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Positive symmetric solutions of a second-order difference equation

Jeffrey T. Neugebauer and Charley L. Seelbach

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Using an extension of the Leggett–Williams fixed-point theorem due to Avery, Henderson, and Anderson, we prove the existence of solutions for a class of second-order difference equations with Dirichlet boundary conditions, and discuss a specific example.

1. Introduction

Many fixed-point theorems have applications in proving the existence of positive solutions of boundary value problems. One class of such theorems, originating with [Krasnoselskii 1964], involves an operator defined on a "wedge" — a portion of a Banach space bounded by level surfaces of positive functionals — and satisfying certain criteria. In Krasnoselskii's original theorem, the functional was the norm; that is, the wedge conditions where $a \le \|x\|$ and $\|x\| \le b$, for $0 < a < b$. A later variant, in [Leggett and Williams 1979], replaced the lower wedge condition by $a \le \alpha(x)$, where α is a concave positive functional with $\alpha(x) \le \|x\|$; this allows more flexibility in the choice of the wedge in applications. The Leggett–Williams theorem was extended by Avery, Henderson, and Anderson [Avery et al. 2009] to allow flexibility also in the upper wedge condition, which gets replaced by $\beta(x) \le b$, where β is a convex positive functional. This is the result of primary interest to this paper; other related results can be found in [Guo 1984; Avery and Henderson 2001; Anderson et al. 2010; Mavridis 2010].

Applications of such fixed-point theorems have been seen in works dealing with ordinary differential equations [Avery et al. 2000; 2010; Erbe and Wang 1994] and dynamic equations on time scales [Erbe et al. 2005; Liu et al. 2012; Prasad and Sreedhar 2011]. Most relevant to this paper, these theorems have been utilized for results that involve finite difference equations [Anderson et al. 2011; Cai and Yu 2006; Henderson et al. 2010].

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Here we give an application of the fixed-point theorem of [Avery et al. 2009], stated below as Theorem 2.1, to obtain at least one positive solution of the difference equation

$$\Delta^2 u(k) + f(u(k)), \quad k \in \{0, \ldots, N-2\}, \tag{1-1}$$

with boundary conditions

$$u(0) = u(N) = 0. \tag{1-2}$$

Here $f : [0, \infty) \to [0, \infty)$ is any continuous function and Δ^2 is the second-difference operator, defined by $(\Delta^2 u)(k) = u(k) - 2u(k+1) + u(k+2)$. In fact we will obtain a *symmetric* solution, in the sense that $u(k) = u(N-k)$ for each k.

In Section 2 we present the fixed-point theorem of Avery et al. Section 3 contains preliminaries needed for our result on the difference equation (1-1), (1-2). That result is stated and proved in Section 4, and applied to a particular case in Section 5.

2. Statement of the fixed-point theorem

Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a *cone* if it contains the origin, is closed under multiplication by positive scalars, and has no overlap with its negative (apart from the origin). In symbols,

$$u \in \mathcal{P},\ \lambda \geq 0 \implies \lambda u \in \mathcal{P} \quad \text{and} \quad u \in \mathcal{P}, -u \in \mathcal{P} \implies u = 0.$$

Let $\mathcal{P}$ be a cone in E. A map $\alpha : \mathcal{P} \to [0, \infty)$ is said to be a *nonnegative continuous concave functional* on $\mathcal{P}$ if it is continuous and satisfies

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v),$$

for all $u, v \in \mathcal{P}$ and $t \in [0, 1]$. Replacing $\geq$ by $\leq$ we obtain the definition of a *nonnegative continuous convex functional* on $\mathcal{P}$.

In the statement of the theorem, there appear two concave functionals, α and ϕ, and two convex ones, β and γ. The functionals α and β delimit the wedge where the operator is defined; the other two ensure additional flexibility in applications, in comparison with the Leggett–Williams theorem.

Theorem 2.1 [Avery et al. 2009]. *Let $\mathcal{P}$ be a cone in a real Banach space E. Suppose that α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$ and that β and δ are nonnegative continuous convex functionals on $\mathcal{P}$. For nonnegative real numbers a, b, c, and d, define*

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d\}, \tag{2-1}$$

and suppose that A is a bounded subset of P. Let $T : A \to \mathcal{P}$ be a completely continuous operator (*that is, it is continuous and maps bounded sets into precompact sets*). *Then T has a fixed point in A provided that the following conditions hold*:

A0. $\{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \varnothing$.

A1. $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \varnothing$.

A2. $\alpha(Tx) \geq a$ *for all* $x \in A$ *with* $\delta(x) \leq b$.

A3. $\alpha(Tx) \geq a$ *for all* $x \in A$ *with* $b < \delta(Tx)$.

A4. $\beta(Tx) \leq d$ *for all* $x \in A$ *with* $c \leq \psi(x)$.

A5. $\beta(Tx) \leq d$ *for all* $x \in A$ *with* $\psi(Tx) < c$.

3. Application of the theorem to a difference equation

In this section we return to the system (1-1), (1-2), stating in Theorem 3.2 sufficient conditions for the existence of a solution. This result is proved in the next section using Theorem 2.1. First, however, we set up some of the objects that appear in the statement of Theorem 2.1. Throughout the discussion we use the abbreviations

$$\underline{N} = \left\lfloor \frac{N}{2} \right\rfloor \quad \text{and} \quad \overline{N} = \left\lceil \frac{N}{2} \right\rceil.$$

Define the Banach space E to be the space of functions $u : \{0, \dots, N\} \to \mathbb{R}$ with the norm

$$\|u\| = \max_{k \in \{0,1,\dots,N\}} |u(k)|.$$

Within E, consider the cone $\mathcal{P}$ consisting of all u that are nonnegative, symmetric, nondecreasing on $\{0, 1, \dots, \underline{N}\}$, and satisfy $wu(y) \geq yu(w)$ for $w \geq y$, where $y, w \in \{0, 1, \dots, \underline{N}\}$.

Set

$$H(k,l) = \frac{1}{N} \begin{cases} k(N-l), & k \in \{0, \dots, l\}, \\ l(N-k), & k \in \{l+1, \dots, N\}. \end{cases}$$

(This is the Green's function for $-\Delta^2$ satisfying the boundary conditions (1-2).) Define the operator T by

$$(Tu)(k) := \sum_{l=1}^{N-1} H(k,l) f(u(l)).$$

By direct checking one sees that the condition $Tu = u$ is equivalent to (1-1) and (1-2). Thus any fixed point of T is a solution of our problem.

Lemma 3.1. *The operator T maps A into $\mathcal{P}$.*

Proof. Let $u \in A$. We first need to show that $Tu(N-k) = Tu(k)$. Notice that $H(N-k, N-l) = H(k,l)$. Now

$$Tu(N-k) = \sum_{l=1}^{N-1} H(N-k,l) f(u(l)).$$

Applying the substitution $r = N - l$, we can write

$$
\begin{aligned}
Tu(N-k) &= \sum_{r=1}^{N-1} H(N-k, N-r) f(u(N-r)) \\
&= \sum_{r=1}^{N-1} H(k, r) f(u(r)) = Tu(k).
\end{aligned}
$$

Next we need to show $Tu(k)$ is nonnegative and nondecreasing on $\{0, 1, \dots, \underline{N}\}$. Since $H(k,l) \geq 0$ for $k, l \in \{0, \dots, N\}$ and f only takes nonnegative values, $Tu(k)$ is nonnegative for all $k \in \{0, \dots, N\}$.

To prove that $Tu(k)$ is nondecreasing on $\{0, 1, \dots, \underline{N}\}$, we show that $\Delta Tu(k) := Tu(k-1) - Tu(k)$ is nonnegative on $\{0, 1, \dots, \underline{N}\}$. Now

$$
H(k+1, l) - H(k, l) = \frac{1}{N} \times \begin{cases} N-l & \text{if } k \in \{0, \dots, l\}, \\ -l & \text{if } k \in \{l, \dots, N-1\}. \end{cases}
$$

So

$$
\begin{aligned}
\Delta Tu(k) &= \sum_{l=1}^{N-1} \big(H(k+1, l) - H(k, l)\big) f(u(l)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N} f(u(l)) + \sum_{l=k}^{N-1} \frac{N-l}{N} f(u(l)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N} f(u(l)) + \sum_{l=k}^{N-1} \frac{N-l}{N} f(u(N-l)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N} f(u(l)) + \sum_{r=1}^{N-k} \frac{r}{N} f(u(r)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N} f(u(l)) + \sum_{l=1}^{N-k} \frac{l}{N} f(u(l)).
\end{aligned}
$$

Since $k \in \{0, 1, \dots, \underline{N}\}$,

$$
\Delta Tu(k) = \sum_{l=1}^{k-1} \frac{-l}{N} f(u(l)) + \sum_{l=1}^{N-k} \frac{l}{N} f(u(l)) = \sum_{l=k}^{N-k} \frac{l}{N} f(u(l)) \geq 0,
$$

as needed.

Lastly, we have $wTu(y) \geq yTu(w)$, since $H(k,l)$ satisfies $\dfrac{H(y,l)}{H(w,l)} \geq \dfrac{y}{w}$ for all l and all $w \geq y$. Thus T maps A into $\mathcal{P}$. □

Theorem 3.2. *Assume that $\tau, \mu, \nu \in \{1, \dots, \underline{N}\}$ are fixed with $\tau \le \mu < \nu$, that d and m are positive real numbers with $0 < m < d\mu/\underline{N}$, and that $f : [0, \infty) \to [0, \infty)$ is a continuous function such that*

(i) $f(w) \ge \dfrac{2Nd}{(\nu-\tau)(2N-1-\tau-\nu)\underline{N}}$ *for* $w \in [\tau d/\underline{N}, \nu d/\underline{N}]$,

(ii) $f(w)$ *is decreasing for* $w \in [0, m]$ *and* $f(m) \ge f(w)$ *for* $w \in [m, d]$, *and*

(iii) $2\displaystyle\sum_{l=1}^{\mu} \frac{l\overline{N}}{N} f\left(\frac{ml}{\mu}\right) \le d - f(m)\frac{1}{N}(\overline{N})(\underline{N}-\mu)(\mu+1+\underline{N})$.

Set $a = \tau d/\underline{N}$. Then (1-1), (1-2) has at least one positive symmetric solution $u^* \in A$, where A is given by (2-1).

4. Proof of Theorem 3.2

Let $a = \tau d/\underline{N}$, $b = \nu d/\underline{N}$, $c = \mu d/\underline{N}$. By Lemma 3.1, T maps A into $\mathcal{P}$. Let $u \in A$. Then $\beta(u) = u(\underline{N}) \le d$. But u achieves its maximum at $\underline{N}$, so A is bounded. By the Arzelà–Ascoli theorem, T is a completely continuous operator.

Now define the functionals appearing in the theorem as follows, where $u \in \mathcal{P}$:

$$\alpha(u) = \min_{k\in\{\tau,\dots,\underline{N}\}} u(k) = u(\tau), \qquad \psi(u) = \min_{k\in\{\mu,\dots,\underline{N}\}} u(k) = u(\mu),$$

$$\delta(u) = \max_{k\in\{0,\dots,\nu\}} u(k) = u(\nu), \qquad \beta(u) = \max_{k\in\{0,\dots,\underline{N}\}} u(k) = u(\underline{N}).$$

It is easy to check that α and ψ are concave and β and δ are convex.

We check conditions A0–A5 in turn. Let $u \in P$ and let $\beta(u) > d$. Then

$$\alpha(u) = u(\tau) \ge \frac{\tau}{\underline{N}} u(\underline{N}) = \frac{\tau}{\underline{N}}\beta(u) > \frac{\tau d}{\underline{N}} = a.$$

So $\{u \in P \ : \ \alpha(u) < a \text{ and } d < \beta(u)\} = \varnothing$, which is A0.

Now let $K \in \left(\dfrac{2d}{\underline{N}(3N-4-\mu)}, \dfrac{2d}{\underline{N}(3N-4-\nu)}\right)$. Define

$$u_K(k) = K\sum_{l=1}^{N-1} H(k,l) = \frac{Kk}{2}(3N-4-k).$$

Then

$$\alpha(u_k) = u_k(\tau) = \frac{K\tau}{2}(3N-4-\tau) > \frac{2d\tau(3N-4-\tau)}{2\underline{N}(3N-4-\mu)} \ge \frac{\tau d}{\underline{N}} = a$$

and

$$\beta(u_k) = u_k(\underline{N}) = \frac{K\underline{N}}{2}(3N-4-\underline{N}) < \frac{2\underline{N}d(3N-4-\underline{N})}{2\underline{N}(3N-4-\nu)} \le \frac{\underline{N}d}{\underline{N}} = d.$$

So $u_k \in A$.

Since

$$\psi(u_k) = u_k(\mu) = \frac{K\mu}{2}(3N-4-\mu) > \frac{2d\mu(3N-4-\mu)}{2\underline{N}(3N-4-\mu)} = \frac{\mu d}{\underline{N}} = c$$

and

$$\delta(u_k) = u_k(\nu) = \frac{K\nu}{2}(3N-4-\nu) < \frac{2d\nu(3N-4-\nu)}{2\underline{N}(3N-4-\nu)} = \frac{\nu d}{\underline{N}} = b,$$

we have $\{u \in A : c < \psi(u) \text{ and } \delta(u) < b\} \neq \varnothing$. Therefore A1 holds.

To show that A2 holds, take $u \in A$ with $\delta(u) < b$. By (i),

$$\begin{aligned}\alpha(Tu) &= \sum_{l=1}^{N-1} H(\tau,l) f(u(l)) \geq \sum_{l=\tau+1}^{\nu} H(\tau,l) f(u(l)) \\ &\geq \frac{2Nd}{(\nu-\tau)(2N-1-\tau-\nu)\underline{N}} \cdot \frac{\tau(\nu-\tau)(2N-1-\tau-\nu)}{2N} \geq \frac{\tau d}{\underline{N}} = a.\end{aligned}$$

To show that A3 holds, let $u \in A$ with $\delta(Tu) > b$. Then

$$\begin{aligned}\alpha(Tu) = Tu(\tau) &= \sum_{l=1}^{N-1} H(\tau,l) f(u(l)) \geq \frac{\tau}{\nu} \sum_{l=1}^{N-1} H(\nu,l) f(u(l)) \\ &= \frac{\tau}{\nu}\delta(Tu) > \frac{\tau}{\nu} b = \frac{d\tau}{\underline{N}} = a.\end{aligned}$$

Now we show that A4 holds. Let $u \in A$ satisfy $c \leq \phi(x)$. By the concavity of u and since $c = \mu d/\underline{N}$, for all $k \in \{0, 1, \ldots, \mu\}$, we have

$$u(k) \geq \frac{ck}{\mu} \geq \frac{mk}{\mu}.$$

So, by (ii) and (iii), we have

$$\begin{aligned}\beta(Tu) &= \sum_{l=1}^{N-1} H(\underline{N},l) f(u(l)) \leq 2\sum_{l=1}^{\underline{N}} \frac{l(N-\underline{N})}{N} f(u(l)) \\ &= 2\sum_{l=1}^{\mu} \frac{l(\overline{N})}{N} f(u(l)) + 2\sum_{l=\mu+1}^{\underline{N}} \frac{l(\overline{N})}{N} f(u(l)) \\ &\leq 2\sum_{l=1}^{\mu} \frac{l(\overline{N})}{N} f(u(ml/\mu)) + 2\sum_{l=\mu+1}^{\underline{N}} \frac{l(\overline{N})}{N} f(m) \\ &\leq d - f(m)\frac{\overline{N}}{N}(\underline{N}-\mu)(\mu+1+\underline{N}) + f(m)\frac{\overline{N}}{N}(\underline{N}-\mu)(\mu+1+\underline{N}) = d.\end{aligned}$$

Thus A4 is satisfied.

Last, we show that A5 is satisfied. Let $u \in A$ with $\psi(Tu) < c$. Then

$$\beta(Tu) = \sum_{l=1}^{N-1} H(\underline{N}, l) f(u(l)) \le \frac{\underline{N}}{\mu} \sum_{l=1}^{N-1} H(\mu, l) f(u(l)) \le \frac{\underline{N}}{\mu} \psi(Tu) < \frac{c\underline{N}}{\mu} = d.$$

Therefore T has a fixed point and (1-1), (1-2) has at least one positive symmetric solution $u^* \in A$.

5. Example

Example 1. Let $N = 20$, $\tau = 1$, $\mu = 9$, $\nu = 10$, $d = 5$, and $m = 4.4$. Notice that $0 < \tau \le \mu < \nu \le 10 = \underline{N}$, and $0 < m = 4.4 \le 4.5 = d\mu/\underline{N}$. Define a continuous function $f : [0, \infty) \to [0, \infty)$ by

$$f(w) = \begin{cases} \frac{1}{500}(45 - w) & \text{if } 0 \le w \le 40, \\ \frac{1}{100} & \text{if } w \ge 40. \end{cases}$$

Then,

(i) for $w \in [\frac{1}{2}, 5]$, $f(w) \ge f(5) = \dfrac{2}{25} > \dfrac{5}{63} = \dfrac{2 \cdot 20 \cdot 5}{(10-1) \cdot (3+2 \cdot 18 - 1 - 10)(10)}$,

(ii) $f(w)$ is decreasing for $w \in [0, 4.4]$ and $f(m) \ge f(w)$ for $w \in [4.4, 5]$, and

(iii) $2 \displaystyle\sum_{l=1}^{9} \frac{10l}{20} f\left(\frac{4.4l}{9}\right) = \frac{5657}{1500} < \frac{1047}{250} = 5 - f(4.4)\left(\tfrac{1}{20}\right)(10)(10-9)(9+1+10)$.

So the hypotheses of Theorem 3.2 are satisfied. Therefore, the difference equation

$$\Delta^2 u(k) + f(u(k)), \quad k \in \{0, 1, \dots, 18\},$$

with boundary conditions

$$u(0) = u(20) = 0,$$

has a positive symmetric solution u^* with $u(1) \ge \frac{1}{2}$ and $u(10) \le 5$.

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