrm{msr}(G) \leq 3$.

From examples, however, it seems that avoiding C_4 in the complement is enough.

Conjecture 3.2. Let G be a connected graph with no duplicate vertices. Then $\mathrm{msr}(G) \leq 3$ if and only if C_4 is not a subgraph of $\overline{G}$.

Remark 3.3. Conjecture 3.2 is not true if the duplicate vertices condition is removed. For example, if G is the graph obtained by identifying an edge of the complete graph on four vertices with an edge of a C_4 (resulting in a graph on six vertices), then a C_4 is a subgraph of $\overline{G}$ but $\mathrm{msr}(G) = 3$.

We now prove several results that are related to this conjecture, including that this result holds for the OS-number.

Lemma 3.4. *Let G be a simple connected graph. If $S = (v_1, v_2, v_3, v_4)$ is an* OS-*set of G, then there is an* OS-*set S' of G of size four such that $G[S']$ has at least two components and each component has at most two vertices.*

Proof. If $G[S]$ has three or four connected components, the conclusion follows. Otherwise, we consider two cases:

Case 1: $G[S]$ has two connected components, $G[\{v_1, v_2, v_3\}]$ and $G[\{v_4\}]$. Then $w_3 \notin N[v_1] \cup N[v_2]$ and $G[\{v_1, v_2, w_3, v_4\}]$ has at least two components with each component having at most two vertices. Also, $S' = (v_1, v_2, v_4, w_3)$ is an OS-set with OS-neighbors (w_1, w_2, w_4, v_3).

Case 2: Suppose $G[S]$ is connected. Then $w_4 \notin \bigcup_{i=1}^{3} N[v_i]$, and therefore $G[\{v_1, v_2, v_3, w_4\}]$ has at least two components. Furthermore, $S_1 = (v_1, v_2, v_3, w_4)$ is an OS-set with OS-neighbors (w_1, w_2, w_3, v_4), reducing the problem to case 1. □

Remark 3.5. If S_1 and S_2 are OS-sets of G such that there are no edges $vw \in E(G)$ with $v \in S_1$ and $w \in S_2$, then $S_1 \cup S_2$ is an OS-set.

Lemma 3.6. *Let G be a connected graph with no duplicate vertices. If an induced subgraph H of G is isomorphic to $sK_2 \cup tK_1$, then the vertices of H form an OS-set.*

Proof. Clearly, K_1 is an OS-set since G is connected. Let $K_2 = \{v, w\}$. Since G has no duplicate vertices, $N[v] \neq N[w]$. Without loss of generality, we can assume there is a vertex u adjacent to v but not adjacent to w. Then (w, v) is an OS-set with neighbors (v, u). □

Proposition 3.7. *Let G be a connected graph with no duplicate vertices. Then* $\mathrm{OS}(G) \geq 4$ *if and only if $\overline{G}$ contains C_4 as a subgraph.*

Proof. Lemma 3.4 and Lemma 3.6 imply that $\mathrm{OS}(G) \geq 4$ if and only if G contains $4K_1$, $2K_1 \cup K_2$, or $2K_2$ as an induced subgraph. However, $\overline{4K_1}$ is K_4, $\overline{2K_1 \cup K_2}$ is K_4 minus an edge, and $\overline{2K_2}$ is C_4, giving the desired result. □

As a consequence of Proposition 3.7, we see the absence of a C_4 subgraph in $\overline{G}$ is necessary for $\mathrm{msr}(G) \leq 3$. We believe that this condition is sufficient and can be shown by proving $\mathrm{OS}(G) = 3$ if and only if $\mathrm{msr}(G) = 3$. We do know, however, that if G is a connected graph without duplicate vertices and $\mathrm{msr}(G) \leq 3$, then $\mathrm{msr}(G) = \mathrm{ts}(G) - 1$ [Booth et al. 2011]. As a result, we have:

Proposition 3.8. *If* $\mathrm{msr}(G) = 3$, *then* $\mathrm{OS}(G) = 3$ (*and* $Z_+(G) = |G| - 3$).

Conjecture 3.9. Suppose G is a connected graph without duplicate vertices. If $\mathrm{OS}(G) = 3$, then $\mathrm{msr}(G) = 3$.

4. Maximum induced trees

Let T be a maximum induced tree of a graph G. For a vertex w in $V(G)$ such that w is not on T, we define $\mathscr{E}(w)$ to be the edge set of all paths in T between every pair of vertices of T that are adjacent to w.

Prior work on minimum semidefinite rank has yielded a sufficient, but not necessary, condition for when $\mathrm{msr}(G) = \mathrm{ts}(G) - 1$ [Booth et al. 2008]:

⊛ There exists a maximum induced tree T such that for u and w not on T, $\mathscr{E}(u) \cap \mathscr{E}(w) \neq \varnothing$ if and only if u and w are adjacent in G.

We now present some sufficient conditions for strict inequality.

Proposition 4.1. *Let T be a maximum induced tree of a graph G. If u and w are vertices not on T such that $uw \notin \mathscr{E}(G)$, $|\mathscr{E}(u) \cap \mathscr{E}(w)| = 1$, and u and w are only adjacent to the longest path P of T that contains $\mathscr{E}(u) \cap \mathscr{E}(w)$, then* $\operatorname{msr}(G) > \operatorname{ts}(G) - 1$.

Proof. The vertices of T not on P belong to an OS-set S. We enlarge S by adding the vertices on P. Let $P = v_1 v_2 \cdots v_i x y v_{i+1} \cdots v_{k-1} v_k$, and without loss of generality assume $xw \in \mathscr{E}(G)$ and $yu \in \mathscr{E}(G)$, where $\{xy\} = \mathscr{E}(u) \cap \mathscr{E}(w)$. We add vertices $v_k, v_{k-1}, \ldots, v_{i+2}, v_{i+1}$ to the set S since we can find OS-neighbors $v_{k-1}, v_{k-2}, \ldots, v_{i+1}, y$, respectively. Then we add w, y, and x in that order to the set followed by $v_i, \ldots, v_2$ since these vertices have OS-neighbors $x, u, v_i, \ldots, v_1$ respectively. The size of this enlarged OS-set is $\operatorname{ts}(G)$. Thus, $\operatorname{msr}(G) \geq \operatorname{OS}(G) > \operatorname{ts}(G) - 1$. □

This leads us to the following result.

Corollary 4.2. *Let T be a maximum induced tree of a graph G. Suppose u and w are vertices not on T such that $uw \notin \mathscr{E}(G)$, $\mathscr{E}(u) \cap \mathscr{E}(w)$ contains only the edge xy where $xw \in \mathscr{E}(G)$, $P = v_1 v_2 \cdots v_i x y v_{i+1} \cdots v_{k-1} v_k$ is the longest path P of T that contains $\mathscr{E}(u) \cap \mathscr{E}(w)$, there exists a path P' on T where $P' = y t_1 t_2 \cdots t_l$ and $t_l u \in \mathscr{E}(G)$, and u and w are adjacent only to vertices of $P \cup P'$. Then* $\operatorname{msr}(G) > \operatorname{ts}(G) - 1$.

Proof. The vertices of T not on P or P' belong to an OS-set S. We enlarge S by adding the vertices of P and P'. We add vertices $v_k, v_{k-1}, \ldots, v_{i+1}$ to the set S since the set of OS-neighbors is $v_{k-1}, v_{k-2}, \ldots, y$, respectively. Then we add $w, y, t_1, \ldots, t_l$ in that order since these vertices have OS-neighbors $x, t_1, t_2, \ldots, t_l, u$, respectively. Also, we add $x, v_i, v_{i-1}, \ldots, v_2$ since the set of OS-neighbors is $v_i, v_{i-1}, \ldots, v_1$, respectively. Thus, by the same argument as in Proposition 4.1, $\operatorname{msr}(G) \geq \operatorname{OS}(G) > \operatorname{ts}(G) - 1$. □

Proposition 4.3. *Let T be a maximum induced tree of a graph G such that T is a star graph. If there exist vertices u and w not on T such that $uw \notin \mathscr{E}(G)$ and $|\mathscr{E}(u) \cap \mathscr{E}(w)| = 1$, then* $\operatorname{msr}(G) > \operatorname{ts}(G) - 1$.

Proof. The vertices of T that are not the center of T and are not adjacent to u or w belong to an OS-set. Let the center vertex of T be x and $\mathscr{E}(u) \cap \mathscr{E}(w) = \{xy\}$. We add vertices of T which are adjacent to u and not on $\mathscr{E}(u) \cap \mathscr{E}(w)$ to the OS-set since all of these vertices have OS-neighbor x. Then we add u and y in that order since they have OS-neighbors y and w. Next, we add vertices that are adjacent to w and not on $\mathscr{E}(u)\cap\mathscr{E}(w)$ to the OS-set since they also have OS-neighbor x. Thus, the size of OS-set is $\mathrm{ts}(G)$, so $\mathrm{msr}(G)\geq\mathrm{OS}(G)>\mathrm{ts}(G)-1$. □

If $\mathscr{E}(u)\cap\mathscr{E}(w)=\varnothing$, we have the following result.

Proposition 4.4. *Let T be a maximum induced tree of a graph G. If there are two vertices $u, w\in V(G)$ such that $u, w\notin V(T)$, $uw\in\mathscr{E}(G)$, and $\mathscr{E}(u)\cap\mathscr{E}(w)=\varnothing$, then* $\mathrm{OS}(G)>\mathrm{ts}(G)-1$. *In particular*, $\mathrm{msr}(G)>\mathrm{ts}(G)-1$.

Proof. Let $G'=G[V(T)\cup\{u, w\}]$. By constructing an OS-set of size $\mathrm{ts}(G)$ in G', we will show that $\mathrm{OS}(G)>\mathrm{ts}(G)-1$. Let $v_1,\dots,v_a\in V(T)$ be vertices of degree one in G'. Then $(v_1,\dots,v_a)$ forms an OS-set of G' with each v_i having corresponding w_i such that w_i is the only vertex adjacent to v_i. Let $F=G[V(G')\backslash\{v_1,\dots,v_a\}]$. If $v_{a+1},\dots,v_l\in V(T)$ such that $\deg_F(v_i)=1$ for all $i\in\{a+1,\dots,l\}$, then $(v_1,\dots,v_a,v_{a+1},\dots,v_l)$ forms an OS-set of G' where, for all $i\in\{a+1,\dots,l\}$, w_i is the unique vertex in F such that $v_iw_i\in\mathscr{E}(F)$. We can repeat this process until all vertices of degree one in $G[V(G')\backslash\{v_1,\dots,v_l\}]$ have been included in an OS-set of G', say $S=(v_1,\dots,v_k)$.

Let $\mathscr{V}(u)=\{v\in V(T): vv'\in\mathscr{E}(u)$ for some $v'\}$ and $\mathscr{V}(w)=\{v\in V(T): vv'\in\mathscr{E}(w)$ for some $v'\}$. Without loss of generality, assume that $|\mathscr{V}(u)|\geq|\mathscr{V}(w)|$. Because $|\mathscr{V}(u)\cap\mathscr{V}(w)|\geq 2$ would imply $\mathscr{E}(u)\cap\mathscr{E}(w)\neq\varnothing$, there are two possibilities:

Case 1: $|\mathscr{V}(u)\cap\mathscr{V}(w)|=1$. Note that if $|\mathscr{V}(u)|=n$ and $|\mathscr{V}(w)|=m$, then $\mathrm{ts}(G)=k+n+m-1$. Suppose $v\in\mathscr{V}(u)\cap\mathscr{V}(w)$. Since $G[\mathscr{V}(u)]$ is a tree, by Proposition 2.2, $\mathscr{V}(u)\backslash\{v\}=(v_{k+1},\dots,v_{k+n-1})$ forms an OS-set. Furthermore, $(v_1,\dots,v_{k+n-1},u)$ forms an OS-set since $uw\in\mathscr{E}(G)$ but $v_iw\notin\mathscr{E}(G)$ for all $i\in\{1,\dots,k+n-1\}$.

Now order vertices $\{x_1,\dots,x_{m-1}\}=\mathscr{V}(w)\backslash\{v\}$ such that $d_H(x_i,u)\leq d_H(x_{i+1},u)$ where $H=G[V(T)\cup\{u\}]$. Since for every $i\leq m-1$ there is a $j>i$ such that $d_H(x_i,u)=d_H(x_j,u)+1$ and where $x_jx_i\in\mathscr{E}(G)$ but x_j is not adjacent to any other vertex in the connected component of $G[\{x_1,\dots,x_{j-1}\}]$, we now have an OS-set $(v_1,\dots,v_{k+n-1},u,x_1,\dots,x_{m-1})$ of size $\mathrm{ts}(G)$.

Case 2: $\mathscr{V}(u)\cap\mathscr{V}(w)=\varnothing$. Begin by ordering vertices $u_i\in\mathscr{V}(u)$ by $d_J(u_i,w)\geq d_J(u_{i+1},w)$ for $i=1,\dots,n-1$ where $J=G[V(T)\cup\{w\}]$.

Let $H=G[V(T)\cup\{u\}]$ and define $\mathscr{V}'(w)=V(T)\backslash(\mathscr{V}(u)\cup S)$. Let v be the unique vertex in $\mathscr{V}'(w)$ such that $d_H(v,u)<d_H(x,u)$ for every $x\in\mathscr{V}'(w)$ where $x\neq v$. If $\mathscr{V}(u)=\{u_1,\dots,u_n\}$, then, because $\{u_1,\dots,u_n,v\}$ induces a tree on G, $(u_1,\dots,u_n)$ forms an OS-set. Moreover, $(v_1,\dots,v_k,u_1,\dots,u_n,u)$ forms an OS-set, as $uw\in\mathscr{E}(G)$ but $u_iw\notin\mathscr{E}(G)$ and $v_jw\notin\mathscr{E}(G)$ for any i, j.

Order the vertices in $\mathscr{V}'(w)=\{x_1,\dots,x_j,v\}$ such that $d_H(x_i,u)\geq d_H(x_{i+1},u)$ for $i=1,\dots,j-1$. Then $S\cup(u_1,\dots,u_n,u,x_1,\dots,x_j)$ is an OS-set that includes u and all vertices on the maximum induced tree except for v. □

5. OS number three

In this final section, we use our work on maximum induced trees, and, in particular, the condition $\circledast$, to prove that $\mathrm{OS}(G) = 3$ implies $\mathrm{msr}(G) = 3$ for certain graphs.

Proposition 5.1. *Let G be a connected graph without duplicate vertices. If $\overline{G}$ does not contain C_4 as a subgraph then* $\mathrm{msr}(G) \leq 3$ *or there exists a connected graph G' without duplicate vertices such that*

(1) *G is an induced subgraph of G',*

(2) *$\overline{G'}$ does not contain C_4 as a subgraph,*

(3) *$K_{1,3}$ is an induced subgraph of G', and*

(4) *G' is not $(|G'| - 3)$-connected.*

Proof. For the last claim, if G' is $(|G'| - 3)$-connected then $\mathrm{msr}(G) \leq 3$ [van der Holst 2008; Lovász et al. 1989; 2000].

Case 1: $\alpha(G) = 3$. If necessary, form G' by adding a new vertex adjacent to all vertices of G.

Case 2: $\alpha(G) = 2$. Let $\{u, v\} \subset V(G)$ induce $2K_1$ in G. Form G' by adding a new vertex adjacent to all vertices of G except for u and v. As $\overline{G}$ does not contain K_3 as an induced subgraph, $\overline{G'}$ does not contain C_4 as a subgraph.

Case 3: $\alpha(G) = 1$. Then G is complete and $\mathrm{msr}(G) \leq 1$. □

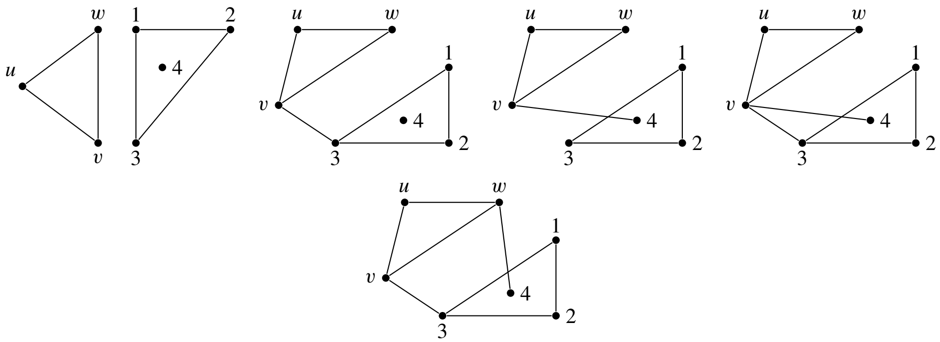
Suppose that G is a connected graph without duplicate vertices such that $\overline{G}$ does not contain C_4 as a subgraph and $\mathrm{OS}(G) = 3$. From Proposition 5.1, we may assume without loss of generality that $K_{1,3}$ is an induced subgraph of G. Therefore $K_{1,3}$ is a maximum induced tree T of G.

Remark 5.2. Since $\overline{G}$ does not contain C_4 as a subgraph, there are at most three vertices in G not belonging to T that are pairwise disjoint.

Remark 5.3. If u and v are not on T and satisfy $\circledast$, then there exists a vector representation of $G[V(T) \cup \{u, v\}]$ of rank three.

Proposition 5.4. *Suppose G is a connected graph without duplicate vertices such that $\overline{G}$ does not contain C_4 as a subgraph and* $\mathrm{OS}(G) = 3$. *Let $T = K_{1,3}$ be a maximum induced tree of G. If u, v, and w are pairwise nonadjacent vertices not on T such that no two of them satisfy $\circledast$, then $H = G[V(T) \cup \{u, v, w\}]$ has minimum semidefinite rank equal to three.*

Proof. If independent vertices u, v, and w are joined to all vertices of $K_{1,3}$, then $H = K_{1,3} \vee 3K_1$. Thus, its complement consists of $2K_3$. From this observation, since $\overline{G}$ does not contain C_4 as a subgraph, the complement of H has to be one of the following graphs:



Since all of these graphs are C_m-free for $m \geq 4$, we can use Proposition 3.1 to conclude that $\mathrm{msr}(H) \leq 3$. Since $\mathrm{OS}(H) = 3$, it follows that the $\mathrm{msr}(H) = 3$. □

Acknowledgements

The authors would like to thank Andrew Zimmer for helpful discussions, and the referee for suggestions that improved the quality of the paper.

References

[AIM 2008] AIM Minimum Rank – Special Graphs Work Group, "Zero forcing sets and the minimum rank of graphs", *Linear Algebra Appl.* **428**:7 (2008), 1628–1648. MR 2008m:05166 Zbl 1135.05035

[Barioli et al. 2009] F. Barioli, S. M. Fallat, H. T. Hall, D. Hershkowitz, L. Hogben, H. van der Holst, and B. Shader, "On the minimum rank of not necessarily symmetric matrices: a preliminary study", *Electron. J. Linear Algebra* **18** (2009), 126–145. MR 2010e:05176 Zbl 1169.05345

[Barioli et al. 2010] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, "Zero forcing parameters and minimum rank problems", *Linear Algebra Appl.* **433**:2 (2010), 401–411. MR 2011g:15002 Zbl 1209.05139

[Berman et al. 2008] A. Berman, S. Friedland, L. Hogben, U. G. Rothblum, and B. Shader, "Minimum rank of matrices described by a graph or pattern over the rational, real and complex numbers", *Electron. J. Combin.* **15**:1 (2008), Research Paper 25, 19. MR 2008k:05124

[Booth et al. 2008] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, K. Steinmetz, B. D. Sutton, and W. Wang, "On the minimum rank among positive semidefinite matrices with a given graph", *SIAM J. Matrix Anal. Appl.* **30**:2 (2008), 731–740. MR 2009g:15003 Zbl 1226.05151

[Booth et al. 2011] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, T. D. Lenker, L. H. Mitchell, S. K. Narayan, A. Pascoe, and B. D. Sutton, "On the minimum semidefinite rank of a simple graph", *Linear Multilinear Algebra* **59**:5 (2011), 483–506. MR 2012e:15004 Zbl 1223.05170

[Burgarth et al. 2011] D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini, and M. Young, "Zero forcing, linear and quantum controllability for systems evolving on networks", preprint, 2011. arXiv 1111.1475

[Chenette et al. 2007] N. L. Chenette, S. V. Droms, L. Hogben, R. Mikkelson, and O. Pryporova, "Minimum rank of a tree over an arbitrary field", *Electron. J. Linear Algebra* **16** (2007), 183–186. MR 2008f:05110 Zbl 1142.05335

[Erdős et al. 1986] P. Erdős, M. Saks, and V. T. Sós, "Maximum induced trees in graphs", *J. Combin. Theory Ser. B* **41**:1 (1986), 61–79. MR 87k:05062

[Hackney et al. 2009] P. Hackney, B. Harris, M. Lay, L. H. Mitchell, S. K. Narayan, and A. Pascoe, "Linearly independent vertices and minimum semidefinite rank", *Linear Algebra Appl.* **431**:8 (2009), 1105–1115. MR 2011a:15016 Zbl 1188.05085

[van der Holst 2003] H. van der Holst, "Graphs whose positive semi-definite matrices have nullity at most two", *Linear Algebra Appl.* **375** (2003), 1–11. MR 2004g:05104 Zbl 1029.05099

[van der Holst 2008] H. van der Holst, "Three-connected graphs whose maximum nullity is at most three", *Linear Algebra Appl.* **429**:2-3 (2008), 625–632. MR 2009g:05104 Zbl 1145.05037

[Horn and Johnson 1990] R. A. Horn and C. R. Johnson, *Matrix analysis*, Corrected reprint of the 1985 original ed., Cambridge University Press, 1990. MR 91i:15001 Zbl 0704.15002

[IMA-ISU 2010] IMA-ISU Research Group on Minimum Rank (Institute for Mathematics and its Applications – Iowa State University), "Minimum rank of skew-symmetric matrices described by a graph", *Linear Algebra Appl.* **432**:10 (2010), 2457–2472. MR 2011h:15001

[Johnson and Duarte 1999] C. R. Johnson and A. L. Duarte, "The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree: Invariant factors", *Linear and Multilinear Algebra* **46**:1-2 (1999), 139–144. MR 2000e:05114 Zbl 0929.15005

[Johnson and Duarte 2006] C. R. Johnson and A. L. Duarte, "Converse to the Parter–Wiener theorem: the case of non-trees", *Discrete Math.* **306**:23 (2006), 3125–3129. MR 2007h:05101 Zbl 1114.05061

[Lovász et al. 1989] L. Lovász, M. Saks, and A. Schrijver, "Orthogonal representations and connectivity of graphs", *Linear Algebra Appl.* **114/115** (1989), 439–454. MR 90k:05095 Zbl 0681.05048

[Lovász et al. 2000] L. Lovász, M. Saks, and A. Schrijver, "A correction: "Orthogonal representations and connectivity of graphs" [Linear Algebra Appl. **114/115** (1989), 439–454; MR0986889 (90k:05095)]", *Linear Algebra Appl.* **313**:1-3 (2000), 101–105. MR 2001g:05070

[Merris 1995] R. Merris, "A survey of graph Laplacians", *Linear and Multilinear Algebra* **39**:1-2 (1995), 19–31. MR 97c:05104 Zbl 0832.05081

[Mitchell et al. 2010] L. H. Mitchell, S. K. Narayan, and A. M. Zimmer, "Lower bounds in minimum rank problems", *Linear Algebra Appl.* **432**:1 (2010), 430–440. MR 2010m:15004 Zbl 1220.05077

[Parsons and Pisanski 1989] T. D. Parsons and T. Pisanski, "Vector representations of graphs", *Discrete Math.* **78**:1-2 (1989), 143–154. MR 90k:05104 Zbl 0693.05058

[West 1996] D. B. West, *Introduction to graph theory*, Prentice Hall, Upper Saddle River, NJ, 1996. MR 96i:05001 Zbl 0845.05001

Received: 2011-05-20 Revised: 2012-06-12 Accepted: 2012-06-13

rachel_cranfill@hmc.edu *Department of Mathematics, Harvey Mudd College, Claremont, CA 91771, United States*

lmitchell2@vcu.edu *Department of Mathematics, Virginia Commonwealth University, Richmond, VA 23284-2014, United States*

sivaram.narayan@cmich.edu *Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859, United States*

tsutsuit@my.hiram.edu *Department of Mathematics, Hiram College, Hiram, OH 44234, United States*

mathematical sciences publishers msp

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2012 is US $105/year for the electronic version, and $145/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2012 vol. 5 no. 4