

involve

a journal of mathematics

New confidence intervals for the AR(1) parameter

Ferebee Tunno and Ashton Erwin



New confidence intervals for the AR(1) parameter

Ferebee Tunno and Ashton Erwin

(Communicated by Robert B. Lund)

This paper presents a new way to construct confidence intervals for the unknown parameter in a first-order autoregressive, or AR(1), time series. Typically, one might construct such an interval by centering it around the ordinary least-squares estimator, but this new method instead centers the interval around a linear combination of a weighted least-squares estimator and the sample autocorrelation function at lag one. When the sample size is small and the parameter has magnitude closer to zero than one, this new approach tends to result in a slightly thinner interval with at least as much coverage.

1. Introduction

Consider the causal stationary AR(1) time series given by

$$X_t = \phi X_{t-1} + \epsilon_t, \quad t = 0, \pm 1, \pm 2, \dots, \quad (1-1)$$

where $|\phi| < 1$, $E(X_t) = 0$ and $\{\epsilon_t\} \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$. We seek a new way to construct confidence intervals for the unknown parameter ϕ .

If X_1, X_2, \dots, X_n are sample observations from this process, then a point estimate for ϕ is found by calculating

$$\tilde{\phi}_p = \frac{\sum_{t=2}^n S_{t-1} |X_{t-1}|^p X_t}{\sum_{t=2}^n |X_{t-1}|^{p+1}},$$

where $p \in \{0, 1, 2, \dots\}$ and S_t is the sign function, defined by

$$S_t = \begin{cases} 1 & \text{if } X_t > 0, \\ 0 & \text{if } X_t = 0, \\ -1 & \text{if } X_t < 0. \end{cases}$$

The estimator $\tilde{\phi}_p$ can be thought of as a weighted least-squares estimator with form

$$\frac{\sum_{t=2}^n W_{t-1} X_{t-1} X_t}{\sum_{t=2}^n W_{t-1} X_{t-1}^2}$$

MSC2010: 60G10, 62F12, 62F99, 62M10.

Keywords: confidence interval, autoregressive parameter, weighted least squares, linear combination.

and weight $W_t = |X_t|^{p-1}$. Note that, when $p = 1$, we get the ordinary (unweighted) least-squares estimator (OLSE), and when $p = 0$, we get what has come to be called the Cauchy estimator:

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^n X_{t-1} X_t}{\sum_{t=2}^n X_{t-1}^2} \quad (\text{OLSE}), \quad \tilde{\phi}_0 = \frac{\sum_{t=2}^n S_{t-1} X_t}{\sum_{t=2}^n |X_{t-1}|} \quad (\text{Cauchy}).$$

The OLSE has been studied since the time of Gauss and its optimal properties for linear models are well known. The eponymously named Cauchy estimator dates back to about the same time and is sometimes used as a surrogate for the OLSE. Traditionally, confidence intervals for ϕ have been centered around the OLSE, although So and Shin [1999] and Phillips, Park and Chang [Phillips et al. 2004] showed that the Cauchy estimator has certain advantages over the OLSE when dealing with a unit root autoregression. Gallagher and Tunno [2008] constructed a confidence interval for ϕ centered around a linear combination of both estimators.

Another point estimate for ϕ comes from the sample autocorrelation function of $\{X_t\}$ at lag one, given by

$$\hat{\rho}(1) = \frac{\sum_{t=2}^n X_{t-1} X_t}{\sum_{t=1}^n X_t^2}.$$

The autocovariance function of $\{X_t\}$ at lag h for an AR(1) series is given by $\gamma(h) = \text{Cov}(X_t, X_{t+h}) = \phi^{|h|} \sigma^2 / (1 - \phi^2)$, which makes the true lag-one autocorrelation function equal to

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\phi \sigma^2 / (1 - \phi^2)}{\sigma^2 / (1 - \phi^2)} = \phi.$$

Observe that the structure of $\hat{\rho}(1)$ is similar to that of the OLSE. In fact, for an AR(1) series, the Yule–Walker, maximum likelihood, and least-squares estimators for ϕ are all approximately the same [Shumway and Stoffer 2006, Section 3.6]. Note also that, in general, if $\{X_t\}$ is not mean-zero, we would subtract \bar{X} from each observation when calculating things like $\hat{\rho}(h)$ and $\tilde{\phi}_p$.

To get a feel for how $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ behave relative to one another, Figure 1 shows their empirical bias and mean squared error (MSE) when $\phi \in (-1, 1)$ and $n = 50$. The Cauchy estimator has the lowest absolute bias, and $\hat{\rho}(1)$ has the smallest MSE for parameter values (roughly) between -0.5 and 0.5 , while the OLSE has the smallest MSE elsewhere. Other simulations not shown here reveal that the MSE and absolute bias of $\tilde{\phi}_p$ keep growing as p gets larger.

The goal of this paper is to construct a confidence interval for ϕ centered around a linear combination of an arbitrary weighted least-squares estimator and the sample autocorrelation function at lag one. That is, the center will take the form

$$a_1 \tilde{\phi}_p + a_2 \hat{\rho}(1), \tag{1-2}$$

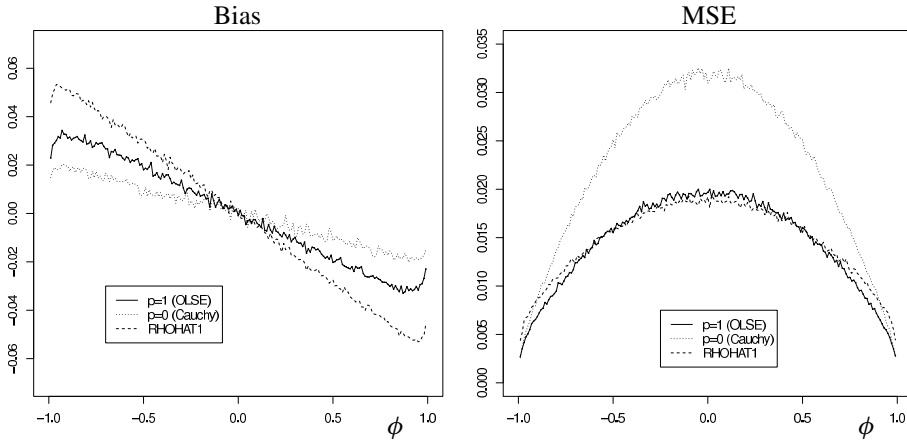


Figure 1. Empirical bias (left) and mean squared error (right) of $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ for $\phi \in (-1, 1)$; 10,000 simulations were run for each parameter value, with distribution $N(0, 1)$ and $n = 50$.

where $a_1 + a_2 = 1$ and $p \neq 1$. We first, however, need to take a brief look at how intervals centered around a single estimator behave in order to find a proper target for our new interval to outperform.

Theorem 2.1 from [Gallagher and Tunno 2008] states that for the AR(1) series given in (1-1), we have

$$\sqrt{n}(\tilde{\phi}_p - \phi) \xrightarrow{D} N\left(0, \frac{\sigma^2 E(X_t^{2p})}{(E|X_t|^{p+1})^2}\right) \tag{1-3}$$

for all p such that $E(|X_t|^r) < \infty$, where $r = \max(2p, p + 1)$. Since the error terms in our series are normal, the X_t 's have finite moments of all orders. Thus, this theorem can be used to create confidence intervals for ϕ centered at $\tilde{\phi}_p$ for any choice of p .

Specifically, if X_1, X_2, \dots, X_n are sample observations from (1-1), then an approximate $(1 - \alpha) \times 100\%$ confidence interval for ϕ has endpoints

$$\tilde{\phi}_p \pm z_{\alpha/2} \sqrt{\widehat{\text{Var}}(\tilde{\phi}_p)},$$

where

$$n\widehat{\text{Var}}(\tilde{\phi}_p) = \frac{\sigma^2 n^{-1} \sum_{t=2}^n X_{t-1}^{2p}}{(n^{-1} \sum_{t=2}^n |X_{t-1}|^{p+1})^2} \xrightarrow{P} \frac{\sigma^2 E(X_{t-1}^{2p})}{(E|X_{t-1}|^{p+1})^2}$$

and $z_{\alpha/2}$ is the standard normal critical value with area $\alpha/2$ to its right.

Similarly, we can create confidence intervals for ϕ centered at $\hat{\rho}(1)$. If we think of $\hat{\rho}(1)$ as being nearly the equivalent of the OLSE, then an approximate $(1 - \alpha) \times 100\%$ confidence interval for ϕ has endpoints

$$\hat{\rho}(1) \pm z_{\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{\rho}(1))},$$

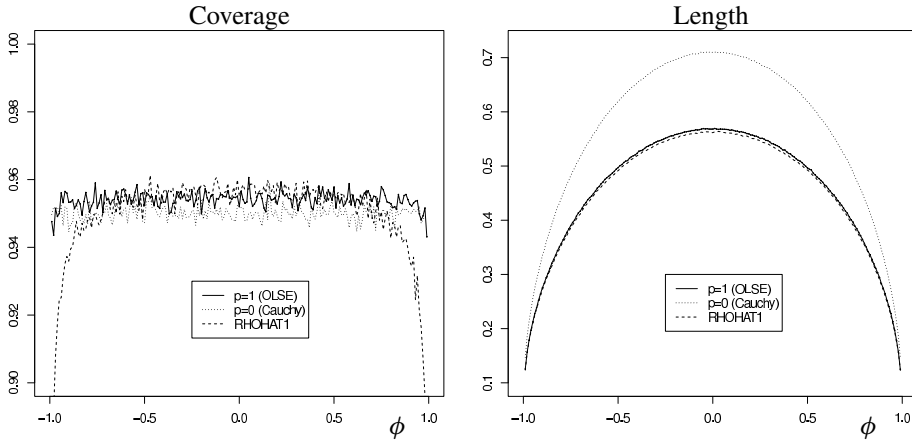


Figure 2. Empirical coverage capability (left) and length (right) of 95% confidence intervals for ϕ centered at $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ for $\phi \in (-1, 1)$; 10,000 simulations were run for each parameter value, with distribution $N(0, 1)$ and $n = 50$.

where

$$n\widehat{\text{Var}}(\hat{\rho}(1)) = \frac{n\sigma^2}{\sum_{t=1}^n X_t^2} \xrightarrow{P} \frac{\sigma^2}{E(X_t^2)}.$$

Figure 2 shows the empirical coverage capability and length of 95% confidence intervals for ϕ centered at $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ when $\phi \in (-1, 1)$ and $n = 50$. The thinnest intervals occur when $\hat{\rho}(1)$ is used, although not by much. The OLSE also has the best overall coverage, except (roughly) for $|\phi| \leq 0.5$, which is where $\hat{\rho}(1)$ once again outperforms the OLSE. Other simulations not shown here reveal that the length of intervals centered at $\tilde{\phi}_p$ keeps growing as p gets larger, while coverage capability starts to break down for $|\phi|$ near 1.

In this paper, we will aim to construct intervals with center (1-2) that outperform those centered at the OLSE. The next section shows the details of this construction, while Section 3 presents some simulations. Section 4 closes the paper with an application and some remarks.

2. Interval construction

Suppose for the moment that we wish to construct a confidence interval for ϕ centered at a linear combination of two weighted least-squares estimators. That is, instead of (1-2), the center would take the form

$$a_1\tilde{\phi}_p + a_2\tilde{\phi}_q, \quad (2-1)$$

where $a_1 + a_2 = 1$ and $p \neq q$. Minimizing the variance of this quantity is equivalent

to minimizing the length of the corresponding interval and occurs when

$$a_1 = \frac{\text{Var}(\tilde{\phi}_q) - \text{Cov}(\tilde{\phi}_p, \tilde{\phi}_q)}{\text{Var}(\tilde{\phi}_p - \tilde{\phi}_q)}. \quad (2-2)$$

Theorem 2.1. *Let $a_1 + a_2 = 1$. If a_1 is given by (2-2), then $\text{Var}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_q)$ is minimized and has upper bound $\text{Var}(\tilde{\phi}_q)$.*

Proof. Let

$$\begin{aligned} f(a_1) &= \text{Var}(a_1\tilde{\phi}_p + (1 - a_1)\tilde{\phi}_q) \\ &= a_1^2 \text{Var}(\tilde{\phi}_p) + (1 - a_1)^2 \text{Var}(\tilde{\phi}_q) + 2a_1(1 - a_1) \text{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) \\ &= a_1^2 \text{Var}(\tilde{\phi}_p - \tilde{\phi}_q) + 2a_1(\text{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) - \text{Var}(\tilde{\phi}_q)) + \text{Var}(\tilde{\phi}_q). \end{aligned}$$

Then $f'(a_1) = 2a_1 \text{Var}(\tilde{\phi}_p - \tilde{\phi}_q) + 2(\text{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) - \text{Var}(\tilde{\phi}_q)) = 0$

$$\Rightarrow a_1 = \frac{\text{Var}(\tilde{\phi}_q) - \text{Cov}(\tilde{\phi}_p, \tilde{\phi}_q)}{\text{Var}(\tilde{\phi}_p - \tilde{\phi}_q)}.$$

Since $f''(a_1) = 2 \text{Var}(\tilde{\phi}_p - \tilde{\phi}_q) > 0$, then this critical value minimizes f . Note that this means

$$a_2 = 1 - a_1 = \frac{\text{Var}(\tilde{\phi}_p) - \text{Cov}(\tilde{\phi}_p, \tilde{\phi}_q)}{\text{Var}(\tilde{\phi}_p - \tilde{\phi}_q)},$$

where the choices of p and q determine the ranges of a_1 and a_2 . Specifically, we have

$$\text{Var}(\tilde{\phi}_p) > \text{Var}(\tilde{\phi}_q) \iff a_1 < 0.5 \text{ and } a_2 > 0.5,$$

$$\text{Var}(\tilde{\phi}_q) > \text{Var}(\tilde{\phi}_p) \iff a_1 > 0.5 \text{ and } a_2 < 0.5,$$

$$\text{Var}(\tilde{\phi}_p) = \text{Var}(\tilde{\phi}_q) \iff a_1 = a_2 = 0.5.$$

Finally, since the critical value found above minimizes f , we have $f(a_1) \leq f(0)$, which is equivalent to saying

$$\text{Var}(a_1\tilde{\phi}_p + (1 - a_1)\tilde{\phi}_q) \leq \text{Var}(\tilde{\phi}_q),$$

where the inequality is strict for $a_1 \neq 0$. □

We would like for the variance of $a_1\tilde{\phi}_p + a_2\tilde{\phi}_q$ to be less than or equal to that of the OLSE. Setting $q = 1$ makes this happen since Theorem 2.1 tells us that

$$\text{Var}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1) \leq \text{Var}(\tilde{\phi}_1).$$

It turns out, however, that the window where these two variances are distinguishable

may be brief since a_1 goes to zero as the sample size increases. This in turn causes $a_1\tilde{\phi}_p + a_2\tilde{\phi}_1$ to be asymptotically normal.

Theorem 2.2. *Let $a_1 + a_2 = 1$. If a_1 is given by (2-2) with $q = 1$, then*

$$\sqrt{n}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1 - \phi) \xrightarrow{D} N\left(0, \frac{\sigma^2}{E(X_t^2)}\right).$$

Proof. First, we note that

$$n \operatorname{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) \xrightarrow{P} \frac{\sigma^2 E|X_t|^{p+q}}{E|X_t|^{p+1} E|X_t|^{q+1}}.$$

Then

$$\begin{aligned} a_1 &= \frac{\operatorname{Var}(\tilde{\phi}_1) - \operatorname{Cov}(\tilde{\phi}_p, \tilde{\phi}_1)}{\operatorname{Var}(\tilde{\phi}_p - \tilde{\phi}_1)} \\ &= \frac{n \operatorname{Var}(\tilde{\phi}_1) - n \operatorname{Cov}(\tilde{\phi}_p, \tilde{\phi}_1)}{n \operatorname{Var}(\tilde{\phi}_p) + n \operatorname{Var}(\tilde{\phi}_1) - 2n \operatorname{Cov}(\tilde{\phi}_p, \tilde{\phi}_1)} \\ &\xrightarrow{P} \frac{\frac{\sigma^2}{E|X_t|^2} - \frac{\sigma^2}{E|X_t|^2}}{\frac{\sigma^2 E|X_t|^{2p}}{(E|X_t|^{p+1})^2} + \frac{\sigma^2}{E|X_t|^2} - \frac{2\sigma^2}{E|X_t|^2}} = \frac{0}{\frac{\sigma^2 E|X_t|^{2p}}{(E|X_t|^{p+1})^2} - \frac{\sigma^2}{E|X_t|^2}} \\ &=: R. \end{aligned}$$

The denominator of R is strictly positive since

$$\operatorname{plim}_{n \rightarrow \infty} n \operatorname{Var}(\tilde{\phi}_p) > \operatorname{plim}_{n \rightarrow \infty} n \operatorname{Var}(\tilde{\phi}_1) \quad \text{for } p \neq 1.$$

Thus, $R = 0$.

Since $a_1 \xrightarrow{P} 0$, we obtain $a_2 \xrightarrow{P} 1$. Hence, $a_1\tilde{\phi}_p + a_2\tilde{\phi}_1$ and $\tilde{\phi}_1$ have the same asymptotic distribution. By (1-3), we have

$$\sqrt{n}(\tilde{\phi}_1 - \phi) \xrightarrow{D} N\left(0, \frac{\sigma^2}{E(X_t^2)}\right).$$

Thus,

$$\sqrt{n}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1 - \phi) \xrightarrow{D} N\left(0, \frac{\sigma^2}{E(X_t^2)}\right)$$

as well. □

If X_1, X_2, \dots, X_n are sample observations from (1-1), then an approximate $(1 - \alpha) \times 100\%$ confidence interval for ϕ centered at $a_1\tilde{\phi}_p + a_2\tilde{\phi}_1$ has endpoints

$$a_1\tilde{\phi}_p + a_2\tilde{\phi}_1 \pm z_{\alpha/2} \sqrt{\widehat{\operatorname{Var}}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1)}.$$

Letting

$$\hat{\sigma}_i^2 = \frac{\sigma^2 \sum_{t=2}^n |X_{t-1}|^{2i}}{\left(\sum_{t=2}^n |X_{t-1}|^{i+1}\right)^2}, \quad \hat{\sigma}_{ij} = \frac{\sigma^2 \sum_{t=2}^n |X_{t-1}|^{i+j}}{\sum_{t=2}^n |X_{t-1}|^{i+1} \sum_{t=2}^n |X_{t-1}|^{j+1}},$$

$$\hat{a}_1 = \frac{\hat{\sigma}_1^2 - \hat{\sigma}_{p1}}{\hat{\sigma}_p^2 + \hat{\sigma}_1^2 - 2\hat{\sigma}_{p1}}, \quad \hat{a}_2 = \frac{\hat{\sigma}_p^2 - \hat{\sigma}_{p1}}{\hat{\sigma}_p^2 + \hat{\sigma}_1^2 - 2\hat{\sigma}_{p1}},$$

we then have

$$\begin{aligned} n\widehat{\text{Var}}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1) &= n(\hat{a}_1^2\widehat{\text{Var}}(\tilde{\phi}_p) + \hat{a}_2^2\widehat{\text{Var}}(\tilde{\phi}_1) + 2\hat{a}_1\hat{a}_2\widehat{\text{Cov}}(\tilde{\phi}_p, \tilde{\phi}_1)) \\ &= \frac{n(\hat{\sigma}_1^2\hat{\sigma}_p^2 - \hat{\sigma}_{p1}^2)}{\hat{\sigma}_1^2 + \hat{\sigma}_p^2 - 2\hat{\sigma}_{p1}} \xrightarrow{P} \frac{\sigma^2}{E(X_t^2)}. \end{aligned}$$

However, observe that $\hat{\sigma}_{p1} = \hat{\sigma}_1^2$ which implies that $\widehat{\text{Var}}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1) = \widehat{\text{Var}}(\tilde{\phi}_1)$. Thus, our choice of asymptotic estimators when $q = 1$ has the unintended consequence of causing our interval to be equivalent to that of the OLSE.

Herein lies the motive to go with center (1-2) in lieu of center (2-1). By replacing $\tilde{\phi}_1$ with $\hat{\rho}(1)$, we avoid this asymptotic equivalence, while preserving some of the desirable properties associated with (2-1). In the upcoming simulations, we also replace $\hat{\sigma}_1^2 = \widehat{\text{Var}}(\tilde{\phi}_1)$ with

$$\widehat{\text{Var}}(\hat{\rho}(1)) = \frac{\sigma^2}{\sum_{t=1}^n X_t^2},$$

but retain $\hat{\sigma}_{p1} = \widehat{\text{Cov}}(\tilde{\phi}_p, \tilde{\phi}_1)$.

3. Simulations

We now look at the length and coverage capability of 95% confidence intervals for ϕ centered at the OLSE and $a_1\tilde{\phi}_p + a_2\hat{\rho}(1)$ for various $p \neq 1$. Each figure reflects 10,000 simulation runs of $n = 50$ independent observations with distribution $N(0, 1)$.

In Figure 3, top, we see that the $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ interval has at least as much coverage as the OLSE interval when (roughly) $|\phi| \leq 0.5$. The $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ interval is also slightly shorter over this same region. In Figure 3, bottom, p has increased to 2, but the coverage of the $a_1\tilde{\phi}_2 + a_2\hat{\rho}(1)$ interval has degenerated with no meaningful difference in interval lengths.

In Figure 4, p has increased to 3 (top two graphs) and 4 (middle row of graphs). The $a_1\tilde{\phi}_3 + a_2\hat{\rho}(1)$ and $a_1\tilde{\phi}_4 + a_2\hat{\rho}(1)$ intervals both return back to the performance level of the $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ interval, with (roughly) $|\phi| \leq 0.5$ again being the domain of interest.

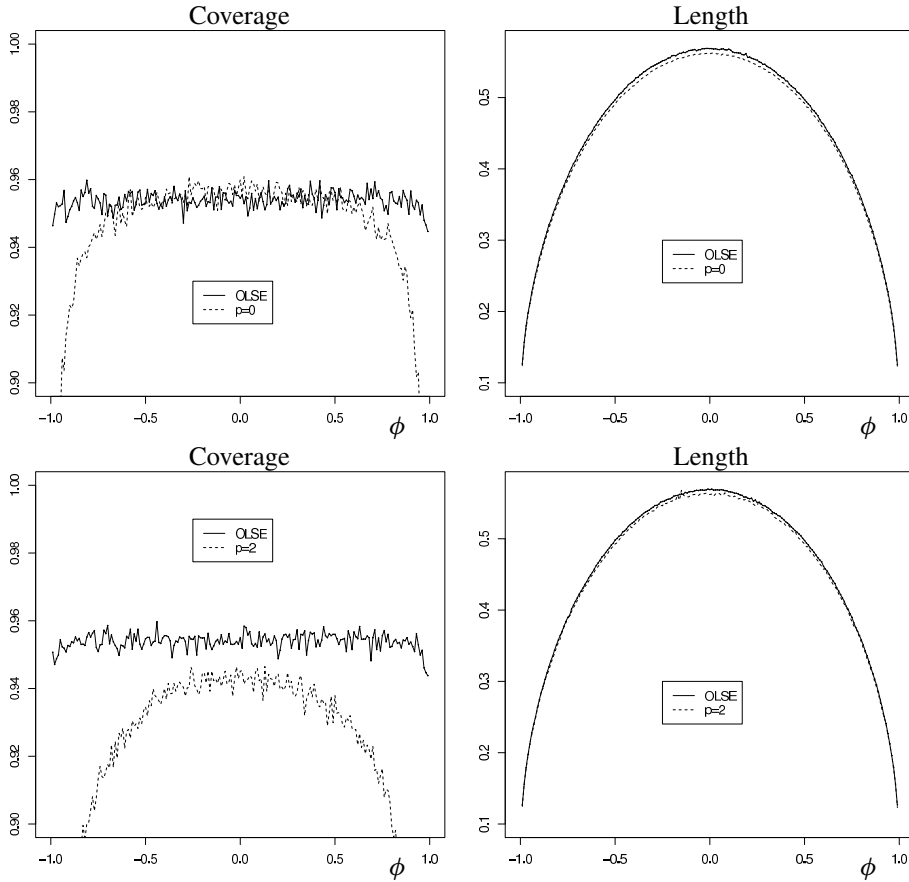


Figure 3. Top row: Empirical coverage capability (left) and length (right) of 95% confidence intervals for ϕ centered at the OLSE and $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ for $\phi \in (-1, 1)$. Bottom row: Same information, for $a_1\tilde{\phi}_2 + a_2\hat{\rho}(1)$.

In Figure 4, bottom, we create intervals whose centers are simply unweighted averages of the OLSE and the sample correlation coefficient. That is, the intervals' endpoints take the form

$$0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1) \pm 1.96\sqrt{\widehat{\text{Var}}(0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1))}.$$

Using the fact that $2\widehat{\text{Cov}}(\tilde{\phi}_1, \hat{\rho}(1)) \approx \widehat{\text{Var}}(\tilde{\phi}_1) + \widehat{\text{Var}}(\hat{\rho}(1))$, this is approximately

$$0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1) \pm 1.96\sqrt{0.5(\widehat{\text{Var}}(\tilde{\phi}_1) + \widehat{\text{Var}}(\hat{\rho}(1)))}.$$

There is no significant difference between the $0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1)$ and OLSE intervals.

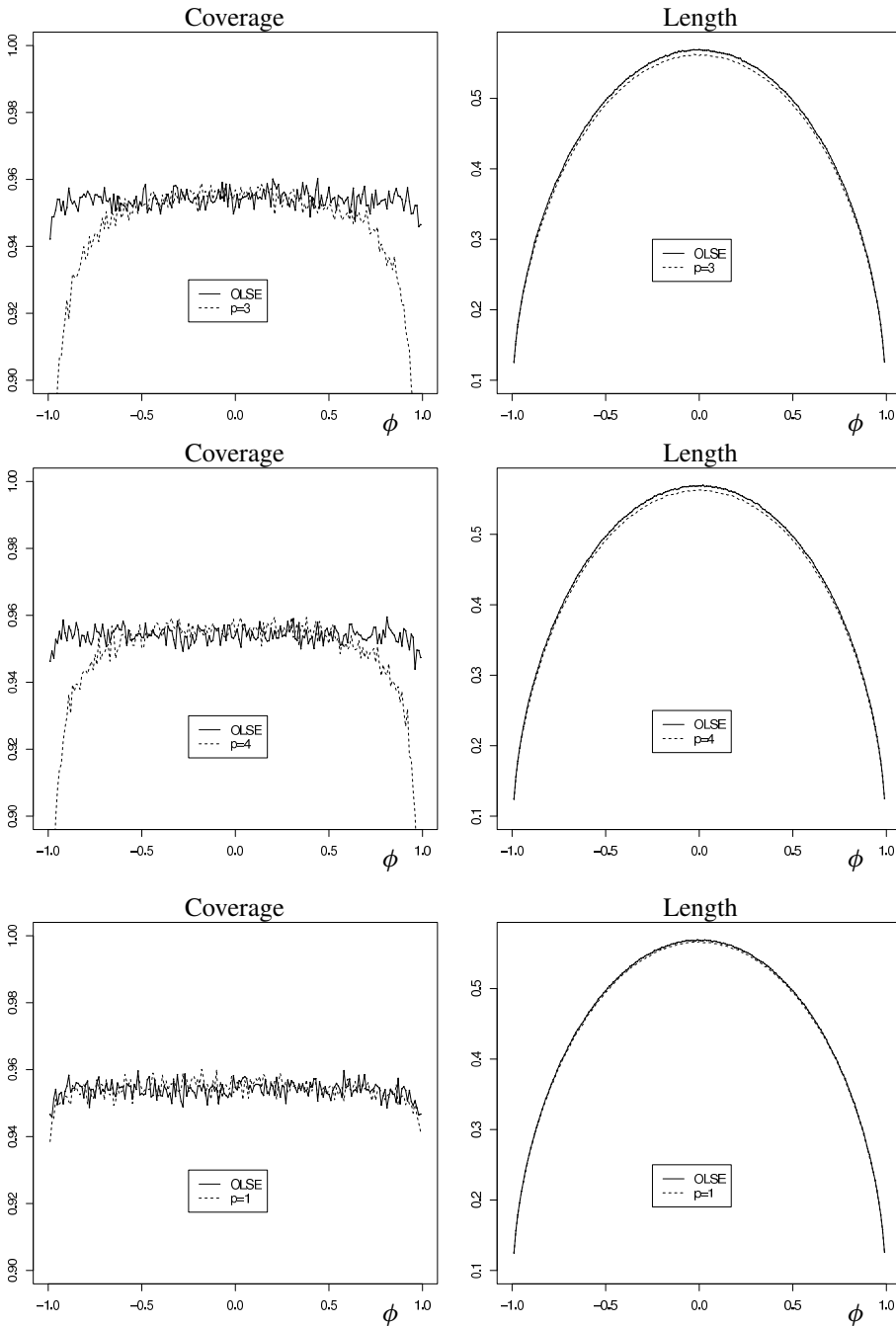


Figure 4. Top row: Empirical coverage capability (left) and length (right) of 95% confidence intervals for ϕ centered at the OLSE and $a_1\tilde{\phi}_3 + a_2\hat{\rho}(1)$ for $\phi \in (-1, 1)$. Middle row: Same, for $a_1\tilde{\phi}_4 + a_2\hat{\rho}(1)$. Bottom row: Same, for $0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1)$.

4. Closing remarks

The performance of the confidence interval centered at $a_1\tilde{\phi}_p + a_2\hat{\rho}(1)$ presented in this paper is modest, but not unimportant. For parameter values (roughly) between -0.5 and 0.5 , its coverage tends to be at least as good as that of the OLSE interval while having a slightly smaller margin of error. This interval also does not require a large sample size, which can be good for certain practical purposes.

For example, consider the daily stock prices for Exxon Mobil Corporation during the fall quarter of 2011 (i.e., September 23 to December 21). A reasonable model for this time series is an ARIMA(1, 1, 0), where $\{X_t\} \sim \text{ARIMA}(p, 1, q)$ implies $\{X_t - X_{t-1}\} \sim \text{ARMA}(p, q)$. Thus, if X_t stands for the price at time t and $Y_t = X_t - X_{t-1}$, it follows that $\{Y_t\} \sim \text{AR}(1)$ with estimated model $Y_t = -0.0444Y_{t-1} + \epsilon_t$. Both the $\{X_t\}$ and $\{Y_t\}$ processes are shown in Figure 5.

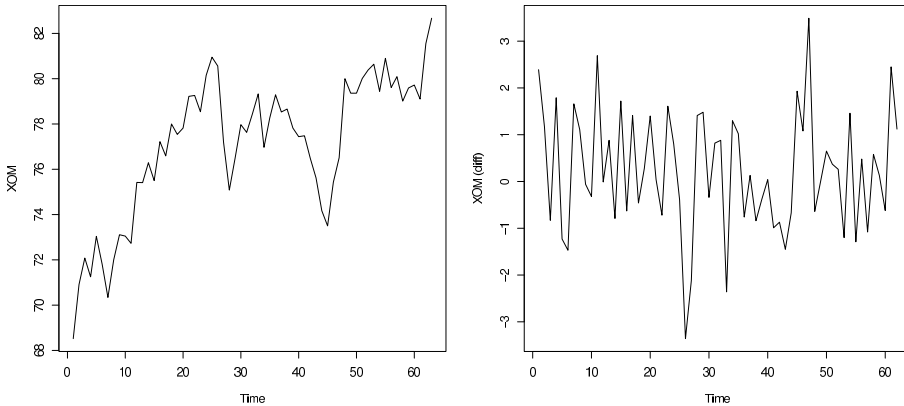


Figure 5. The original (left) and differenced (right) stock prices for Exxon Mobil (XOM) from 9/23/11 to 12/21/11. The sample sizes are 63 and 62, respectively.

If we supplement this model with 95% confidence intervals for ϕ , we get the following:

Center	Interval	Length
$\tilde{\phi}_1$	$(-0.2089871, 0.1719117)$	0.3808988
$\hat{\rho}(1)$	$(-0.2076522, 0.1710108)$	0.3786630
$0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1)$	$(-0.2083205, 0.1714621)$	0.3797825
$a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$	$(-0.2036185, 0.1749948)$	0.3786133
$a_1\tilde{\phi}_2 + a_2\hat{\rho}(1)$	$(-0.2127095, 0.1658015)$	0.3785110
$a_1\tilde{\phi}_3 + a_2\hat{\rho}(1)$	$(-0.2100079, 0.1686098)$	0.3786177
$a_1\tilde{\phi}_4 + a_2\hat{\rho}(1)$	$(-0.2092005, 0.1694378)$	0.3786383

All seven intervals contain the point estimate $\hat{\phi} = -0.0444$, but the last four are slightly thinner than the first three.

One extension of the research presented in this paper would be to create confidence intervals centered around a linear combination of an arbitrary number of weighted least-squares estimators. For example, it can be shown that the variance of $a_1\tilde{\phi}_p + a_2\tilde{\phi}_q + a_3\tilde{\phi}_r$ is minimized when

$$a_1 = \frac{\sigma_{qr}^*(\sigma_r^2 - \sigma_{pr}) + M(\sigma_r^2 - \sigma_{qr})}{\sigma_{pr}^*\sigma_{qr}^* - M^2}, \quad a_2 = \frac{\sigma_{pr}^*(\sigma_r^2 - \sigma_{qr}) + M(\sigma_r^2 - \sigma_{pr})}{\sigma_{pr}^*\sigma_{qr}^* - M^2},$$

and $a_3 = 1 - a_1 - a_2$, where $\sigma_i^2 = \text{Var}(\tilde{\phi}_i)$, $\sigma_{ij} = \text{Cov}(\tilde{\phi}_i, \tilde{\phi}_j)$, $\sigma_{ij}^* = \text{Var}(\tilde{\phi}_i - \tilde{\phi}_j)$, and $M = \sigma_{pr} + \sigma_{qr} - \sigma_{pq} - \sigma_r^2$. However, once the number of estimators in the center goes beyond two, the work required to construct and analyze the interval may outweigh any benefits it would bestow.

Another (less tedious) extension would be to find a new sequence $\{a_{1,n}\}$ that converges to zero while yielding a linear combination of estimators with smaller MSE than the OLSE. This new combination would still have the same distributional limit as the OLSE and could then serve as the center for another competitive interval for ϕ . Specifically, if we simply set the standard error equal to the square root of the asymptotic variance of the OLSE, the resulting interval should have length equal to that of the OLSE, but with better coverage capability.

References

- [Gallagher and Tunno 2008] C. Gallagher and F. Tunno, "A small sample confidence interval for autoregressive parameters", *J. Statist. Plann. Inference* **138**:12 (2008), 3858–3868. MR 2009m:62257 Zbl 1146.62065
- [Phillips et al. 2004] P. C. B. Phillips, J. Y. Park, and Y. Chang, "Nonlinear instrumental variable estimation of an autoregression", *J. Econometrics* **118**:1-2 (2004), 219–246. MR 2004j:62186 Zbl 1033.62085
- [Shumway and Stoffer 2006] R. H. Shumway and D. S. Stoffer, *Time series analysis and its applications*, 2nd ed., Springer, New York, 2006. MR 2007a:62003 Zbl 1096.62088
- [So and Shin 1999] B. S. So and D. W. Shin, "Cauchy estimators for autoregressive processes with applications to unit root tests and confidence intervals", *Econometric Theory* **15**:2 (1999), 165–176. MR 2000f:62228 Zbl 0985.62072

Received: 2011-12-31 Revised: 2012-04-26 Accepted: 2012-05-10

ftunno@astate.edu

Department of Mathematics and Statistics,
Arkansas State University, P.O. Box 70,
State University, AK 72467, United States

ashton.erwin@smail.astate.edu

Department of Mathematics and Statistics,
Arkansas State University, P.O. Box 70,
State University, AK 72467, United States

involve

msp.org/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsteam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2013 is US \$105/year for the electronic version, and \$145/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

involve

2013

vol. 6

no. 1

Refined inertias of tree sign patterns of orders 2 and 3 D. D. OLESKY, MICHAEL F. REMPEL AND P. VAN DEN DRIESSCHE	1
The group of primitive almost pythagorean triples NIKOLAI A. KRYLOV AND LINDSAY M. KULZER	13
Properties of generalized derangement graphs HANNAH JACKSON, KATHRYN NYMAN AND LES REID	25
Rook polynomials in three and higher dimensions FERYAL ALAYONT AND NICHOLAS KRZYWONOS	35
New confidence intervals for the AR(1) parameter FEREBEE TUNNO AND ASHTON ERWIN	53
Knots in the canonical book representation of complete graphs DANA ROWLAND AND ANDREA POLITANO	65
On closed modular colorings of rooted trees BRYAN PHINEZY AND PING ZHANG	83
Iterations of quadratic polynomials over finite fields WILLIAM WORDEN	99
Positive solutions to singular third-order boundary value problems on purely discrete time scales COURTNEY DEHOET, CURTIS KUNKEL AND ASHLEY MARTIN	113



1944-4176(2013)6:1;1-A