

involve

a journal of mathematics

On the zeros of $\zeta(s) - c$

Adam Boseman and Sebastian Pauli



On the zeros of $\zeta(s) - c$

Adam Boseman and Sebastian Pauli

(Communicated by Filip Saidak)

Let $\zeta(s)$ be the Riemann zeta function and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ a zero of $\zeta(s)$. We investigate the graphs of the implicit functions $z : [0, 1) \rightarrow \mathbb{C}$, with $z(0) = z_0$ given by

$$\zeta(z(c)) - c = 0.$$

We give zero-free regions for $\zeta(s) - c$ where $c \in [0, 1)$.

1. Introduction

For $\sigma = \Re(s) > 1$, the Riemann zeta function can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (1)$$

By analytic continuation, $\zeta(s)$ may be extended to the whole complex plane, with the exception of the simple pole $s = 1$. This analytic continuation is characterized by the functional equation

$$\zeta(1-s) = 2\Gamma(s)\zeta(s)(2\pi)^{-s} \cos \frac{s\pi}{2}. \quad (2)$$

The existence of a class of zeros of the form $-2n$, $n \in \mathbb{N}$, follows directly from the functional equation. These zeros are called trivial. The Riemann hypothesis states that all nontrivial zeros of $\zeta(s)$ are located on the critical line $\sigma = \frac{1}{2}$.

In order to understand the Riemann zeta function better, various mathematicians have investigated the behavior of its derivatives. Speiser [1935] showed that the Riemann hypothesis is equivalent to $\zeta'(s)$ having no zeros for $0 < \Re(s) < \frac{1}{2}$.

Spira [1965] computed zeros of the first and second derivative of $\zeta(s)$ and noticed that they occur in pairs. Skorokhodov [2003] went further in his computation and noticed that the zeros of derivatives seem to form chains; that is, for each zero s_k of $\zeta^{(k)}(s)$ there is a corresponding zero s_{k+1} of $\zeta^{(k+1)}(s)$. For sufficiently large k , the existence of these chains is a direct consequence of the following theorem.

MSC2010: 11M26.

Keywords: Riemann zeta function.

Theorem 1 [Binder, Pauli and Saidak 2013]. *Let $u \in \mathbb{R}^{>0}$ be a solution of*

$$1 - \frac{1}{e^u - 1} - \frac{1}{e^u} \left(1 + \frac{1}{u}\right) \geq 0.$$

Let $M \in \mathbb{N}$, $M \geq 2$, and $j \in \mathbb{Z}$. Let

$$q_M := \log \frac{\log M}{\log(M+1)} \bigg/ \log \frac{M}{M+1}.$$

If there is $k \in \mathbb{N}$ with

$$q_{M+1}k + (M+2)u \leq q_M k - (M+1)u,$$

then each rectangle $R_j \subset S_M^k$, consisting of all $s = \sigma + it$ with

$$q_M k - (M+1)u < \sigma < q_M k + (M+1)u$$

and

$$\frac{2\pi j}{\log(M+1) - \log M} < t < \frac{2\pi(j+1)}{\log(M+1) - \log M},$$

contains exactly one zero of $\zeta^{(k)}(s)$. This zero is simple.

The existence of the chains of zeros of derivatives can be seen as follows. For a given $M \in \mathbb{N}$, $M \geq 2$ there is $K \in \mathbb{N}$ such that $q_{M+1}k + (M+2)u \leq q_M k - (M+1)u$ for all $k \geq K$. By [Theorem 1](#), for each $k \geq K$ and each $j \in \mathbb{Z}$ there is exactly one zero in a rectangular region given by M , k , and j . Again by [Theorem 1](#) there exists a unique corresponding zero of $\zeta^{(k+1)}(s)$ in the rectangular region given by M , $k+1$, and j , which can be obtained by shifting the first region to the right (and stretching it horizontally). This shows the existence of a chain of zeros of $\zeta^{(K)}(s)$, $\zeta^{(K+1)}(s)$, $\zeta^{(K+2)}(s)$, \dots .

Skorokhodov also noticed that the zeros of $\zeta(s) - 1$ can be regarded as the first points in these chains, and that there are curves from some zeros of $\zeta(s)$ to these points given by the zeros of $\zeta(s) - c$ for $c \in [0, 1)$ (see [Figure 1](#)).

The curves of zeros $s(c)$ of $\zeta(s) - c$ for $c \in [0, 1)$ either end at a zero of $\zeta(s) - 1$ or go off to the left approaching their asymptote

$$t = \Re(s) = \frac{(2m+1)\pi}{\log 2},$$

for some $m \in \mathbb{Z}$ as $\sigma = \Re(s)$ approaches infinity. If each zero of $\zeta(s) - 1$ indeed corresponded to a zero of $\zeta'(s)$, $\zeta''(s)$, $\zeta'''(s)$, \dots , then some zeros of $\zeta(s)$ would not correspond to zeros with derivatives, namely those from which the paths of zeros of $\zeta(s) - c$ for $c \in [0, 1)$ go off to the right.

This agrees with the formulas for the number of nontrivial zeros of $\zeta(s)$ and $\zeta^{(k)}(s)$. Namely, let $N(T)$ and $N_k(T)$ denote the number of such zeros ρ with

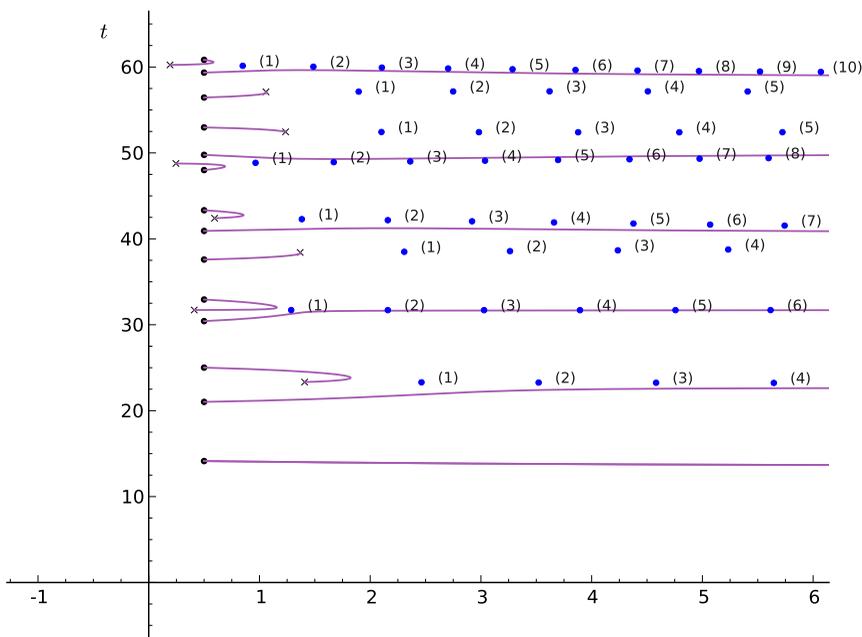


Figure 1. Zeros of derivatives of $\zeta^{(k)}(s)$ (denoted by $\bullet^{(k)}$) and the paths from zeros of $\zeta(s)$ (denoted by \bullet) to the zeros of $\zeta(s) - 1$ (denoted by \times).

$0 \leq \Im(\rho) \leq T$ of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively. The classical Riemann–von Mangoldt formula [Landau 1974] states that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{3}$$

and according to Berndt [1970], we have

$$N_k(T) = N(T) - \frac{T \log 2}{2\pi} + O(\log T). \tag{4}$$

So there are about $(T \log 2)/2\pi$ fewer zeros of $\zeta^{(k)}(s)$ with imaginary part less than T than there are of $\zeta(s)$, which is also about the number of paths of zeros of $\zeta(s) - c$ with imaginary part less than T that go off to the right.

The aim of this paper is to describe better the behavior of paths of zeros of $\zeta(s) - c = 0$ for $c \in [0, 1)$ by finding new zero-free regions for the functions $\zeta(s) - c$. Our results are summarized in Figure 2. Clearly, the zeros of $\zeta(s) - c$ lie on the real lines of $\zeta(s)$, that is, the lines on which $\Im(\zeta(s)) = 0$. A review of some results about these lines in Section 2 is followed by the derivation of the zero-free

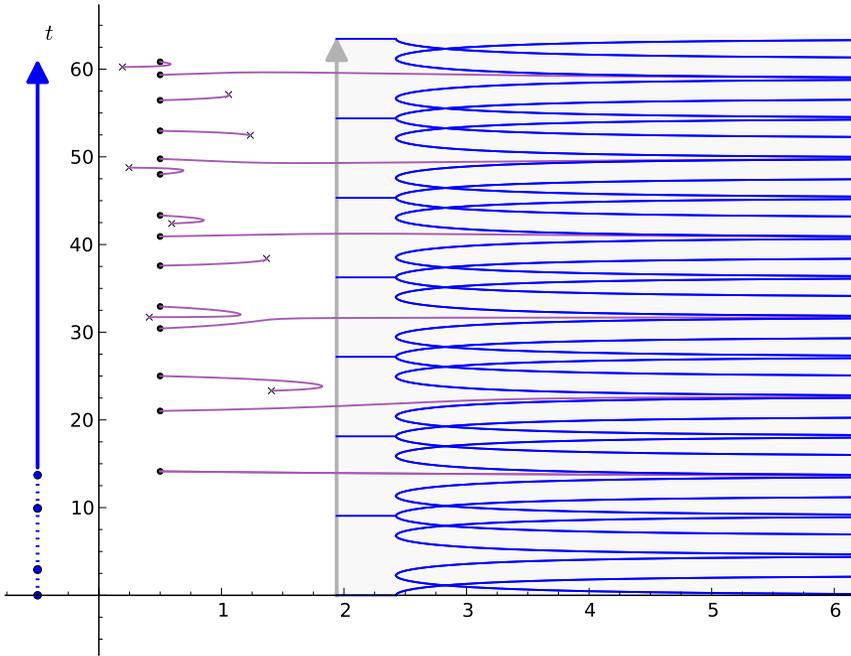


Figure 2. The paths from zeros of $\zeta(s)$ (denoted by \bullet) to the zeros of $\zeta(s) - 1$ (denoted by \times), the barrier on the left (denoted by \uparrow), the zeros of $\Im(\zeta(-\frac{1}{2} + it))$ with $0 \leq t < 13.7$ (denoted by \bullet), the borders of zero-free regions of $\zeta(s) - c$ for $c \in [0, 1)$ (denoted by blue lines), and the zero-free region of $\zeta(s) - 1$ on the right in gray.

regions for $\zeta(s) - c$ on the right half-plane (Section 3) and the vertical boundary for the zeros of $\zeta(s) - 1$ for $\Re(s) = \frac{1}{2}$ (Section 4).

2. Real lines

Obviously the solutions of the equations $\zeta(s) - c = 0$ where $c \in [0, 1)$ are on the level lines with $\Im(\zeta(s)) = 0$, called real lines. Most of the results described here go back to the work of Speiser and his student Utzinger [Speiser 1935]. Plots of the behavior of the real (and imaginary) lines and some further discussion can be found in [Arias-de-Reyna 2005].

Because the term $1 + 2^{-s}$ dominates the infinite series $\zeta(s) = \sum_{i=0}^{\infty} (1/n^s)$ for $\sigma = \Re(s) > 3$, the real lines have asymptotes $t = j\pi/\log 2$ for $j \in \mathbb{Z}$. On the real lines with asymptote $t = 2m\pi/\log 2$ ($m \in \mathbb{Z}$) the function $\zeta(s)$ approaches 1 from above, while on the real lines with asymptote $t = (2m + 1)\pi/\log 2$ ($m \in \mathbb{Z}$) the function $\zeta(s)$ approaches 1 from below. The zero-free regions for $\zeta(s) - c = 0$

where $c \in [0, 1)$ narrow around these asymptotes as σ increases — see [Lemma 4](#) and [Lemma 3](#).

As $\zeta(s)$ is a meromorphic function, no two of these real lines can cross where $\zeta'(s) \neq 0$. Zero-free regions for $\zeta'(s)$ have been found on the left of the critical line for $\Im(s) \neq 0$ and $\Re(s) < 0$ [[Levinson and Montgomery 1974](#), Theorem 9] ($\Re(s) < \frac{1}{2}$ under the Riemann hypothesis [[Speiser 1935](#)]) and on the right of the critical line for $\sigma > 2.94$ [[Skorokhodov 2003](#), Theorem 2]. Indeed, the only point where two real lines coming from the right cross is the first real zero of $\zeta'(s)$ at $s \approx -2.7172628292$ [[Speiser 1935](#)]. Here the lines with asymptotes $t = 2\pi/\log 2$ and $t = -2\pi/\log 2$ intersect the real axis.

The lines coming from the right continue to the left at least until $\sigma = 1.95$ (compare [Lemma 5](#)). If one of the lines coming from the right did not cross the strip $-1 \leq \sigma \leq 2$, it would have go up towards infinity. Because no two real lines coming from the right intersect, all following lines would have to do the same. This would contradict the estimate

$$\Im \left(\int_{2+Ti}^{-1+Ti} \frac{\zeta'(s)}{\zeta(s)} ds \right) = O(\log T)$$

used in the proof of the Riemann–von Mangoldt formula (3). Thus all real lines coming from the right cross the strip $-1 \leq \sigma \leq 2$ [[Speiser 1935](#)].

Hence the zeros of $\zeta(s) - c = 0$, where $c \in [0, 1)$, are either on the real lines described above or on real lines that enter the critical strip from the left half-plane and then curve back to the left half-plane. The lines coming from the left half-plane are the lines on which $\zeta(s) - 1$ is 0. By [Proposition 7](#), we have $|\zeta(-\frac{1}{2} + it)| > 1$ for $t \geq 13.7$. Furthermore, for $0 < t < 13.7$, there are only two points where $\Re(\zeta(-\frac{1}{2} + it)) = 0$, that is, where the real lines with asymptote $t = 2\pi/\log 2$ and $t = 3\pi/\log 2$ cross the line $\sigma = -\frac{1}{2}$ (see [Remark 8](#)). It follows that each of these lines coming from the left contains a zero of $\zeta(s)$ and a zero of $\zeta(s) - 1$ on the left of $\sigma = -\frac{1}{2}$. It is well-known that the real part of the zeros of $\zeta(s)$ is between 0 and 1, and equals $\frac{1}{2}$ if one assumes the Riemann hypothesis. An upper bound for the real part zeros of $\zeta(s) - 1$ was given by Skorokhodov [[2003](#)]; see [Lemma 2](#) below.

3. Zero-free regions for $\zeta(s) - c$ on the right

A right bound $\sigma = 3$ for the zeros of $\zeta(s) - 1$ can easily be obtained with the triangle inequality and an estimate for $\zeta(\sigma) - 1/2^\sigma - 1$. Skorokhodov was able to get a better bound by applying the triangle inequality to a real-valued function that only considers terms of the zeta function with n odd.

Lemma 2 [Skorokhodov 2003]. *The function $\zeta(s)$ is distinct from unity at $\sigma \in (\sigma_0, \infty)$, where*

$$\sigma_0 = 1.940101683745 \dots$$

is the zero of the function

$$f(\sigma) = 1 + 2^{-\sigma} - (1 - 2^{-\sigma})\zeta(\sigma), \quad \sigma > 1.$$

For $c \in [0, 1)$ we find zero-free regions of $\zeta(s) - c$ that depend on t . We obtain them by considering the real and imaginary parts of $\zeta(s) - c$ separately.

Lemma 3. *If $c \in [0, 1)$ and $|\sin(t \log 2)| \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1$, then $\zeta(\sigma + it) - c \neq 0$.*

Proof. We consider the imaginary part of $\zeta(s) - c$ and obtain

$$\begin{aligned} |\Im(\zeta(s) - c)| &\geq \left| \frac{1}{2^\sigma} \sin(t \log 2) \right| - \left| \sum_{n=3}^{\infty} \frac{1}{n^\sigma} \right| \\ &= \left| \frac{1}{2^\sigma} \sin(t \log 2) \right| - \left| \zeta(\sigma) - 1 - \frac{1}{2^\sigma} \right|, \end{aligned} \quad (5)$$

which is greater than 0 when

$$|\sin(t \log 2)| \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1. \quad \square$$

Lemma 4. *If $c \in [0, 1)$ and $\cos(t \log 2) \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1$, then $\zeta(\sigma + it) - c \neq 0$.*

Proof. For the real part of $\zeta(s) - c$ we obtain

$$\begin{aligned} \Re(\zeta(s) - c) &= 1 - c + \frac{1}{2^\sigma} \cos(t \log 2) + \dots \\ &\geq \frac{1}{2^\sigma} \cos(t \log 2) - \left(\zeta(\sigma) - 1 - \frac{1}{2^\sigma} \right) \quad \text{assuming } c = 1, \end{aligned}$$

which is greater than 0 when

$$\cos(t \log 2) \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1. \quad \square$$

These regions can be extended a bit if we restrict ourselves to certain values of t .

Lemma 5. *If $c \in [0, 1)$, $m \in \mathbb{Z}$, and t is fixed at $2\pi m / \log 2$, then $\Re(\zeta(s) - c) \neq 0$ for $\sigma \geq 1.95$.*

Proof. $\Re(\zeta(s) - c) = 1 - c + (1/2^\sigma) \cos(t \log 2) + (1/3^\sigma) \cos(t \log 3) + \dots$ When t is fixed and $t \log 2 = 2\pi m$, we get

$$\begin{aligned} \Re(\zeta(s) - c) &\geq 1 - c + \sum_{\nu=0}^{\infty} \frac{1}{(2^\nu)^\sigma} - \left(\sum_{n=2}^{\infty} \frac{1}{n^\sigma} - \sum_{\nu=0}^{\infty} \frac{1}{(2^\nu)^\sigma} \right) \\ &= 2 \sum_{\nu=1}^{\infty} \left(\frac{1}{2^\sigma} \right)^\nu - \zeta(\sigma) = \frac{2}{1 - 1/2^\sigma} - \zeta(\sigma), \end{aligned}$$

which is greater than 1 for $\sigma \geq 1.95$. □

4. Zero-free barrier for $\zeta(s) - c$ on the left

On the left, instead of finding a zero-free region, we find a horizontal line where $|\zeta(s)| > 1$. The line $\sigma = -\frac{1}{2}$ fulfills this condition with the exception of one point.

First we find a lower bound for the absolute value of $\zeta(s)$ where $\sigma = \frac{3}{2}$.

Lemma 6. $|\zeta(\frac{3}{2} + it)| > 0.46$ for all $t \in \mathbb{R}$.

Proof. To get a lower bound for $|\zeta(s)|$, we use the Euler product. Let P be the set of the first million prime numbers, and consider the expression $\prod_{p \in P} |1 - p^{-s}| |\zeta(s)|$. We have

$$\begin{aligned} \prod_{p \in P} |1 - p^{-s}| |\zeta(s)| &= \left| 1 + \sum_{\substack{p|n \\ p \in P}} \frac{1}{n^s} \right| \geq \left| 1 - \sum_{\substack{p|n \\ p \in P}} \frac{1}{n^s} \right| \\ &\geq 1 - \sum_{\substack{p|n \\ p \in P}} \frac{1}{n^\sigma} = 2 - \prod_{p \in P} (1 + p^{-\sigma}). \end{aligned}$$

We also have from the triangle inequality that $|1 - p^{-s}| \leq 1 + p^{-\sigma}$, and thus

$$|\zeta(s)| \geq \frac{2 - \prod_{p \in P} (1 + p^{-\sigma}) \zeta(\sigma)}{\prod_{p \in P} (1 + p^{-\sigma})} \geq 0.46 \quad \text{for } \sigma = \frac{3}{2}.$$

So we get $|\zeta(s)| \geq \delta > 0$ for $\sigma = \frac{3}{2}$ and $\delta = 0.46$. □

Now we can use δ and the functional equation to obtain a barrier for the zeros of $\zeta(s) - c$ on the left.

Proposition 7. $|\zeta(-\frac{1}{2} + it)| > 1$ for $t \geq 13.7$.

Proof. By the functional equation,

$$\begin{aligned} \zeta(1-s) &= 2^{1-s} \pi^{-s} \sin\left(\frac{\pi}{2}(1-s)\right) \Gamma(s) \zeta(s) \\ &= 2^{1-s} \pi^{-s} \cos \frac{s\pi}{2} \Gamma(s) \zeta(s). \end{aligned}$$

Taking the absolute value of both sides gives

$$|\zeta(1-s)| = 2^{1-\sigma} \pi^{-\sigma} \left| \cos \frac{s\pi}{2} \right| |\Gamma(s)| |\zeta(s)|.$$

But

$$\begin{aligned}
 \left| \cos \frac{s\pi}{2} \right| &= \frac{1}{2} \left| e^{-\pi(\sigma i - t)/2} + e^{\pi(t - \sigma i)/2} \right| \\
 &= \frac{1}{2} \left| e^{-t\pi/2} (\cos \sigma + i \sin \sigma) + e^{t\pi/2} (\cos \sigma - i \sin \sigma) \right| \\
 &= \frac{1}{2} \left| \cos \sigma (e^{t\pi/2} + e^{-t\pi/2}) + i \sin \sigma (e^{-t\pi/2} - e^{t\pi/2}) \right| \\
 &= \frac{1}{2} (\cos^2 \sigma (e^{\pi t} + e^{-\pi t} + 2) + \sin^2 \sigma (e^{\pi t} + e^{-\pi t} - 2))^{\frac{1}{2}} \\
 &= \frac{1}{2} (e^{\pi t} + e^{-\pi t} + 2(\cos^2 \sigma - \sin^2 \sigma))^{\frac{1}{2}}.
 \end{aligned}$$

As $\Gamma(z + 1) = z\Gamma(z)$ for $z \in \mathbb{C}$ and as

$$\left| \Gamma\left(\frac{1}{2} + it\right) \right| = \sqrt{\pi \operatorname{sech}(\pi t)} = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}}$$

for $t \in \mathbb{R}$, we get

$$\left| \Gamma\left(\frac{3}{2} + it\right) \right| = \left| \left(\frac{1}{2} + it\right) \Gamma\left(\frac{1}{2} + it\right) \right| = \sqrt{\frac{1}{4} + t^2} \cdot \sqrt{\pi} \cdot \sqrt{\frac{2}{e^{\pi t} + e^{-\pi t}}}.$$

For $\sigma = \frac{3}{2}$ we obtain

$$\left| \zeta\left(-\frac{1}{2} + it\right) \right| \geq 2^{-0.5} \pi^{-1} \frac{1}{\sqrt{2}} \left(1 + \frac{4 \cos^2\left(\frac{3}{2}\right) - 2}{e^{\pi t} + e^{-\pi t}} \right) \cdot \sqrt{\frac{1}{4} + t^2} \cdot \delta,$$

where the right-hand side is obviously increasing in t . With $\delta > 0.46$, this gives $\left| \zeta\left(\frac{1}{2} + it\right) \right| > 1$ for $t \geq 13.7$ by [Lemma 6](#). \square

Remark 8. The zeros of $\Im(\zeta(-\frac{1}{2} + it))$ with $0 \leq t < 13.7$ are $t_0 = 0$, $t_1 \approx 2.93$, and $t_2 \approx 9.92$, where

$$\zeta\left(-\frac{1}{2} + it_0\right) \approx -0.21, \quad \zeta\left(-\frac{1}{2} + it_1\right) \approx 0.35, \quad \zeta\left(-\frac{1}{2} + it_2\right) \approx 2.03.$$

So the only hole in the barrier is $-\frac{1}{2} + it_1$. This is where the real line with asymptote $\pi/\log 2$ crosses the line $\sigma = -\frac{1}{2}$.

5. Outlook

In our work, we investigated the behavior of the graphs of the continuous functions $s : [0, 1) \rightarrow \mathbb{C}$ defined by the equation $\zeta(s(c)) - c = 0$ and an initial point $s(0)$ (a zero of the zeta function). If $s(1)$ exists, such a graph connects a zero of $\zeta(s)$ to a zero of $\zeta(s) - 1$. The latter zeros are the first points on the conjectured chains of zeros of derivatives.

A similar approach could also be used to investigate the conjectured chains of zeros of the derivatives of $\zeta(s)$. For each zero s_0 of

$$\zeta(s) - 1 = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

one would consider the implicit function $s : [0, \infty) \rightarrow \mathbb{C}$ given by

$$\zeta^{(k)}(s(k)) = (-1)^k \sum_{n=1}^{\infty} \frac{\log^k n}{n^{s(k)}} = 0,$$

with $s(0) = s_0$. This function $s(k)$ should yield the correspondence of zeros of $\zeta^{(k)}(s)$ and $\zeta^{(k+1)}(s)$ for $k \in \mathbb{Z}$, $k \geq 0$ for two zeros which would be connected by $\{s(x) \mid k \leq x \leq k+1\}$.

Together, the two implicit functions could give more detailed insight into the distribution of the zeros of $\zeta(s)$ by relating it to the distribution of higher derivatives (see [Theorem 1](#)). Furthermore it will be interesting to see how the conjectured chains of zeros of the derivatives of $\zeta(s)$ fit in with the universality of $\zeta(s)$ found by Voronin [[1975](#)].

Acknowledgements

Most of the work on this paper was carried out as a project in the REU Interdisciplinary Quantitative Science at UNCG in the summer of 2009 supported by NSF grant 080465. All computations were conducted in the computer algebra system Sage [[Stein et al. 2009](#)].

References

- [Arias-de-Reyna 2005] J. Arias-de-Reyna, “X-ray of Riemann zeta-function”, preprint, 2005. [arXiv math/0309433v1](#).
- [Berndt 1970] B. C. Berndt, “The number of zeros for $\zeta^{(k)}(s)$ ”, *J. London Math. Soc.* (2) **2**:4 (1970), 577–580. [MR 42 #1776](#) [Zbl 0203.35503](#)
- [Binder et al. 2013] T. Binder, S. Pauli, and F. Saidak, “Zeros of high derivatives of the Riemann zeta function”, *Rocky Mountain J. of Math.* (2013). To appear.
- [Landau 1974] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 3rd ed., Chelsea, New York, 1974. [MR 16,904d](#) [Zbl 0051.28007](#)
- [Levinson and Montgomery 1974] N. Levinson and H. L. Montgomery, “Zeros of the derivatives of the Riemann zeta-function”, *Acta Math.* **133** (1974), 49–65. [MR 54 #5135](#) [Zbl 0287.10025](#)
- [Skorokhodov 2003] S. L. Skorokhodov, “Аппроксимации Паде и численный анализ дзета-функции Римана”, *Zh. Vychisl. Mat. Mat. Fiz.* **43**:9 (2003), 1330–1352. Translated as “Padé approximation and numerical analysis for the Riemann ζ -function” in *Comput. Math. Math. Phys.* **43**:9 (2003), 1277–1298. [MR 2004h:11113](#) [Zbl 1079.41012](#)
- [Speiser 1935] A. Speiser, “Geometrisches zur Riemannschen Zetafunktion”, *Math. Ann.* **110**:1 (1935), 514–521. [MR 1512953](#) [Zbl 0010.16401](#)

[Spira 1965] R. Spira, “Zero-free regions of $\zeta^{(k)}(s)$ ”, *J. London Math. Soc.* **40** (1965), 677–682. [MR 31 #5849](#) [Zbl 0147.30503](#)

[Stein et al. 2009] W. Stein et al., “Sage: open-source mathematics software”, 2009, Available at <http://www.sagemath.org>.

[Voronin 1975] S. M. Voronin, “Теорема об ‘универсальности’ дзета-функции Римана”, *Izv. Akad. Nauk SSSR Ser. Mat.* **39**:3 (1975), 475–486. Translated as “Theorem on the ‘universality’ of the Riemann zeta-function” in *Math. USSR Izv.* **9**:3 (1975), 443–453. [MR 57 #12419](#) [Zbl 0315.10037](#)

Received: 2012-03-08

Revised: 2012-05-31

Accepted: 2013-05-15

a_bosema@uncg.edu

*Joint School of Nanoscience and Nanoengineering,
The University of North Carolina at Greensboro,
Greensboro, North Carolina 27402, United States*

s_pauli@uncg.edu

*Department of Mathematics and Statistics,
The University of North Carolina at Greensboro,
Greensboro, North Carolina 27402, United States*

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tbriell@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisys@potsdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA plemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA kgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew.andrew@umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sgupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakill@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION

Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2013 is US \$105/year for the electronic version, and \$145/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

involve

2013

vol. 6

no. 2

The influence of education in reducing the HIV epidemic RENEE MARGEVICIUS AND HEM RAJ JOSHI	127
On the zeros of $\zeta(s) - c$ ADAM BOSEMAN AND SEBASTIAN PAULI	137
Dynamic impact of a particle JEONGHO AHN AND JARED R. WOLF	147
Magic polygrams AMANDA BIENZ, KAREN A. YOKLEY AND CRISTA ARANGALA	169
Trading cookies in a gambler's ruin scenario KUEJAI JUNGJATURAPIT, TIMOTHY PLUTA, REZA RASTEGAR, ALEXANDER ROITERSHTEIN, MATTHEW TEMBA, CHAD N. VIDDEN AND BRIAN WU	191
Decomposing induced characters of the centralizer of an n -cycle in the symmetric group on $2n$ elements JOSEPH RICCI	221
On the geometric deformations of functions in $L^2[D]$ LUIS CONTRERAS, DEREK DESANTIS AND KATHRYN LEONARD	233
Spectral characterization for von Neumann's iterative algorithm in \mathbb{R}^n RUDY JOLY, MARCO LÓPEZ, DOUGLAS MUPASIRI AND MICHAEL NEWSOME	243
The 3-point Steiner problem on a cylinder DENISE M. HALVERSON AND ANDREW E. LOGAN	251