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Refined inertias of tree sign patterns of orders 2 and 3

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Sign patterns are matrices with only the sign of each entry specified. The refined inertia of a matrix categorizes the eigenvalues as positive, negative, zero or nonzero imaginary, and the refined inertia of a sign pattern is the set of all refined inertias allowed by real matrices with that sign pattern. The complete sets of allowed refined inertias for all tree sign patterns of orders 2 and 3 (up to equivalence and negation) are determined.

1. Introduction

The *inertia* of an $n \times n$ real matrix A, denoted by $\mathrm{i}(A)$, is the triple (n_+, n_-, n_0), where n_+, n_- and n_0 are the numbers of eigenvalues with positive, negative and zero real part, respectively. Note that $n_+ + n_- + n_0 = n$. The *refined inertia* of A, $\mathrm{ri}(A)$, is the 4-tuple $(n_+, n_-, n_z, 2n_p)$ with n_+ and n_- as above, n_z the number of zero eigenvalues and n_p the number of nonzero imaginary complex conjugate pairs of eigenvalues; see [Kim et al. 2009]. The refined inertia of A distinguishes between zero and nonzero imaginary eigenvalues, which is important for linear dynamical systems.

A *sign pattern* $\mathcal{A} = [\alpha_{ij}]$ of order n is an $n \times n$ matrix with entries in $\{+, -, 0\}$. A real matrix A is a *realization* of $\mathcal{A}$ if the signs of the entries in A correspond to the entries in $\mathcal{A}$. The *sign pattern class* of $\mathcal{A}$ is $Q(\mathcal{A}) = \{A \mid A \text{ is a realization of } \mathcal{A}\}$. A sign pattern $\mathcal{B} = [\beta_{ij}]$ is a *superpattern* of $\mathcal{A}$ if $\beta_{ij} = \alpha_{ij}$ for all $\alpha_{ij} \in \{+, -\}$. The inertia of a sign pattern $\mathcal{A}$ is $\mathrm{i}(\mathcal{A}) = \{\mathrm{i}(A) \mid A \in Q(\mathcal{A})\}$ and the refined inertia of $\mathcal{A}$ is $\mathrm{ri}(\mathcal{A}) = \{\mathrm{ri}(A) \mid A \in Q(\mathcal{A})\}$. A sign pattern $\mathcal{A}$ *allows* a refined inertia $(n_+, n_-, n_z, 2n_p)$ if there exists some $A \in Q(\mathcal{A})$ having this refined inertia. See, for example, [Catral et al. 2009; Johnson and Summers 1989] for related allow problems on sign patterns.

Sign patterns have applications in areas where dynamical systems arise (see, for example, [Logofet 1993]), but characterizing sign patterns that have a particular

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property can be challenging since each nonzero entry is free to take on any value in one half of the real line. Section 2 introduces more definitions and concepts required for our analysis of sign patterns. Sections 3 and 4 identify the refined inertias allowed by all tree sign patterns of orders 2 and 3, respectively, and these are listed in Appendices A and B.

2. Fundamentals

Given an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, the *transpose* of $\mathcal{A}$ is $\mathcal{A}^T = [\alpha_{ji}]$. A *permutation similarity transformation* is $\mathcal{A} \mapsto P\mathcal{A}P^T$ where P is an $n \times n$ permutation matrix. A *signature similarity transformation* is $\mathcal{A} \mapsto D\mathcal{A}D^{-1}$ where $D = D^{-1}$ is an $n \times n$ diagonal matrix with each diagonal entry equal to ± 1. The refined inertia of a sign pattern $\mathcal{A}$ is preserved by each of these three transformations, which define equivalence classes of sign patterns. Two sign patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*, and therefore in the same equivalence class, if $\mathcal{B}$ can be derived from $\mathcal{A}$ by some sequence of the three transformations. One other important transformation is *negation*. Since $\operatorname{ri}(-\mathcal{A}) = \{(n_-, n_+, n_z, 2n_p) \mid (n_+, n_-, n_z, 2n_p) \in \operatorname{ri}(\mathcal{A})\}$, the refined inertia (and inertia) of $-\mathcal{A}$ is easily obtained from that of $\mathcal{A}$.

An $n \times n$ sign pattern $\mathcal{A}$ is a *spectrally arbitrary* pattern (SAP) if, given any set of n complex numbers closed under complex conjugation, there exists a realization $A \in Q(\mathcal{A})$ having these n numbers as its eigenvalues [Drew et al. 2000], and $\mathcal{A}$ is a *refined inertially arbitrary* pattern (rIAP) if, given any 4-tuple $(n_+, n_-, n_z, 2n_p)$ with $n_+ + n_- + n_z + 2n_p = n$, there exists a realization of $\mathcal{A}$ having this 4-tuple as its refined inertia [Kim et al. 2009]. An inertially arbitrary pattern (IAP) is defined similarly [Drew et al. 2000]. The properties of being an IAP, rIAP and SAP are invariant with respect to equivalence and negation. Any SAP is obviously an rIAP, and any rIAP is also an IAP. Conversely, for $n = 2$ and 3, $\mathcal{A}$ is a SAP if it is an rIAP [Kim et al. 2009], but it is not known if this holds for larger n.

A *tree sign pattern* $\mathcal{A} = [\alpha_{ij}]$ is a sign pattern that is combinatorially symmetric (i.e., $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$) and has $\alpha_{i_1 i_2}\alpha_{i_2 i_3} \cdots \alpha_{i_k i_1} = 0$ for all $k \geq 3$. As in [Johnson and Summers 1989], associated with an $n \times n$ tree sign pattern $\mathcal{A}$ is a signed tree graph with n vertices labeled $1, 2, \ldots, n$ and an edge between vertices i and $j \neq i$ if and only if α_{ij} is not 0; the sign on the edge is the sign of the product $\alpha_{ij}\alpha_{ji}$. In addition, if $\alpha_{ii} \neq 0$ the graph has a loop at vertex i, signed as the sign of α_{ii}. The negation of the sign pattern changes the signs of the loops of the graph, but not the signs of its edges. Every graph in the appendices uniquely represents those sign patterns that are the same up to equivalence, with one such sign pattern shown for each class. For $n = 2$ every irreducible sign pattern is a tree sign pattern. For $n = 3$, a tree sign pattern is equivalent to an irreducible tridiagonal sign pattern. We consider only irreducible sign patterns because the refined inertia of a reducible sign pattern is the sumset of the refined inertias of its irreducible components.

3. Sign patterns of order 2

For completeness, the refined inertias and graphs of all irreducible sign patterns of order 2 up to equivalence and negation are given in Appendix A. These can be determined simply by considering the trace and determinant of a real matrix with each sign pattern. The trace of a matrix is equal to the sum of its eigenvalues, and its determinant is equal to the product of its eigenvalues. For the 2×2 case the possible signs of the trace and determinant provide complete information on the refined inertia. For example, for an order 2 sign pattern, if the trace must be positive and the determinant must be negative, the only allowed refined inertia is $(1, 1, 0, 0)$, with the positive eigenvalue larger in magnitude than the negative one. Note that only one sign pattern of order 2 is an rIAP.

4. Tree sign patterns of order 3

Since there are a total of 3^9 sign patterns of order 3, a computer program was written to identify the set of all irreducible order 3 sign patterns up to equivalence and negation. This set was determined by examining in turn every possible 3×3 irreducible sign pattern and checking for equivalence or negation with each sign pattern already in the set. The program identified 187 sign patterns, of which 34 were tree sign patterns. The equivalence classes corresponding to the 34 tree sign patterns are each represented by a graph in Appendix B, along with their refined inertias, a representative sign pattern and its associated characteristic equation, and references to techniques used for finding the refined inertias.

We now present our techniques and methods for finding the exact set of refined inertias allowed by each sign pattern. For every sign pattern, each refined inertia was either proved to not be allowed or shown to be allowed either by a proof or a numerical realization, respectively.

A tree sign pattern can be represented by a tridiagonal matrix with (real) variables for each nonzero entry. The generic 3×3 tridiagonal matrix is

$$\begin{bmatrix} \pm e & \pm a & 0 \\ \pm b & \pm f & \pm c \\ 0 & \pm d & \pm g \end{bmatrix}. \tag{4-1}$$

Here $a, b, c, d > 0$ for an irreducible tree sign pattern and $e, f, g \geq 0$. For the moment we let $a' = \pm a$, and similarly for the other variables, but when working with specific sign patterns, we always use strictly positive variables.

We can normalize the matrix in (4-1) to reduce the number of unknowns by up to three, setting them to ± 1. If $e \neq 0$ we set $e = 1$ by multiplying the matrix by $1/e$, and if $e = 0$ and $f \neq 0$ we multiply by $1/f$ to set $f = 1$. By Lemma 2.3 in [Britz et al. 2004] we can also set $a = c = 1$, and since each 3×3 tree sign pattern has a', $c' > 0$ up to equivalence, $a' = c' = 1$. Thus the characteristic polynomial can be simplified to

$$x^3 - (e' + f' + g')x^2 + (e'f' + e'g' + f'g' - b' - d')x + e'd' + b'g' - e'f'g', \quad (4\text{-}2)$$

with one of e', f' equal to 1 or -1, or $e' = f' = g' = 0$.

In terms of the characteristic polynomial, the trace is the negative of the coefficient of x^2 and the determinant is the negative of the constant term. Each refined inertia corresponds to a unique product of three linear factors, i.e., to a unique factorization of a monic cubic polynomial. The 13 possible refined inertias for 3×3 matrices are listed in Table 1 with their factorizations. After expanding each factorization, the coefficients of the resulting polynomial can be compared directly to the coefficients of the characteristic polynomial in (4-2). If there is no solution to the resulting set of equations, then the corresponding refined inertia is not allowed by the sign pattern with the given characteristic polynomial.

The real parts of α, β are positive and $\gamma > 0$. Note that for refined inertias (a)–(d), (g) and (i) in Table 1, the unknowns α and β may be complex conjugate pairs. However, since $\alpha\beta$ and $\alpha + \beta$ are always real and positive, when used in these combinations the fact that α and β are possibly complex can be ignored.

The following list summarizes the techniques that we used to determine the refined inertias. Each sign pattern in Appendix B references the techniques used for that sign pattern.

T1. The rIAPs (equivalently SAPs for $n = 3$) were found by determining which of the 34 tree sign patterns are equivalent to one of the two 3×3 SAPs that are

	refined inertia	factorization	characteristic polynomial
(a)	(3,0,0,0)	$(x-\alpha)(x-\beta)(x-\gamma)$	$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+(\alpha+\beta)\gamma)x-\alpha\beta\gamma$
(b)	(2,1,0,0)	$(x-\alpha)(x-\beta)(x+\gamma)$	$x^3-(\alpha+\beta-\gamma)x^2+(\alpha\beta-(\alpha+\beta)\gamma)x+\alpha\beta\gamma$
(c)	(1,2,0,0)	$(x+\alpha)(x+\beta)(x-\gamma)$	$x^3+(\alpha+\beta-\gamma)x^2+(\alpha\beta-(\alpha+\beta)\gamma)x-\alpha\beta\gamma$
(d)	(0,3,0,0)	$(x+\alpha)(x+\beta)(x+\gamma)$	$x^3+(\alpha+\beta+\gamma)x^2+(\alpha\beta+(\alpha+\beta)\gamma)x+\alpha\beta\gamma$
(e)	(1,0,0,2)	$(x-\alpha)(x^2+\beta)$	$x^3-\alpha x^2+\beta x-\alpha\beta$
(f)	(0,1,0,2)	$(x+\alpha)(x^2+\beta)$	$x^3+\alpha x^2+\beta x+\alpha\beta$
(g)	(2,0,1,0)	$x(x-\alpha)(x-\beta)$	$x^3-(\alpha+\beta)x^2+\alpha\beta x$
(h)	(1,1,1,0)	$x(x+\alpha)(x-\beta)$	$x^3+(\alpha-\beta)x^2-\alpha\beta x$
(i)	(0,2,1,0)	$x(x+\alpha)(x+\beta)$	$x^3+(\alpha+\beta)x^2+\alpha\beta x$
(j)	(0,0,1,2)	$x(x^2+\alpha)$	$x^3+\alpha x$
(k)	(1,0,2,0)	$x^2(x-\alpha)$	$x^3-\alpha x^2$
(l)	(0,1,2,0)	$x^2(x+\alpha)$	$x^3+\alpha x^2$
(m)	(0,0,3,0)	x^3	x^3

Table 1. All refined inertias for matrices of order 3.

trees, namely $\mathscr{T}_3$ and $\mathscr{U}_3$ in [Britz et al. 2004] (see also [Cavers and Vander Meulen 2005]), or are equivalent to a superpattern of one of them.

T2. If $e = g = 0$, then the determinant must be zero and the trace is ± 1 or 0. The characteristic polynomial factors into a zero root and a quadratic, and the possible refined inertias are easily determined.

T3. In order to find a realization of a given sign pattern that has a refined inertia with $n_z = n_p = 0$, a random matrix with that sign pattern was generated in MATLAB, and its eigenvalues were computed. This ad hoc technique was used for many sign patterns to find an example of refined inertias (a)–(d) in Table 1 that each allows, although it does not show the nonexistence of those that are not allowed.

T4. If a tridiagonal sign pattern $\mathscr{A}$ is symmetric, then any $A \in Q(\mathscr{A})$ is diagonally similar to a symmetric matrix, so its eigenvalues are real. Thus the refined inertias (e)–(f) and (j) in Table 1 are not allowed for such sign patterns.

T5. If the determinant must be positive or must be negative, then the sign pattern does not allow (g)–(m) in Table 1, as well as either (b), (d), (f) if positive, or (a), (c), (e) if negative.

T6. If the trace must be positive, then the sign pattern does not allow (d), (f), (i), (j), (l) and (m) in Table 1, and if negative it does not allow (a), (e), (g), (j), (k) and (m).

T7. If the sign pattern is such that the coefficient of the x term in (4-2) is negative, then only refined inertias (b), (c) and (h) in Table 1 can be allowed.

Note. The next four techniques are algebraic, involving the characteristic polynomial (4-2) and equations in Table 1. There are many possible ways of proceeding; however, we found the following techniques to be the most straightforward.

T8. To show that one of (e) or (f) in Table 1 is not allowed, equate the coefficients of the characteristic polynomial (4-2) to the coefficients of the polynomial corresponding to the refined inertia being considered. If this leads to a contradiction, then that refined inertia is not allowed.

For example, consider sign pattern (4e) in Appendix B. Equating its characteristic polynomial to the polynomial associated with (0, 1, 0, 2) ((f) in Table 1) gives

$$1 - f = \alpha \implies \alpha < 1, \quad d - f - b = \beta \Leftrightarrow d = \beta + b + f, \quad d = \alpha\beta.$$

A contradiction is immediate since $0 < \alpha < 1$ implies $d < \beta$ from the last equation, but the second equation implies $d > \beta$. Therefore this sign pattern does not allow refined inertia (0, 1, 0, 2).

A more complicated argument shows that sign pattern (5d) in Appendix B does not allow refined inertia (1, 0, 0, 2). In this case

$$1 + f + g = \alpha \implies \alpha > g \text{ and } \alpha > 1,$$

while

$$f+g+fg+b+d=\beta \implies \beta > fg+d+b$$

and $d+bg+fg=\alpha\beta$. From the inequality for β it follows that $\alpha\beta > \alpha(fg+d+b) = \alpha(fg+d)+\alpha b$ and from the inequalities for α it follows that $\alpha(fg+d)+\alpha b > fg+d+gb=\alpha\beta$, which is a contradiction. This method can also be used to show that (a) or (d) in Table 1 are not allowed, but it is easier to invoke continuity when appropriate (see T12).

T9. This technique is for eliminating or identifying allowed refined inertias with at least one eigenvalue equal to zero (refined inertias (g)–(m) in Table 1) when the determinant is not necessarily zero (hence differing from T2). The objective is to determine inequalities between unknowns in the characteristic equation that require the coefficient of x to have a certain sign.

If the coefficient of x must be positive, only (g), (i) and/or (j) in Table 1 can be allowed, while if it must be negative, only (h) is allowed. This argument can also be viewed in terms of the discriminant of the quadratic that arises after factoring out the zero root from the characteristic polynomial.

As a simple example, consider sign pattern (10a) in Appendix B with the coefficient of x equal to $f+g+fg-b-d$. Here we are interested in the case $d+bg-fg=0$ (i.e., at least one zero eigenvalue). This equation gives

$$fg=d+bg \implies fg>d \text{ and } f>b.$$

These imply that $f-b+fg-d+g>0$. Since the trace $1+f+g$ must be positive, (2, 0, 1, 0) is the only refined inertia with a zero eigenvalue that sign pattern (10a) allows, eliminating (h)–(m) in Table 1.

A more complex use of this technique is needed for sign pattern (11d) in Appendix B, with the coefficient of x equal to $g-f-fg+b+d$. Again we set the determinant to zero: $fg-d-bg=0$. Considering the case with trace not positive ($f \geq 1+g$) gives $f>1$, $f>g$, so:

- If $g \leq 1$, then

  $$fg=bg+d \implies fg \geq bg+dg \implies f \geq b+d$$

  and $fg>g$. Therefore the coefficient of x is negative.
- If $g>1$, then

  $$fg=bg+d \implies fg>b+d$$

  and $f>g$. Therefore the coefficient of x is negative.

Thus, when the trace is negative or zero and the determinant is zero, only (1, 1, 1, 0) is allowed, eliminating (i)–(j) and (l)–(m) in Table 1.

For certain sign patterns, this technique can also be used to show that when the determinant and trace are taken to be negative, then the coefficient of x is also negative, and therefore (d) and (f) in Table 1 are not allowed. For sign pattern (11d) the argument follows as in the above example, except that the equality $fg = bg + d$ is replaced by the inequality $fg > bg + d$.

Note. The next two techniques are for finding realizations with certain refined inertias when continuity cannot be obviously invoked. This could be done simply by trial and error, but it is easier to do some algebra first.

T10. The first technique is for finding realizations with a zero eigenvalue. First fix the trace to be either positive or negative, and then find inequalities that ensure that the coefficient of x is positive, negative and/or zero.

As an example, consider sign pattern (11c) in Appendix B with the coefficient of x equal to $f - g - fg - b + d$. For this sign pattern $fg - d - bg = 0$ ensures a zero eigenvalue while $1 + f > g$ implies a positive trace.

To find a realization with refined inertia (k) in Table 1, set

$$f + d = g + b + fg = g + b + d + bg \implies f - g = b(g + 1).$$

This reduces the number of unknowns by one, and further note that a solution to the last equation ensures that the trace is positive; therefore that condition can be ignored. Also, a solution implies $f > b$, and since $d = g(f - b)$, d is positive for all solutions. An obvious solution is then $b = 1$, $g = 1$, $f = 3$.

Similarly, to show that (g) in Table 1 is allowed, a realization is required with positive trace, positive coefficient of x and zero determinant, which imply that $f - g > b(g + 1)$. Thus, in the above solution, increase the value of f. Similarly, to show that (h) in Table 1 is allowed, increase b so that $f - g < b(g + 1)$ while maintaining $b < f$.

T11. In order to determine a realization with refined inertia (e) or (f) in Table 1, begin as in T8, but now instead of trying to reach a contradiction, the goal is to find a reduced set of expressions that will ensure that all unknowns are positive, and therefore show that the refined inertia exists.

For example consider sign pattern (11c) as in the previous example. Equating the coefficients of the characteristic polynomial with those of (e) in Table 1 gives

$$1 + f - g = \alpha \implies f - g = \alpha - 1,$$
$$f - g - fg - b + d = \beta, \quad d + bg - fg = \alpha\beta.$$

From the last equation, choosing $f > b$ ensures $d > 0$. Combining all three equations to eliminate d and f gives

$$b(1 + g) = \alpha\beta + \alpha - \beta - 1 = (\alpha - 1)(\beta + 1).$$

Therefore, by choosing $g > \beta$, necessarily $b < \alpha - 1 < \alpha - 1 + g = f$, which gives two simple expressions, namely $\alpha > 1$ and $g > \beta > 0$, that ensure all variables remain positive. A solution is, for example, $\alpha = 2$, $\beta = 1$, $g = 3$ giving $f = 4$, $b = 0.5$, $d = 12.5$.

T12. Since the eigenvalues are continuous functions of the matrix entries, continuity can require that a sign pattern allow or not allow a particular refined inertia. This can often be used after the set of possible refined inertias is narrowed down.

For an example of the first case, if a sign pattern has been shown to allow $(3, 0, 0, 0)$ and $(1, 2, 0, 0)$ and the determinant is nonzero, then the sign pattern must also allow $(1, 0, 0, 2)$.

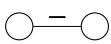
For an example of the second case, if a sign pattern allows $(3, 0, 0, 0)$ but no others except possibly $(2, 1, 0, 0)$, as in Appendix B(5), by continuity $(2, 1, 0, 0)$ is also not possible since an eigenvalue would have to cross the imaginary axis, and therefore $(2, 0, 1, 0)$ would also have to be allowed.

For each tree sign pattern of order 3 (up to equivalence and negation), the above techniques determine the set of all allowed refined inertias. Appendix B contains a list of sign patterns arranged according to these sets, and also includes the graph corresponding to each equivalence class. This list suggests some open questions. Given a list of refined inertias, what classes of sign patterns allow exactly these refined inertias, and which lists have at least one such sign pattern?

Appendix A. Refined inertias of all 2 × 2 irreducible sign patterns up to equivalence and negation

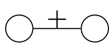
(1) $\mathrm{ri}(\mathscr{A}) = \{(0, 0, 0, 2)\}$

(1a)

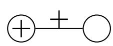
$\begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}$ 

(2) $\mathrm{ri}(\mathscr{A}) = \{(1, 1, 0, 0)\}$

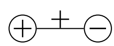
(2a)

$\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ 

(2b)

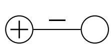
$\begin{bmatrix} + & + \\ + & 0 \end{bmatrix}$ 

(2c)

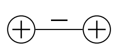
$\begin{bmatrix} + & + \\ + & - \end{bmatrix}$ 

(3) $\mathrm{ri}(\mathscr{A}) = \{(2, 0, 0, 0)\}$

(3a)

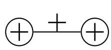
$\begin{bmatrix} + & + \\ - & 0 \end{bmatrix}$ 

(3b)

$\begin{bmatrix} + & + \\ - & + \end{bmatrix}$ 

(4) $\mathrm{ri}(\mathscr{A}) = \{(2, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0)\}$

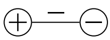
(4a)

$\begin{bmatrix} + & + \\ + & + \end{bmatrix}$ 

(5) rIAP (allows all 7 refined inertias)

(5a)

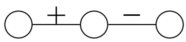
$$\begin{bmatrix} + & + \\ - & - \end{bmatrix}$$



Appendix B. Refined inertias of all 3 × 3 tree sign patterns up to equivalence and negation

(1) $\mathrm{ri}(\mathcal{A}) = \{(1,1,1,0), (0,0,3,0), (0,0,1,2)\}$

(1a) $$\begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & - & 0 \end{bmatrix}$$

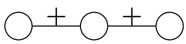


$x^3 - (b-d)x$

Technique: T2

(2) $\mathrm{ri}(\mathcal{A}) = \{(1,1,1,0)\}$

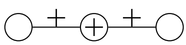
(2a) $$\begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}$$



$x^3 - (b+d)x$

Technique: T2

(2b) $$\begin{bmatrix} 0 & + & 0 \\ + & + & + \\ 0 & + & 0 \end{bmatrix}$$

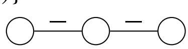


$x^3 - x^2 - (b+d)x$

Technique: T2

(3) $\mathrm{ri}(\mathcal{A}) = \{(0,0,1,2)\}$

(3a) $$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{bmatrix}$$

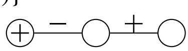


$x^3 + (b+d)x$

Technique: T2

(4) $\mathrm{ri}(\mathcal{A}) = \{(2,1,0,0)\}$

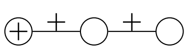
(4a) $$\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix}$$



$x^3 - x^2 + (b-d)x + d$

Techniques: T3, T5, T6

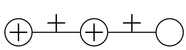
(4b) $$\begin{bmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}$$



$x^3 - x^2 - (b+d)x + d$

Techniques: T5, T7

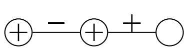
(4c) $$\begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & 0 \end{bmatrix}$$



$x^3 - (1+f)x^2 + (f-b-d)x + d$

Techniques: T3, T5, T6

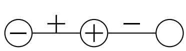
(4d) $$\begin{bmatrix} + & + & 0 \\ - & + & + \\ 0 & + & 0 \end{bmatrix}$$



$x^3 - (1+f)x^2 + (f+b-d)x + d$

Techniques: T3, T5, T6

(4e) $$\begin{bmatrix} - & + & 0 \\ + & + & + \\ 0 & - & 0 \end{bmatrix}$$



$x^3 + (1-f)x^2 - (f+b-d)x + d$

Techniques: T3, T5, T8, T12

(4f) $\begin{bmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & + \end{bmatrix}$ $\oplus \overset{+}{\text{———}} \bigcirc \overset{+}{\text{———}} \oplus$
$x^3-(1+g)x^2+(g-b-d)x+d+bg$
Techniques: T3, T5, T6

(4g) $\begin{bmatrix} + & + & 0 \\ + & - & + \\ 0 & + & + \end{bmatrix}$ $\oplus \overset{+}{\text{———}} \ominus \overset{+}{\text{———}} \oplus$
$x^3-(1-f+g)x^2-(f-g+fg+b+d)x+d+bg+fg$
Techniques: T3, T4, T5, T12

(4h) $\begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & - & 0 \end{bmatrix}$ $\ominus \overset{+}{\text{———}} \bigcirc \overset{-}{\text{———}} \bigcirc$
$x^3+x^2-(b-d)x+d$
Techniques: T3, T5, T8, T12

(4i) $\begin{bmatrix} + & + & 0 \\ + & - & + \\ 0 & + & 0 \end{bmatrix}$ $\oplus \overset{+}{\text{———}} \ominus \overset{+}{\text{———}} \bigcirc$
$x^3-(1-f)x^2-(f+b+d)x+d$
Techniques: T5, T7

(4j) $\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & + & - \end{bmatrix}$ $\oplus \overset{-}{\text{———}} \bigcirc \overset{+}{\text{———}} \ominus$
$x^3-(1-g)x^2-(g-b+d)x+d+bg$
Techniques: T3, T5, T8, T12

(4k) $\begin{bmatrix} + & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}$ $\oplus \overset{+}{\text{———}} \ominus \overset{-}{\text{———}} \oplus$
$x^3-(1+f-g)x^2+(f-g-fg+b-d)x+d+bg+fg$
Techniques: T3, T5, T8, T12

(5) $\mathrm{ri}(\mathscr{A})=\{(3,0,0,0)\}$

(5a) $\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{bmatrix}$ $\oplus \overset{-}{\text{———}} \bigcirc \overset{-}{\text{———}} \bigcirc$
$x^3-x^2+(b+d)x-d$
Techniques: T3, T5, T8, T12

(5b) $\begin{bmatrix} + & + & 0 \\ - & + & + \\ 0 & - & 0 \end{bmatrix}$ $\oplus \overset{-}{\text{———}} \oplus \overset{-}{\text{———}} \bigcirc$
$x^3-(1+f)x^2+(f+b+d)x-d$
Techniques: T3, T5, T8, T12

(5c) $\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}$ $\oplus \overset{-}{\text{———}} \bigcirc \overset{-}{\text{———}} \oplus$
$x^3-(1+g)x^2+(g+b+d)x-d-bg$
Techniques: T3, T5, T8, T12

(5d) $\begin{bmatrix} + & + & 0 \\ - & + & + \\ 0 & - & + \end{bmatrix}$ $\oplus \overset{-}{\text{———}} \oplus \overset{-}{\text{———}} \oplus$
$x^3-(1+f+g)x^2+(f+g+fg+b+d)x-d-bg-fg$
Techniques: T3, T5, T8, T12

(6) $\mathrm{ri}(\mathscr{A})=\{(2,0,1,0)\}$

(6a) $\begin{bmatrix} 0 & + & 0 \\ - & + & + \\ 0 & - & 0 \end{bmatrix}$ $\bigcirc \overset{-}{\text{———}} \oplus \overset{-}{\text{———}} \bigcirc$
$x^3-x^2+(b+d)x$
Technique: T2

(7) $\mathrm{ri}(\mathscr{A})=\{(1,0,2,0),(1,1,1,0),(2,0,1,0)\}$

(7a) $\begin{bmatrix} 0 & + & 0 \\ + & + & + \\ 0 & - & 0 \end{bmatrix}$

$\bigcirc \overset{+}{\text{——}} \oplus \overset{-}{\text{——}} \bigcirc$

$x^3 - x^2 - (b-d)x$

Technique: T2

(8) $\mathrm{ri}(\mathscr{A}) = \{(3,0,0,0), (1,2,0,0), (1,0,0,2)\}$

(8a) $\begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & 0 \end{bmatrix}$

$\ominus \overset{-}{\text{——}} \oplus \overset{+}{\text{——}} \bigcirc$

$x^3 + (1-f)x^2 - (f-b+d)x - d$

Techniques: T3, T5, T12

(8b) $\begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & - & 0 \end{bmatrix}$

$\oplus \overset{+}{\text{——}} \oplus \overset{-}{\text{——}} \bigcirc$

$x^3 - (1+f)x^2 + (f-b+d)x - d$

Techniques: T3, T5, T12

(8c) $\begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & - & 0 \end{bmatrix}$

$\oplus \overset{-}{\text{——}} \ominus \overset{-}{\text{——}} \bigcirc$

$x^3 - (1-f)x^2 - (f-b-d)x - d$

Techniques: T3, T5, T12

(9) $\mathrm{ri}(\mathscr{A}) = \{(2,1,0,0), (1,2,0,0), (1,1,1,0)\}$

(9a) $\begin{bmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & - \end{bmatrix}$

$\oplus \overset{+}{\text{——}} \bigcirc \overset{+}{\text{——}} \ominus$

$x^3 - (1-g)x^2 - (g+b+d)x + d - bg$

Techniques: T3, T7, T12

(9b) $\begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & - \end{bmatrix}$

$\oplus \overset{+}{\text{——}} \oplus \overset{+}{\text{——}} \ominus$

$x^3-(1+f-g)x^2+(f-g-fg-b-d)x+d-bg+fg$

Techniques: T3, T4, T9, T12

(10) $\mathrm{ri}(\mathscr{A}) = \{(3,0,0,0), (2,1,0,0), (2,0,1,0)\}$

(10a) $\begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & + \end{bmatrix}$

$\oplus \overset{+}{\text{——}} \oplus \overset{+}{\text{——}} \oplus$

$x^3-(1+f+g)x^2+(f+g+fg-b-d)x+d+bg-fg$

Techniques: T3, T4, T5, T9, T12

(11) $\mathrm{ri}(\mathscr{A}) = \{(3,0,0,0), (1,2,0,0), (2,1,0,0), (2,0,1,0), (1,0,2,0), (1,1,1,0), (1,0,0,2)\}$

(11a) $\begin{bmatrix} + & + & 0 \\ + & 0 & + \\ 0 & - & + \end{bmatrix}$

$\oplus \overset{+}{\text{——}} \bigcirc \overset{-}{\text{——}} \oplus$

$x^3 - (1+g)x^2 + (g-b+d)x - d + bg$

Techniques: T3, T6, T10, T11

(11b) $\begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & - & + \end{bmatrix}$

$\oplus \overset{+}{\text{——}} \oplus \overset{-}{\text{——}} \oplus$

$x^3-(1+f+g)x^2+(f+g+fg-b+d)x-d+bg-fg$

Techniques: T3, T6, T10, T11

(11c) $\begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & - & - \end{bmatrix}$

$\oplus \overset{+}{\text{——}} \oplus \overset{-}{\text{——}} \ominus$

$x^3-(1+f-g)x^2+(f-g-fg-b+d)x-d-bg+fg$

Techniques: T3, T9, T10, T11

(11d) $\begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & - & + \end{bmatrix}$

$\oplus \overset{-}{\text{——}} \ominus \overset{-}{\text{——}} \oplus$

$x^3-(1-f+g)x^2-(f-g+fg-b-d)x-d-bg+fg$

Techniques: T3, T9, T10, T11

(12) rIAP (allows all 13 refined inertias)

(12a) $\begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}$ $\ominus \overset{-}{\text{——}} \bigcirc \overset{-}{\text{——}} \oplus$
$x^3+(1-g)x^2-(g-b-d)x+d-bg$
Technique: T1

(12b) $\begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}$ $\ominus \overset{-}{\text{——}} \oplus \overset{+}{\text{——}} \ominus$
$x^3+(1-f+g)x^2-(f-g+fg-b+d)x-d+bg-fg$
Technique: T1

(12c) $\begin{bmatrix} - & + & 0 \\ - & - & + \\ 0 & - & + \end{bmatrix}$ $\ominus \overset{-}{\text{——}} \ominus \overset{-}{\text{——}} \oplus$
$x^3+(1+f-g)x^2+(f-g-fg+b+d)x+d-bg-fg$
Technique: T1

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