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# New confidence intervals for the AR(1) parameter

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This paper presents a new way to construct confidence intervals for the unknown parameter in a first-order autoregressive, or AR(1), time series. Typically, one might construct such an interval by centering it around the ordinary least-squares estimator, but this new method instead centers the interval around a linear combination of a weighted least-squares estimator and the sample autocorrelation function at lag one. When the sample size is small and the parameter has magnitude closer to zero than one, this new approach tends to result in a slightly thinner interval with at least as much coverage.

## 1. Introduction

Consider the causal stationary AR(1) time series given by

$$X_t = \phi X_{t-1} + \epsilon_t, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{1-1}$$

where $|\phi| < 1$, $E(X_t) = 0$ and $\{\epsilon_t\} \overset{\text{iid}}{\sim} N(0, \sigma^2)$. We seek a new way to construct confidence intervals for the unknown parameter $\phi$.

If $X_1, X_2, \ldots, X_n$ are sample observations from this process, then a point estimate for $\phi$ is found by calculating

$$\tilde{\phi}_p = \frac{\sum_{t=2}^{n} S_{t-1} |X_{t-1}|^p X_t}{\sum_{t=2}^{n} |X_{t-1}|^{p+1}},$$

where $p \in \{0, 1, 2, \ldots\}$ and $S_t$ is the sign function, defined by

$$S_t = \begin{cases} 1 & \text{if } X_t > 0, \\ 0 & \text{if } X_t = 0, \\ -1 & \text{if } X_t < 0. \end{cases}$$

The estimator $\tilde{\phi}_p$ can be thought of as a weighted least-squares estimator with form

$$\frac{\sum_{t=2}^{n} W_{t-1} X_{t-1} X_t}{\sum_{t=2}^{n} W_{t-1} X_{t-1}^2}$$

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and weight $W_t = |X_t|^{p-1}$. Note that, when $p = 1$, we get the ordinary (unweighted) least-squares estimator (OLSE), and when $p = 0$, we get what has come to be called the Cauchy estimator:

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^{n} X_{t-1}X_t}{\sum_{t=2}^{n} X_{t-1}^2} \quad \text{(OLSE)}, \qquad \tilde{\phi}_0 = \frac{\sum_{t=2}^{n} S_{t-1}X_t}{\sum_{t=2}^{n} |X_{t-1}|} \quad \text{(Cauchy)}.$$

The OLSE has been studied since the time of Gauss and its optimal properties for linear models are well known. The eponymously named Cauchy estimator dates back to about the same time and is sometimes used as a surrogate for the OLSE. Traditionally, confidence intervals for $\phi$ have been centered around the OLSE, although So and Shin [1999] and Phillips, Park and Chang [Phillips et al. 2004] showed that the Cauchy estimator has certain advantages over the OLSE when dealing with a unit root autoregression. Gallagher and Tunno [2008] constructed a confidence interval for $\phi$ centered around a linear combination of both estimators.

Another point estimate for $\phi$ comes from the sample autocorrelation function of $\{X_t\}$ at lag one, given by

$$\hat{\rho}(1) = \frac{\sum_{t=2}^{n} X_{t-1}X_t}{\sum_{t=1}^{n} X_t^2}.$$

The autocovariance function of $\{X_t\}$ at lag $h$ for an AR(1) series is given by $\gamma(h) = \mathrm{Cov}(X_t, X_{t+h}) = \phi^{|h|}\sigma^2/(1-\phi^2)$, which makes the true lag-one autocorrelation function equal to

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\phi\sigma^2/(1-\phi^2)}{\sigma^2/(1-\phi^2)} = \phi.$$

Observe that the structure of $\hat{\rho}(1)$ is similar to that of the OLSE. In fact, for an AR(1) series, the Yule–Walker, maximum likelihood, and least-squares estimators for $\phi$ are all approximately the same [Shumway and Stoffer 2006, Section 3.6]. Note also that, in general, if $\{X_t\}$ is not mean-zero, we would subtract $\overline{X}$ from each observation when calculating things like $\hat{\rho}(h)$ and $\tilde{\phi}_p$.

To get a feel for how $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ behave relative to one another, Figure 1 shows their empirical bias and mean squared error (MSE) when $\phi \in (-1, 1)$ and $n = 50$. The Cauchy estimator has the lowest absolute bias, and $\hat{\rho}(1)$ has the smallest MSE for parameter values (roughly) between $-0.5$ and $0.5$, while the OLSE has the smallest MSE elsewhere. Other simulations not shown here reveal that the MSE and absolute bias of $\tilde{\phi}_p$ keep growing as $p$ gets larger.

The goal of this paper is to construct a confidence interval for $\phi$ centered around a linear combination of an arbitrary weighted least-squares estimator and the sample autocorrelation function at lag one. That is, the center will take the form

$$a_1\tilde{\phi}_p + a_2\hat{\rho}(1), \tag{1-2}$$

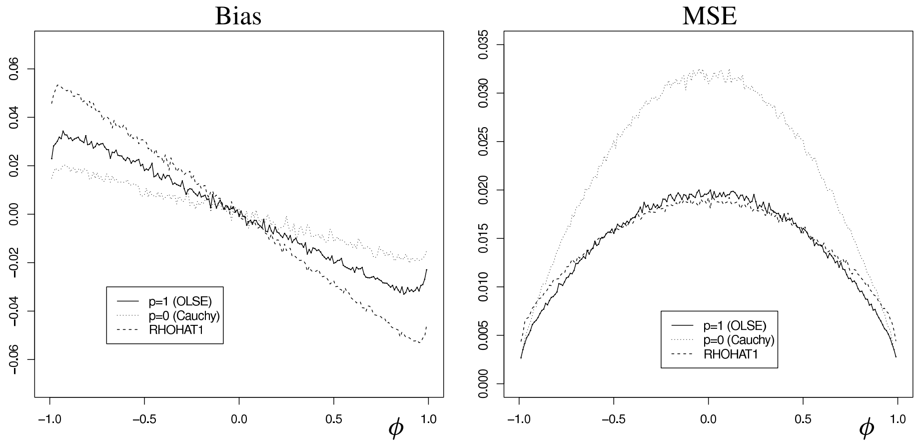


**Figure 1.** Empirical bias (left) and mean squared error (right) of $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ for $\phi \in (-1, 1)$; 10,000 simulations were run for each parameter value, with distribution $N(0, 1)$ and $n = 50$.

where $a_1 + a_2 = 1$ and $p \neq 1$. We first, however, need to take a brief look at how intervals centered around a single estimator behave in order to find a proper target for our new interval to outperform.

Theorem 2.1 from [Gallagher and Tunno 2008] states that for the AR(1) series given in (1-1), we have

$$\sqrt{n}(\tilde{\phi}_p - \phi) \xrightarrow{D} N\left(0, \frac{\sigma^2 E(X_t^{2p})}{(E|X_t|^{p+1})^2}\right) \tag{1-3}$$

for all $p$ such that $E(|X_t|^r) < \infty$, where $r = \max(2p, p+1)$. Since the error terms in our series are normal, the $X_t$'s have finite moments of all orders. Thus, this theorem can be used to create confidence intervals for $\phi$ centered at $\tilde{\phi}_p$ for any choice of $p$.

Specifically, if $X_1, X_2, \ldots, X_n$ are sample observations from (1-1), then an approximate $(1-\alpha) \times 100\%$ confidence interval for $\phi$ has endpoints

$$\tilde{\phi}_p \pm z_{\alpha/2}\sqrt{\widehat{\text{Var}}(\tilde{\phi}_p)},$$

where

$$n\widehat{\text{Var}}(\tilde{\phi}_p) = \frac{\sigma^2 n^{-1} \sum_{t=2}^n X_{t-1}^{2p}}{\left(n^{-1} \sum_{t=2}^n |X_{t-1}|^{p+1}\right)^2} \xrightarrow{P} \frac{\sigma^2 E(X_{t-1}^{2p})}{\left(E|X_{t-1}|^{p+1}\right)^2}$$

and $z_{\alpha/2}$ is the standard normal critical value with area $\alpha/2$ to its right.

Similarly, we can create confidence intervals for $\phi$ centered at $\hat{\rho}(1)$. If we think of $\hat{\rho}(1)$ as being nearly the equivalent of the OLSE, then an approximate $(1-\alpha) \times 100\%$ confidence interval for $\phi$ has endpoints

$$\hat{\rho}(1) \pm z_{\alpha/2}\sqrt{\widehat{\text{Var}}(\hat{\rho}(1))},$$

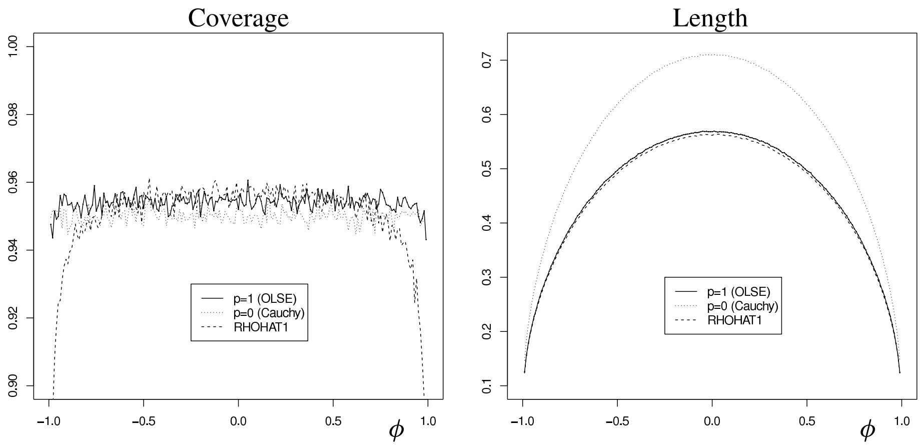


**Figure 2.** Empirical coverage capability (left) and length (right) of 95% confidence intervals for $\phi$ centered at $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ for $\phi \in (-1, 1)$; 10,000 simulations were run for each parameter value, with distribution $N(0, 1)$ and $n = 50$.

where

$$n\widehat{\mathrm{Var}}(\hat{\rho}(1)) = \frac{n\sigma^2}{\sum_{t=1}^{n} X_t^2} \xrightarrow{P} \frac{\sigma^2}{E(X_t^2)}.$$

Figure 2 shows the empirical coverage capability and length of 95% confidence intervals for $\phi$ centered at $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\hat{\rho}(1)$ when $\phi \in (-1, 1)$ and $n = 50$. The thinnest intervals occur when $\hat{\rho}(1)$ is used, although not by much. The OLSE also has the best overall coverage, except (roughly) for $|\phi| \le 0.5$, which is where $\hat{\rho}(1)$ once again outperforms the OLSE. Other simulations not shown here reveal that the length of intervals centered at $\tilde{\phi}_p$ keeps growing as $p$ gets larger, while coverage capability starts to break down for $|\phi|$ near 1.

In this paper, we will aim to construct intervals with center (1-2) that outperform those centered at the OLSE. The next section shows the details of this construction, while Section 3 presents some simulations. Section 4 closes the paper with an application and some remarks.

## 2. Interval construction

Suppose for the moment that we wish to construct a confidence interval for $\phi$ centered at a linear combination of two weighted least-squares estimators. That is, instead of (1-2), the center would take the form

$$a_1\tilde{\phi}_p + a_2\tilde{\phi}_q, \tag{2-1}$$

where $a_1 + a_2 = 1$ and $p \ne q$. Minimizing the variance of this quantity is equivalent to minimizing the length of the corresponding interval and occurs when

$$a_1 = \frac{\mathrm{Var}(\tilde{\phi}_q) - \mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_q)}{\mathrm{Var}(\tilde{\phi}_p - \tilde{\phi}_q)}. \tag{2-2}$$

**Theorem 2.1.** *Let $a_1 + a_2 = 1$. If $a_1$ is given by* (2-2), *then* $\mathrm{Var}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_q)$ *is minimized and has upper bound* $\mathrm{Var}(\tilde{\phi}_q)$.

*Proof.* Let

$$\begin{aligned} f(a_1) &= \mathrm{Var}\big(a_1\tilde{\phi}_p + (1-a_1)\tilde{\phi}_q\big) \\ &= a_1^2 \mathrm{Var}(\tilde{\phi}_p) + (1-a_1)^2 \mathrm{Var}(\tilde{\phi}_q) + 2a_1(1-a_1)\,\mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) \\ &= a_1^2 \mathrm{Var}(\tilde{\phi}_p - \tilde{\phi}_q) + 2a_1\big(\mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) - \mathrm{Var}(\tilde{\phi}_q)\big) + \mathrm{Var}(\tilde{\phi}_q). \end{aligned}$$

Then $f'(a_1) = 2a_1 \mathrm{Var}(\tilde{\phi}_p - \tilde{\phi}_q) + 2\big(\mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) - \mathrm{Var}(\tilde{\phi}_q)\big) = 0$

$$\Rightarrow \quad a_1 = \frac{\mathrm{Var}(\tilde{\phi}_q) - \mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_q)}{\mathrm{Var}(\tilde{\phi}_p - \tilde{\phi}_q)}.$$

Since $f''(a_1) = 2\,\mathrm{Var}(\tilde{\phi}_p - \tilde{\phi}_q) > 0$, then this critical value minimizes $f$. Note that this means

$$a_2 = 1 - a_1 = \frac{\mathrm{Var}(\tilde{\phi}_p) - \mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_q)}{\mathrm{Var}(\tilde{\phi}_p - \tilde{\phi}_q)},$$

where the choices of $p$ and $q$ determine the ranges of $a_1$ and $a_2$. Specifically, we have

$$\begin{aligned} \mathrm{Var}(\tilde{\phi}_p) > \mathrm{Var}(\tilde{\phi}_q) &\iff a_1 < 0.5 \text{ and } a_2 > 0.5, \\ \mathrm{Var}(\tilde{\phi}_q) > \mathrm{Var}(\tilde{\phi}_p) &\iff a_1 > 0.5 \text{ and } a_2 < 0.5, \\ \mathrm{Var}(\tilde{\phi}_p) = \mathrm{Var}(\tilde{\phi}_q) &\iff a_1 = a_2 = 0.5. \end{aligned}$$

Finally, since the critical value found above minimizes $f$, we have $f(a_1) \le f(0)$, which is equivalent to saying

$$\mathrm{Var}\big(a_1\tilde{\phi}_p + (1-a_1)\tilde{\phi}_q\big) \le \mathrm{Var}(\tilde{\phi}_q),$$

where the inequality is strict for $a_1 \neq 0$. □

We would like for the variance of $a_1\tilde{\phi}_p + a_2\tilde{\phi}_q$ to be less than or equal to that of the OLSE. Setting $q = 1$ makes this happen since Theorem 2.1 tells us that

$$\mathrm{Var}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1) \le \mathrm{Var}(\tilde{\phi}_1).$$

It turns out, however, that the window where these two variances are distinguishable may be brief since $a_1$ goes to zero as the sample size increases. This in turn causes $a_1\tilde{\phi}_p + a_2\tilde{\phi}_1$ to be asymptotically normal.

**Theorem 2.2.** *Let $a_1 + a_2 = 1$. If $a_1$ is given by* (2-2) *with $q = 1$, then*

$$\sqrt{n}\big(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1 - \phi\big) \xrightarrow{D} N\left(0, \frac{\sigma^2}{E(X_t^2)}\right).$$

*Proof.* First, we note that

$$n\,\mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_q) \xrightarrow{P} \frac{\sigma^2 E|X_t|^{p+q}}{E|X_t|^{p+1}E|X_t|^{q+1}}.$$

Then

$$\begin{aligned} a_1 &= \frac{\mathrm{Var}(\tilde{\phi}_1) - \mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_1)}{\mathrm{Var}(\tilde{\phi}_p - \tilde{\phi}_1)} \\ &= \frac{n\,\mathrm{Var}(\tilde{\phi}_1) - n\,\mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_1)}{n\,\mathrm{Var}(\tilde{\phi}_p) + n\,\mathrm{Var}(\tilde{\phi}_1) - 2n\,\mathrm{Cov}(\tilde{\phi}_p, \tilde{\phi}_1)} \\ &\xrightarrow{P} \frac{\dfrac{\sigma^2}{E|X_t|^2} - \dfrac{\sigma^2}{E|X_t|^2}}{\dfrac{\sigma^2 E|X_t|^{2p}}{(E|X_t|^{p+1})^2} + \dfrac{\sigma^2}{E|X_t|^2} - \dfrac{2\sigma^2}{E|X_t|^2}} = \frac{0}{\dfrac{\sigma^2 E|X_t|^{2p}}{(E|X_t|^{p+1})^2} - \dfrac{\sigma^2}{E|X_t|^2}} \\ &=: R. \end{aligned}$$

The denominator of $R$ is strictly positive since

$$\operatorname*{plim}_{n\to\infty} n\,\mathrm{Var}(\tilde{\phi}_p) > \operatorname*{plim}_{n\to\infty} n\,\mathrm{Var}(\tilde{\phi}_1) \quad \text{for } p \neq 1.$$

Thus, $R = 0$.

Since $a_1 \xrightarrow{P} 0$, we obtain $a_2 \xrightarrow{P} 1$. Hence, $a_1\tilde{\phi}_p + a_2\tilde{\phi}_1$ and $\tilde{\phi}_1$ have the same asymptotic distribution. By (1-3), we have

$$\sqrt{n}(\tilde{\phi}_1 - \phi) \xrightarrow{D} N\left(0, \frac{\sigma^2}{E(X_t^2)}\right).$$

Thus,

$$\sqrt{n}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1 - \phi) \xrightarrow{D} N\left(0, \frac{\sigma^2}{E(X_t^2)}\right)$$

as well. □

If $X_1, X_2, \dots, X_n$ are sample observations from (1-1), then an approximate $(1-\alpha) \times 100\%$ confidence interval for $\phi$ centered at $a_1\tilde{\phi}_p + a_2\tilde{\phi}_1$ has endpoints

$$a_1\tilde{\phi}_p + a_2\tilde{\phi}_1 \pm z_{\alpha/2}\sqrt{\widehat{\mathrm{Var}}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1)}.$$

Letting

$$\hat{\sigma}_i^2 = \frac{\sigma^2 \sum_{t=2}^n |X_{t-1}|^{2i}}{\left(\sum_{t=2}^n |X_{t-1}|^{i+1}\right)^2}, \quad \hat{\sigma}_{ij} = \frac{\sigma^2 \sum_{t=2}^n |X_{t-1}|^{i+j}}{\sum_{t=2}^n |X_{t-1}|^{i+1} \sum_{t=2}^n |X_{t-1}|^{j+1}},$$

$$\hat{a}_1 = \frac{\hat{\sigma}_1^2 - \hat{\sigma}_{p1}}{\hat{\sigma}_p^2 + \hat{\sigma}_1^2 - 2\hat{\sigma}_{p1}}, \quad \hat{a}_2 = \frac{\hat{\sigma}_p^2 - \hat{\sigma}_{p1}}{\hat{\sigma}_p^2 + \hat{\sigma}_1^2 - 2\hat{\sigma}_{p1}},$$

we then have

$$\begin{aligned} n\widehat{\mathrm{Var}}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1) &= n\big(\hat{a}_1^2\widehat{\mathrm{Var}}(\tilde{\phi}_p) + \hat{a}_2^2\widehat{\mathrm{Var}}(\tilde{\phi}_1) + 2\hat{a}_1^2\hat{a}_2^2\widehat{\mathrm{Cov}}(\tilde{\phi}_p, \tilde{\phi}_1)\big) \\ &= \frac{n(\hat{\sigma}_1^2\hat{\sigma}_p^2 - \hat{\sigma}_{p1}^2)}{\hat{\sigma}_1^2 + \hat{\sigma}_p^2 - 2\hat{\sigma}_{p1}} \xrightarrow{P} \frac{\sigma^2}{E(X_t^2)}. \end{aligned}$$

However, observe that $\hat{\sigma}_{p1} = \hat{\sigma}_1^2$ which implies that $\widehat{\mathrm{Var}}(a_1\tilde{\phi}_p + a_2\tilde{\phi}_1) = \widehat{\mathrm{Var}}(\tilde{\phi}_1)$. Thus, our choice of asymptotic estimators when $q = 1$ has the unintended consequence of causing our interval to be equivalent to that of the OLSE.

Herein lies the motive to go with center (1-2) in lieu of center (2-1). By replacing $\tilde{\phi}_1$ with $\hat{\rho}(1)$, we avoid this asymptotic equivalence, while preserving some of the desirable properties associated with (2-1). In the upcoming simulations, we also replace $\hat{\sigma}_1^2 = \widehat{\mathrm{Var}}(\tilde{\phi}_1)$ with

$$\widehat{\mathrm{Var}}(\hat{\rho}(1)) = \frac{\sigma^2}{\sum_{t=1}^n X_t^2},$$

but retain $\hat{\sigma}_{p1} = \widehat{\mathrm{Cov}}(\tilde{\phi}_p, \tilde{\phi}_1)$.

## 3. Simulations

We now look at the length and coverage capability of 95% confidence intervals for $\phi$ centered at the OLSE and $a_1\tilde{\phi}_p + a_2\hat{\rho}(1)$ for various $p \neq 1$. Each figure reflects 10,000 simulation runs of $n = 50$ independent observations with distribution $N(0, 1)$.

In Figure 3, top, we see that the $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ interval has at least as much coverage as the OLSE interval when (roughly) $|\phi| \leq 0.5$. The $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ interval is also slightly shorter over this same region. In Figure 3, bottom, $p$ has increased to 2, but the coverage of the $a_1\tilde{\phi}_2 + a_2\hat{\rho}(1)$ interval has degenerated with no meaningful difference in interval lengths.

In Figure 4, $p$ has increased to 3 (top two graphs) and 4 (middle row of graphs). The $a_1\tilde{\phi}_3 + a_2\hat{\rho}(1)$ and $a_1\tilde{\phi}_4 + a_2\hat{\rho}(1)$ intervals both return back to the performance level of the $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ interval, with (roughly) $|\phi| \leq 0.5$ again being the domain of interest.

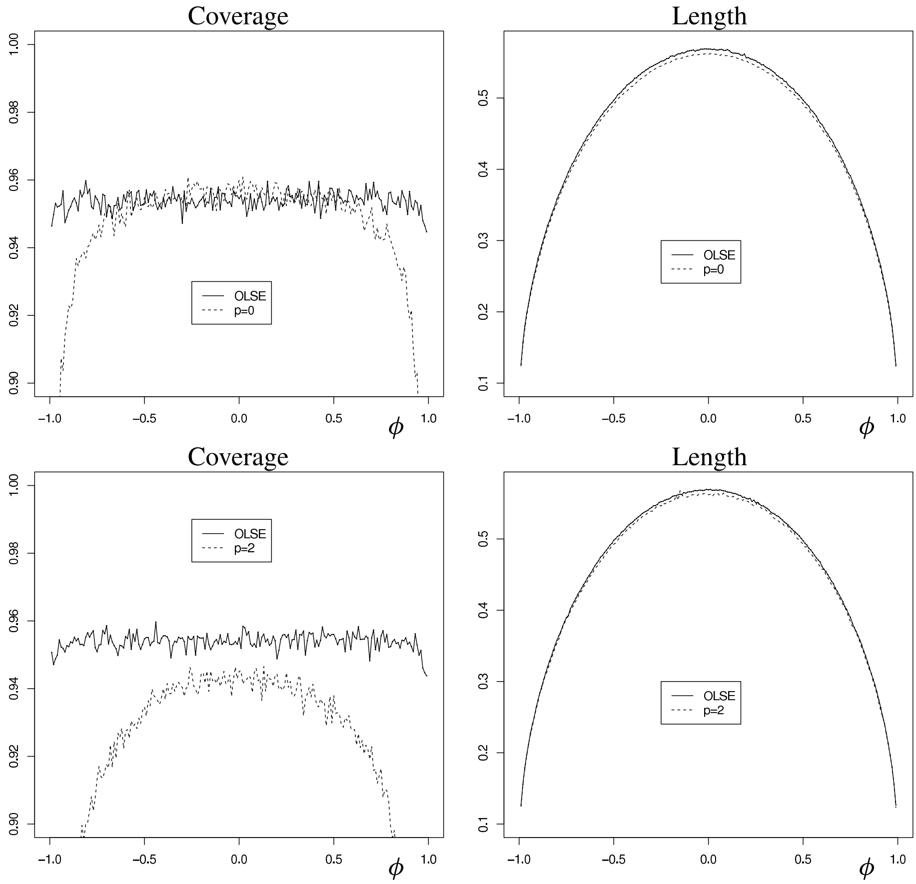


**Figure 3.** Top row: Empirical coverage capability (left) and length (right) of 95% confidence intervals for $\phi$ centered at the OLSE and $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ for $\phi \in (-1, 1)$. Bottom row: Same information, for $a_1\tilde{\phi}_2 + a_2\hat{\rho}(1)$.

In Figure 4, bottom, we create intervals whose centers are simply unweighted averages of the OLSE and the sample correlation coefficient. That is, the intervals' endpoints take the form

$$0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1) \pm 1.96\sqrt{\widehat{\mathrm{Var}}\big(0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1)\big)}.$$

Using the fact that $2\widehat{\mathrm{Cov}}(\tilde{\phi}_1, \hat{\rho}(1)) \approx \widehat{\mathrm{Var}}(\tilde{\phi}_1) + \widehat{\mathrm{Var}}(\hat{\rho}(1))$, this is approximately

$$0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1) \pm 1.96\sqrt{0.5\big(\widehat{\mathrm{Var}}(\tilde{\phi}_1) + \widehat{\mathrm{Var}}(\hat{\rho}(1))\big)}.$$

There is no significant difference between the $0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1)$ and OLSE intervals.

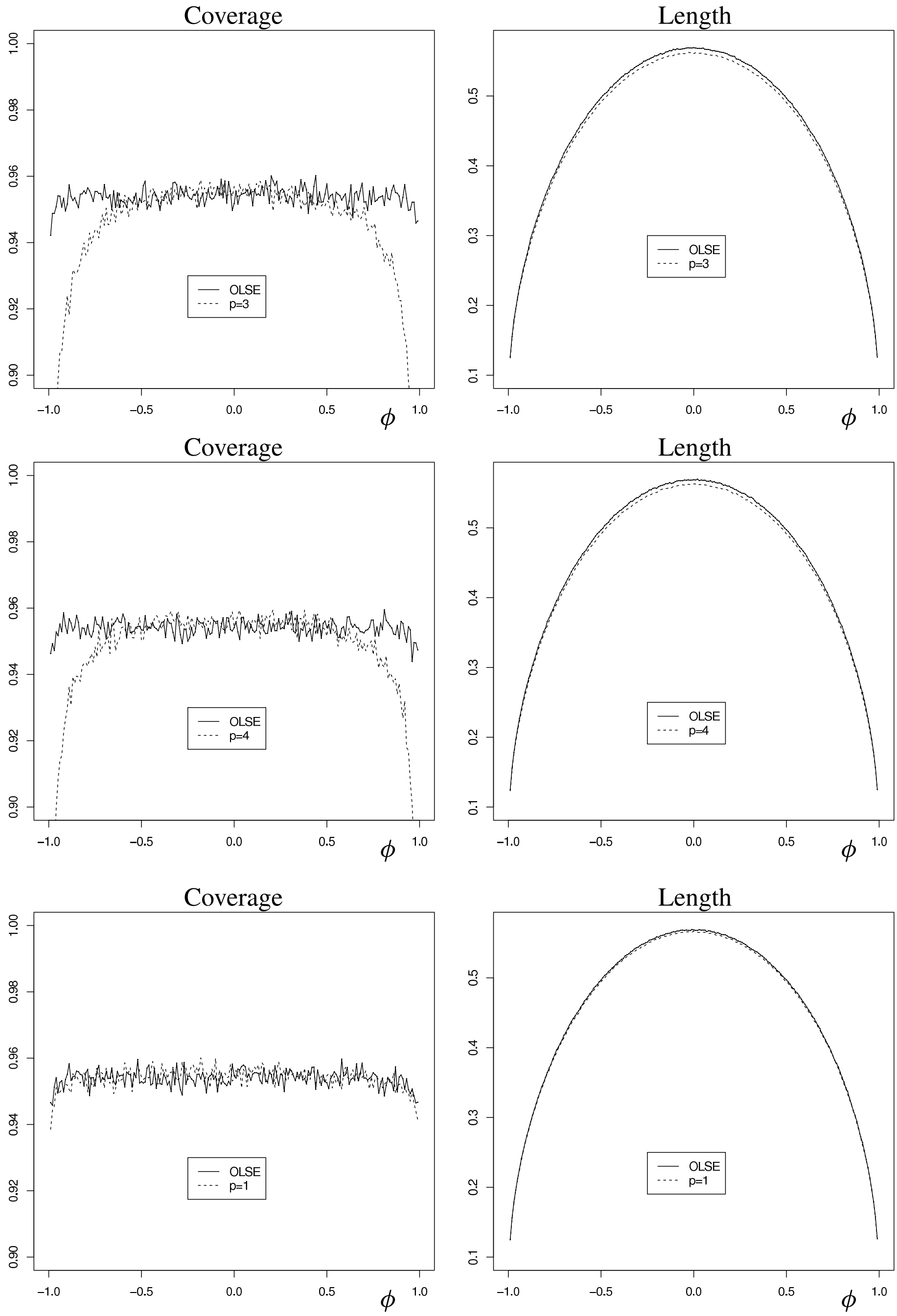


**Figure 4.** Top row: Empirical coverage capability (left) and length (right) of 95% confidence intervals for $\phi$ centered at the OLSE and $a_1\tilde{\phi}_3 + a_2\hat{\rho}(1)$ for $\phi \in (-1, 1)$. Middle row: Same, for $a_1\tilde{\phi}_4 + a_2\hat{\rho}(1)$. Bottom row: Same, for $0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1)$.

## 4. Closing remarks

The performance of the confidence interval centered at $a_1\tilde{\phi}_p + a_2\hat{\rho}(1)$ presented in this paper is modest, but not unimportant. For parameter values (roughly) between $-0.5$ and 0.5, its coverage tends to be at least as good as that of the OLSE interval while having a slightly smaller margin of error. This interval also does not require a large sample size, which can be good for certain practical purposes.

For example, consider the daily stock prices for Exxon Mobil Corporation during the fall quarter of 2011 (i.e., September 23 to December 21). A reasonable model for this time series is an ARIMA(1, 1, 0), where $\{X_t\} \sim \text{ARIMA}(p, 1, q)$ implies $\{X_t - X_{t-1}\} \sim \text{ARMA}(p, q)$. Thus, if $X_t$ stands for the price at time $t$ and $Y_t = X_t - X_{t-1}$, it follows that $\{Y_t\} \sim \text{AR}(1)$ with estimated model $Y_t = -0.0444Y_{t-1} + \epsilon_t$. Both the $\{X_t\}$ and $\{Y_t\}$ processes are shown in Figure 5.

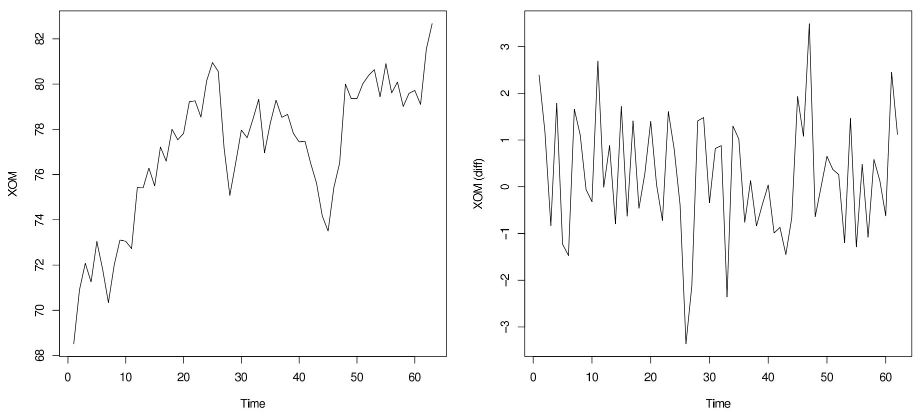


**Figure 5.** The original (left) and differenced (right) stock prices for Exxon Mobil (XOM) from 9/23/11 to 12/21/11. The sample sizes are 63 and 62, respectively.

If we supplement this model with 95% confidence intervals for $\phi$, we get the following:

| Center | Interval | Length |
|---|---|---|
| $\tilde{\phi}_1$ | $(-0.2089871, 0.1719117)$ | 0.3808988 |
| $\hat{\rho}(1)$ | $(-0.2076522, 0.1710108)$ | 0.3786630 |
| $0.5\tilde{\phi}_1 + 0.5\hat{\rho}(1)$ | $(-0.2083205, 0.1714621)$ | 0.3797825 |
| $a_1\tilde{\phi}_0 + a_2\hat{\rho}(1)$ | $(-0.2036185, 0.1749948)$ | 0.3786133 |
| $a_1\tilde{\phi}_2 + a_2\hat{\rho}(1)$ | $(-0.2127095, 0.1658015)$ | 0.3785110 |
| $a_1\tilde{\phi}_3 + a_2\hat{\rho}(1)$ | $(-0.2100079, 0.1686098)$ | 0.3786177 |
| $a_1\tilde{\phi}_4 + a_2\hat{\rho}(1)$ | $(-0.2092005, 0.1694378)$ | 0.3786383 |

All seven intervals contain the point estimate $\hat{\phi} = -0.0444$, but the last four are slightly thinner than the first three.

One extension of the research presented in this paper would be to create confidence intervals centered around a linear combination of an arbitrary number of weighted least-squares estimators. For example, it can be shown that the variance of $a_1\tilde{\phi}_p + a_2\tilde{\phi}_q + a_3\tilde{\phi}_r$ is minimized when

$$a_1 = \frac{\sigma^*_{qr}(\sigma_r^2 - \sigma_{pr}) + M(\sigma_r^2 - \sigma_{qr})}{\sigma^*_{pr}\sigma^*_{qr} - M^2}, \quad a_2 = \frac{\sigma^*_{pr}(\sigma_r^2 - \sigma_{qr}) + M(\sigma_r^2 - \sigma_{pr})}{\sigma^*_{pr}\sigma^*_{qr} - M^2},$$

and $a_3 = 1 - a_1 - a_2$, where $\sigma_i^2 = \mathrm{Var}(\tilde{\phi}_i)$, $\sigma_{ij} = \mathrm{Cov}(\tilde{\phi}_i, \tilde{\phi}_j)$, $\sigma^*_{ij} = \mathrm{Var}(\tilde{\phi}_i - \tilde{\phi}_j)$, and $M = \sigma_{pr} + \sigma_{qr} - \sigma_{pq} - \sigma_r^2$. However, once the number of estimators in the center goes beyond two, the work required to construct and analyze the interval may outweigh any benefits it would bestow.

Another (less tedious) extension would be to find a new sequence $\{a_{1,n}\}$ that converges to zero while yielding a linear combination of estimators with smaller MSE than the OLSE. This new combination would still have the same distributional limit as the OLSE and could then serve as the center for another competitive interval for $\phi$. Specifically, if we simply set the standard error equal to the square root of the asymptotic variance of the OLSE, the resulting interval should have length equal to that of the OLSE, but with better coverage capability.

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