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Adam Boseman and Sebastian Pauli

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On the zeros of $\zeta(s) - c$

Adam Boseman and Sebastian Pauli

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Let $\zeta(s)$ be the Riemann zeta function and $z_0 \in \mathbb{C} \backslash \mathbb{R}$ a zero of $\zeta(s)$. We investigate the graphs of the implicit functions $z : [0, 1) \to \mathbb{C}$, with $z(0) = z_0$ given by

$$\zeta(z(c)) - c = 0.$$

We give zero-free regions for $\zeta(s) - c$ where $c \in [0, 1)$.

1. Introduction

For $\sigma = \Re(s) > 1$, the Riemann zeta function can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

By analytic continuation, $\zeta(s)$ may be extended to the whole complex plane, with the exception of the simple pole $s = 1$. This analytic continuation is characterized by the functional equation

$$\zeta(1 - s) = 2\Gamma(s)\zeta(s)(2\pi)^{-s} \cos \frac{s\pi}{2}. \tag{2}$$

The existence of a class of zeros of the form $-2n$, $n \in \mathbb{N}$, follows directly from the functional equation. These zeros are called trivial. The Riemann hypothesis states that all nontrivial zeros of $\zeta(s)$ are located on the critical line $\sigma = \frac{1}{2}$.

In order to understand the Riemann zeta function better, various mathematicians have investigated the behavior of its derivatives. Speiser [1935] showed that the Riemann hypothesis is equivalent to $\zeta'(s)$ having no zeros for $0 < \Re(s) < \frac{1}{2}$.

Spira [1965] computed zeros of the first and second derivative of $\zeta(s)$ and noticed that they occur in pairs. Skorokhodov [2003] went further in his computation and noticed that the zeros of derivatives seem to form chains; that is, for each zero s_k of $\zeta^{(k)}(s)$ there is a corresponding zero s_{k+1} of $\zeta^{(k+1)}(s)$. For sufficiently large k, the existence of these chains is a direct consequence of the following theorem.

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Theorem 1 [Binder, Pauli and Saidak 2013]. *Let $u \in \mathbb{R}^{>0}$ be a solution of*

$$1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) \geq 0.$$

Let $M \in \mathbb{N}$, $M \geq 2$, and $j \in \mathbb{Z}$. Let

$$q_M := \log \frac{\log M}{\log(M+1)} \Big/ \log \frac{M}{M+1}.$$

If there is $k \in \mathbb{N}$ with

$$q_{M+1}k + (M+2)u \leq q_M k - (M+1)u,$$

then each rectangle $R_j \subset S_M^k$, consisting of all $s = \sigma + it$ with

$$q_M k - (M+1)u < \sigma < q_M k + (M+1)u$$

and

$$\frac{2\pi j}{\log(M+1) - \log M} < t < \frac{2\pi(j+1)}{\log(M+1) - \log M},$$

contains exactly one zero of $\zeta^{(k)}(s)$. This zero is simple.

The existence of the chains of zeros of derivatives can be seen as follows. For a given $M \in \mathbb{N}$, $M \geq 2$ there is $K \in \mathbb{N}$ such that $q_{M+1}k + (M+2)u \leq q_M k - (M+1)u$ for all $k \geq K$. By Theorem 1, for each $k \geq K$ and each $j \in \mathbb{Z}$ there is exactly one zero in a rectangular region given by M, k, and j. Again by Theorem 1 there exists a unique corresponding zero of $\zeta^{(k+1)}(s)$ in the rectangular region given by M, $k+1$, and j, which can be obtained by shifting the first region to the right (and stretching it horizontally). This shows the existence of a chain of zeros of $\zeta^{(K)}(s), \zeta^{(K+1)}(s), \zeta^{(K+2)}(s), \ldots$.

Skorokhodov also noticed that the zeros of $\zeta(s) - 1$ can be regarded as the first points in these chains, and that there are curves from some zeros of $\zeta(s)$ to these points given by the zeros of $\zeta(s) - c$ for $c \in [0, 1)$ (see Figure 1).

The curves of zeros $s(c)$ of $\zeta(s) - c$ for $c \in [0, 1)$ either end at a zero of $\zeta(s) - 1$ or go off to the left approaching their asymptote

$$t = \Re(s) = \frac{(2m+1)\pi}{\log 2},$$

for some $m \in \mathbb{Z}$ as $\sigma = \Re(s)$ approaches infinity. If each zero of $\zeta(s) - 1$ indeed corresponded to a zero of $\zeta'(s), \zeta''(s), \zeta'''(s), \ldots$, then some zeros of $\zeta(s)$ would not correspond to zeros with derivatives, namely those from which the paths of zeros of $\zeta(s) - c$ for $c \in [0, 1)$ go off to the right.

This agrees with the formulas for the number of nontrivial zeros of $\zeta(s)$ and $\zeta^{(k)}(s)$. Namely, let $N(T)$ and $N_k(T)$ denote the number of such zeros ρ with

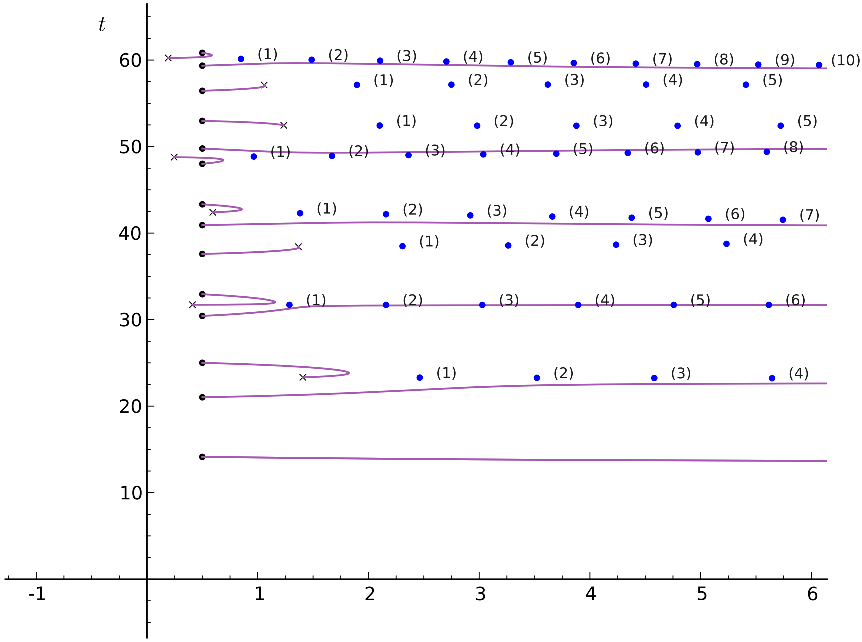

Figure 1. Zeros of derivatives of $\zeta^{(k)}(s)$ (denoted by $\bullet^{(k)}$) and the paths from zeros of $\zeta(s)$ (denoted by $\bullet$) to the zeros of $\zeta(s) - 1$ (denoted by $\times$).

$0 \le \Im(\rho) \le T$ of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively. The classical Riemann–von Mangoldt formula [Landau 1974] states that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{3}$$

and according to Berndt [1970], we have

$$N_k(T) = N(T) - \frac{T \log 2}{2\pi} + O(\log T). \tag{4}$$

So there are about $(T \log 2)/2\pi$ fewer zeros of $\zeta^{(k)}(s)$ with imaginary part less than T than there are of $\zeta(s)$, which is also about the number of paths of zeros of $\zeta(s) - c$ with imaginary part less than T that go off to the right.

The aim of this paper is to describe better the behavior of paths of zeros of $\zeta(s) - c = 0$ for $c \in [0, 1)$ by finding new zero-free regions for the functions $\zeta(s) - c$. Our results are summarized in Figure 2. Clearly, the zeros of $\zeta(s) - c$ lie on the real lines of $\zeta(s)$, that is, the lines on which $\Im(\zeta(s)) = 0$. A review of some results about these lines in Section 2 is followed by the derivation of the zero-free

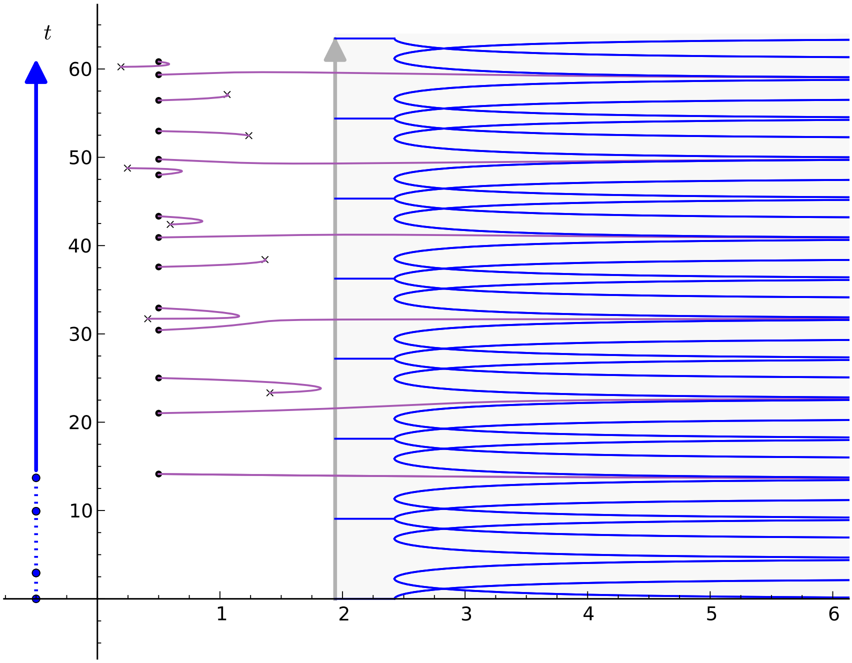

Figure 2. The paths from zeros of $\zeta(s)$ (denoted by $\bullet$) to the zeros of $\zeta(s)-1$ (denoted by $\times$), the barrier on the left (denoted by $\uparrow$), the zeros of $\Im\bigl(\zeta\bigl(-\frac{1}{2}+it\bigr)\bigr)$ with $0 \le t < 13.7$ (denoted by $\bullet$), the borders of zero-free regions of $\zeta(s)-c$ for $c \in [0, 1)$ (denoted by blue lines), and the zero-free region of $\zeta(s)-1$ on the right in gray.

regions for $\zeta(s)-c$ on the right half-plane (Section 3) and the vertical boundary for the zeros of $\zeta(s)-1$ for $\Re(s)=\frac{1}{2}$ (Section 4).

2. Real lines

Obviously the solutions of the equations $\zeta(s)-c=0$ where $c \in [0, 1)$ are on the level lines with $\Im(\zeta(s))=0$, called real lines. Most of the results described here go back to the work of Speiser and his student Utzinger [Speiser 1935]. Plots of the behavior of the real (and imaginary) lines and some further discussion can be found in [Arias-de-Reyna 2005].

Because the term $1+2^{-s}$ dominates the infinite series $\zeta(s)=\sum_{i=0}^{\infty}(1/n^s)$ for $\sigma=\Re(s)>3$, the real lines have asymptotes $t=j\pi/\log 2$ for $j\in\mathbb{Z}$. On the real lines with asymptote $t=2m\pi/\log 2$ ($m\in\mathbb{Z}$) the function $\zeta(s)$ approaches 1 from above, while on the real lines with asymptote $t=(2m+1)\pi/\log 2$ ($m\in\mathbb{Z}$) the function $\zeta(s)$ approaches 1 from below. The zero-free regions for $\zeta(s)-c=0$

where $c \in [0, 1)$ narrow around these asymptotes as σ increases — see Lemma 4 and Lemma 3.

As $\zeta(s)$ is a meromorphic function, no two of these real lines can cross where $\zeta'(s) \neq 0$. Zero-free regions for $\zeta'(s)$ have been found on the left of the critical line for $\Im(s) \neq 0$ and $\Re(s) < 0$ [Levinson and Montgomery 1974, Theorem 9] ($\Re(s) < \frac{1}{2}$ under the Riemann hypothesis [Speiser 1935]) and on the right of the critical line for $\sigma > 2.94$ [Skorokhodov 2003, Theorem 2]. Indeed, the only point where two real lines coming from the right cross is the first real zero of $\zeta'(s)$ at $s \approx -2.7172628292$ [Speiser 1935]. Here the lines with asymptotes $t = 2\pi/\log 2$ and $t = -2\pi/\log 2$ intersect the real axis.

The lines coming from the right continue to the left at least until $\sigma = 1.95$ (compare Lemma 5). If one of the lines coming from the right did not cross the strip $-1 \leq \sigma \leq 2$, it would have go up towards infinity. Because no two real lines coming from the right intersect, all following lines would have to do the same. This would contradict the estimate

$$\Im\left(\int_{2+Ti}^{-1+Ti} \frac{\zeta'(s)}{\zeta(s)}\, ds\right) = O(\log T)$$

used in the proof of the Riemann–von Mangoldt formula (3). Thus all real lines coming from the right cross the strip $-1 \leq \sigma \leq 2$ [Speiser 1935].

Hence the zeros of $\zeta(s) - c = 0$, where $c \in [0, 1)$, are either on the real lines described above or on real lines that enter the critical strip from the left half-plane and then curve back to the left half-plane. The lines coming from the left half-plane are the lines on which $\zeta(s) - 1$ is 0. By Proposition 7, we have $\left|\zeta\left(-\frac{1}{2} + it\right)\right| > 1$ for $t \geq 13.7$. Furthermore, for $0 < t < 13.7$, there are only two points where $\Re\left(\zeta\left(-\frac{1}{2} + it\right)\right) = 0$, that is, where the real lines with asymptote $t = 2\pi/\log 2$ and $t = 3\pi/\log 2$ cross the line $\sigma = -\frac{1}{2}$ (see Remark 8). It follows that each of these lines coming from the left contains a zero of $\zeta(s)$ and a zero of $\zeta(s) - 1$ on the left of $\sigma = -\frac{1}{2}$. It is well-known that the real part of the zeros of $\zeta(s)$ is between 0 and 1, and equals $\frac{1}{2}$ if one assumes the Riemann hypothesis. An upper bound for the real part zeros of $\zeta(s) - 1$ was given by Skorokhodov [2003]; see Lemma 2 below.

3. Zero-free regions for $\zeta(s) - c$ on the right

A right bound $\sigma = 3$ for the zeros of $\zeta(s) - 1$ can easily be obtained with the triangle inequality and an estimate for $\zeta(\sigma) - 1/2^{\sigma} - 1$. Skorokhodov was able to get a better bound by applying the triangle inequality to a real-valued function that only considers terms of the zeta function with n odd.

Lemma 2 [Skorokhodov 2003]. *The function* $\zeta(s)$ *is distinct from unity at* $\sigma \in (\sigma_0, \infty)$, *where*

$$\sigma_0 = 1.940101683745\ldots$$

is the zero of the function

$$f(\sigma) = 1 + 2^{-\sigma} - (1 - 2^{-\sigma})\zeta(\sigma), \quad \sigma > 1.$$

For $c \in [0, 1)$ we find zero-free regions of $\zeta(s) - c$ that depend on t. We obtain them by considering the real and imaginary parts of $\zeta(s) - c$ separately.

Lemma 3. *If* $c \in [0, 1)$ *and* $|\sin(t \log 2)| \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1$, *then* $\zeta(\sigma + it) - c \neq 0$.

Proof. We consider the imaginary part of $\zeta(s) - c$ and obtain

$$\begin{aligned} \left|\Im(\zeta(s) - c)\right| &\geq \left|\frac{1}{2^\sigma} \sin(t \log 2)\right| - \left|\sum_{n=3}^{\infty} \frac{1}{n^\sigma}\right| \\ &= \left|\frac{1}{2^\sigma} \sin(t \log 2)\right| - \left|\zeta(\sigma) - 1 - \frac{1}{2^\sigma}\right|, \end{aligned} \tag{5}$$

which is greater than 0 when

$$\left|\sin(t \log 2)\right| \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1. \qquad \square$$

Lemma 4. *If* $c \in [0, 1)$ *and* $\cos(t \log 2) \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1$, *then* $\zeta(\sigma + it) - c \neq 0$.

Proof. For the real part of $\zeta(s) - c$ we obtain

$$\begin{aligned} \Re(\zeta(s) - c) &= 1 - c + \frac{1}{2^\sigma} \cos(t \log 2) + \cdots \\ &\geq \frac{1}{2^\sigma} \cos(t \log 2) - \left(\zeta(\sigma) - 1 - \frac{1}{2^\sigma}\right) \quad \text{assuming } c = 1, \end{aligned}$$

which is greater than 0 when

$$\cos(t \log 2) \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1. \qquad \square$$

These regions can be extended a bit if we restrict ourselves to certain values of t.

Lemma 5. *If* $c \in [0, 1)$, $m \in \mathbb{Z}$, *and* t *is fixed at* $2\pi m / \log 2$, *then* $\Re(\zeta(s) - c) \neq 0$ *for* $\sigma \geq 1.95$.

Proof. $\Re(\zeta(s) - c) = 1 - c + (1/2^\sigma)\cos(t \log 2) + (1/3^\sigma)\cos(t \log 3) + \cdots$ When t is fixed and $t \log 2 = 2\pi m$, we get

$$\begin{aligned} \Re(\zeta(s) - c) &\geq 1 - c + \sum_{\nu=0}^{\infty} \frac{1}{(2^\nu)^\sigma} - \left(\sum_{n=2}^{\infty} \frac{1}{n^\sigma} - \sum_{\nu=0}^{\infty} \frac{1}{(2^\nu)^\sigma}\right) \\ &= 2\sum_{\nu=1}^{\infty} \left(\frac{1}{2^\sigma}\right)^\nu - \zeta(\sigma) = \frac{2}{1 - 1/2^\sigma} - \zeta(\sigma), \end{aligned}$$

which is greater than 1 for $\sigma \geq 1.95$. □

4. Zero-free barrier for $\zeta(s)-c$ on the left

On the left, instead of finding a zero-free region, we find a horizontal line where $|\zeta(s)| > 1$. The line $\sigma = -\frac{1}{2}$ fulfills this condition with the exception of one point.

First we find a lower bound for the absolute value of $\zeta(s)$ where $\sigma = \frac{3}{2}$.

Lemma 6. $\left|\zeta\left(\frac{3}{2}+it\right)\right| > 0.46$ *for all* $t \in \mathbb{R}$.

Proof. To get a lower bound for $|\zeta(s)|$, we use the Euler product. Let P be the set of the first million prime numbers, and consider the expression $\prod_{p\in P}|1-p^{-s}||\zeta(s)|$. We have

$$\prod_{p\in P}|1-p^{-s}||\zeta(s)| = \left|1+\sum_{\substack{p\nmid n\\ p\in P}}\frac{1}{n^s}\right| \geq \left|1-\left|\sum_{\substack{p\nmid n\\ p\in P}}\frac{1}{n^s}\right|\right|$$
$$\geq 1-\sum_{\substack{p\nmid n\\ p\in P}}\frac{1}{n^\sigma} = 2-\prod_{p\in P}(1-p^{-\sigma})\zeta(\sigma).$$

We also have from the triangle inequality that $|1-p^{-s}| \leq 1+p^{-\sigma}$, and thus

$$|\zeta(s)| \geq \frac{2-\prod_{p\in P}(1-p^{-\sigma})\zeta(\sigma)}{\prod_{p\in P}(1+p^{-\sigma})} \geq 0.46 \quad \text{for } \sigma = \tfrac{3}{2}.$$

So we get $|\zeta(s)| \geq \delta > 0$ for $\sigma = \frac{3}{2}$ and $\delta = 0.46$. □

Now we can use δ and the functional equation to obtain a barrier for the zeros of $\zeta(s)-c$ on the left.

Proposition 7. $|\zeta(-\frac{1}{2}+it)| > 1$ *for* $t \geq 13.7$.

Proof. By the functional equation,

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma(s)\zeta(s)$$
$$= 2^{1-s}\pi^{-s}\cos\frac{s\pi}{2}\,\Gamma(s)\zeta(s).$$

Taking the absolute value of both sides gives

$$|\zeta(1-s)| = 2^{1-\sigma}\pi^{-\sigma}\left|\cos\frac{s\pi}{2}\right||\Gamma(s)||\zeta(s)|.$$

But

$$\begin{aligned}\left|\cos\frac{s\pi}{2}\right| &= \tfrac{1}{2}\left|e^{-\pi(\sigma i-t)/2}+e^{\pi(t-\sigma i)/2}\right| \\ &= \tfrac{1}{2}\left|e^{-t\pi/2}(\cos\sigma+i\sin\sigma)+e^{t\pi/2}(\cos\sigma-i\sin\sigma)\right| \\ &= \tfrac{1}{2}\left|\cos\sigma\,(e^{t\pi/2}+e^{-t\pi/2})+i\sin\sigma\,(e^{-t\pi/2}-e^{t\pi/2})\right| \\ &= \tfrac{1}{2}\left(\cos^2\sigma\,(e^{\pi t}+e^{-\pi t}+2)+\sin^2\sigma\,(e^{\pi t}+e^{-\pi t}-2)\right)^{\frac{1}{2}} \\ &= \tfrac{1}{2}\left(e^{\pi t}+e^{-\pi t}+2(\cos^2\sigma-\sin^2\sigma)\right)^{\frac{1}{2}}.\end{aligned}$$

As $\Gamma(z+1)=z\Gamma(z)$ for $z\in\mathbb{C}$ and as

$$\left|\Gamma\left(\tfrac{1}{2}+it\right)\right|=\sqrt{\pi\,\mathrm{sech}(\pi t)}=\sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}$$

for $t\in\mathbb{R}$, we get

$$\left|\Gamma\left(\tfrac{3}{2}+it\right)\right|=\left|\left(\tfrac{1}{2}+it\right)\Gamma\left(\tfrac{1}{2}+it\right)\right|=\sqrt{\tfrac{1}{4}+t^2}\cdot\sqrt{\pi}\cdot\sqrt{\frac{2}{e^{\pi t}+e^{-\pi t}}}.$$

For $\sigma=\frac{3}{2}$ we obtain

$$\left|\zeta\left(-\tfrac{1}{2}+it\right)\right|\geq 2^{-0.5}\pi^{-1}\frac{1}{\sqrt{2}}\left(1+\frac{4\cos^2\left(\frac{3}{2}\right)-2}{e^{\pi t}+e^{-\pi t}}\right)\cdot\sqrt{\tfrac{1}{4}+t^2}\cdot\delta,$$

where the right-hand side is obviously increasing in t. With $\delta>0.46$, this gives $\left|\zeta\left(\frac{1}{2}+it\right)\right|>1$ for $t\geq 13.7$ by Lemma 6. □

Remark 8. The zeros of $\Im\left(\zeta\left(-\frac{1}{2}+it\right)\right)$ with $0\leq t<13.7$ are $t_0=0$, $t_1\approx 2.93$, and $t_2\approx 9.92$, where

$$\zeta\left(-\tfrac{1}{2}+it_0\right)\approx -0.21,\quad \zeta\left(-\tfrac{1}{2}+it_1\right)\approx 0.35,\quad \zeta\left(-\tfrac{1}{2}+it_2\right)\approx 2.03.$$

So the only hole in the barrier is $-\frac{1}{2}+it_1$. This is where the real line with asymptote $\pi/\log 2$ crosses the line $\sigma=-\frac{1}{2}$.

5. Outlook

In our work, we investigated the behavior of the graphs of the continuous functions $s:[0,1)\to\mathbb{C}$ defined by the equation $\zeta(s(c))-c=0$ and an initial point $s(0)$ (a zero of the zeta function). If $s(1)$ exists, such a graph connects a zero of $\zeta(s)$ to a zero of $\zeta(s)-1$. The latter zeros are the first points on the conjectured chains of zeros of derivatives.

A similar approach could also be used to investigate the conjectured chains of zeros of the derivatives of $\zeta(s)$. For each zero s_0 of

$$\zeta(s) - 1 = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

one would consider the implicit function $s : [0, \infty) \to \mathbb{C}$ given by

$$\zeta^{(k)}(s(k)) = (-1)^k \sum_{n=1}^{\infty} \frac{\log^k n}{n^{s(k)}} = 0,$$

with $s(0) = s_0$. This function $s(k)$ should yield the correspondence of zeros of $\zeta^{(k)}(s)$ and $\zeta^{(k+1)}(s)$ for $k \in \mathbb{Z}$, $k \geq 0$ for two zeros which would be connected by $\{s(x) \mid k \leq x \leq k+1\}$.

Together, the two implicit functions could give more detailed insight into the distribution of the zeros of $\zeta(s)$ by relating it to the distribution of higher derivatives (see Theorem 1). Furthermore it will be interesting to see how the conjectured chains of zeros of the derivatives of $\zeta(s)$ fit in with the universality of $\zeta(s)$ found by Voronin [1975].

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a_bosema@uncg.edu *Joint School of Nanoscience and Nanoengineering, The University of North Carolina at Greensboro, Greensboro, North Carolina 27402, United States*

s_pauli@uncg.edu *Department of Mathematics and Statistics, The University of North Carolina at Greensboro, Greensboro, North Carolina 27402, United States*

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