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On the geometric deformations of functions in $L^2[D]$

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We derive a formal relationship between the coefficients of a function expanded in either the Legendre basis or Haar wavelet basis, before and after a polynomial deformation of the function's domain. We compute the relationship of coefficients explicitly in three cases: linear deformation with Haar basis, linear deformation with Legendre basis, and polynomial deformation with Legendre basis.

1. Introduction

This paper explores the relationship between Schauder coefficients of a function before and after the domain of that function has been deformed in some reasonably well-behaved manner. As an analogy, one may think of a function as a melody recorded on an LP, and its domain as the position in the groove on the LP. The groove will become deformed if the LP is left in the sun, but the melody played on the LP after deformation will be related to the original melody. We are interested in understanding that relationship. Our results are a preliminary step toward addressing the inverse question of how to recover information about the undeformed function given the deformed function and an unknown deformation.

More formally, let $\mathcal{W} = \{w : D \to D \mid w \text{ is a diffeomorphism}\}$ be a class of diffeomorphisms defined on a closed subinterval $D \subset \mathbb{R}$. Then each $w \in \mathcal{W}$ defines a function F_w on $L^2[D]$, where $F_w(f) = f \circ w$. Below, we provide necessary background information to pose our question in terms of coefficients of elements in $L^2[D]$. In Section 2, we derive a general relationship between the coefficients of f, w, and $g = F_w(f)$. In Section 3, we compute precise relationships between coefficients of f and g in the Legendre and Haar wavelet bases for linear deformations, and in Section 4, in the Legendre basis for polynomial deformations.

MSC2010: 26.

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1.1. ***Background.*** For the Hilbert space $L^2[D] = \{f : D \to \mathbb{R} \mid \int_D f^2 < \infty\}$, recall that the inner product is given by $\langle f, g\rangle = \int_D fg$. Therefore, given an orthonormal basis $\{\phi_i(x)\}_{i=0}^{\infty}$ for $L^2[D]$, the Schauder coefficients $\{a_i\}$ corresponding to a function expanded in that basis, $f(x) = \sum_{i=0}^{\infty} a_i\phi_i(x)$, can be computed by $a_i = \int_D f(x)\phi_i(x)\,dx$ [Kreyszig 1989].

We will be exploring two orthonormal bases in our work: the Legendre basis, which is a basis of polynomials, and the Haar wavelet basis, a basis that localizes in scale and location. As noted above, domain deformation corresponds to composition of functions. The Legendre basis has the advantage that computations involving composition with polynomial deformations are straightforward. On the other hand, because the support of each basis function is the entire domain D, localized deformations will produce changes in every Legendre coefficient. The Haar wavelet basis has the opposite problem: local deformations will change only the subset of coefficients corresponding to that locale, but composing basis functions with polynomial deformations is computationally intimidating. Examined together, however, these two bases provide a wide view of possible behaviors. We now define each basis formally.

1.1.1. *Legendre basis for* $L^2[-1, 1]$. The Legendre basis arises by applying the Gram–Schmidt orthonormalization process to the simplest basis for $L^2[-1, 1]$, the monomials $\{x^i\}_{i=0}^{\infty}$. For $D = [-1, 1]$, the resulting basis is as below (though choosing a different D will produce a different normalizing constant K):

$$\psi_i(x) = \begin{cases} \sqrt{\dfrac{2i+1}{2}} \displaystyle\sum_{n=0}^{N} (-1)^n \frac{(2i-2n)!}{2^i n!(i-n)!(i-2n)!} x^{i-2n} & \text{for } -1 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $N = i/2$ when i is even, and $N = (i-1)/2$ when i is odd [Jackson 2004]. Rewriting the normalizing constant

$$K_{in} = \sqrt{\frac{2i+1}{2}}(-1)^n \frac{(2i-2n)!}{2^i n!(i-n)!(i-2n)!},$$

our basis becomes

$$\psi_i(x) = \sum_{n=0}^{N} K_{in} x^{i-2n}. \tag{1}$$

A function $f(x) \in L^2[-1, 1]$ can therefore be written as

$$f(x) = \sum_{i=0}^{\infty} a_i \sum_{n=0}^{N} K_{in} x^{i-2n} = \sum_i \sum_n a_i K_{in} x^{i-2n}.$$

1.1.2. *Haar basis for* $L^2[0, 1]$. The Haar wavelet basis is generated by shifting and scaling the simplest mother wavelet,

$$\psi(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

which can be thought of as a coarse piecewise constant approximation to a sine curve. After scaling and shifting, the resulting orthonormal basis is given by

$$\psi_{ij}(x) = \begin{cases} 2^{i/2} & \text{for } \frac{j}{2^i} \le x < \frac{j+1/2}{2^i}, \\ -2^{i/2} & \text{for } \frac{j+1/2}{2^i} \le x < \frac{j+1}{2^i}, \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in \mathbb{N}$ and $0 \le j \le 2^i - 1$ [Radunović 2009].

2. General relationships of coefficients

Our first result presents a general relationship between Schauder coefficients of f and those of g.

Theorem 1. *Consider $f(x) \in L^2[D]$, where $D \subset \mathbb{R}$ is a a closed interval, and let $w(x) = h^{-1}(x) : D \to D$ be a diffeomorphism. Set $g(x) = f \circ w(x)$. Then for $f(x) = \sum_i a_i \psi_i(x)$, where $\{\psi_i\}$ is an orthonormal basis for $L^2[D]$,*

$$g(x) = \sum_i c_i \psi_i(x) = \sum_i \sum_j \alpha_{ij} a_j \psi_i(x),$$

where $\alpha_{ij} = \langle \psi_j \circ w(x), \psi_i(x) \rangle_{L^2}$.

Proof. We claim that $g \in L^2(D)$. Because w is a diffeomorphism, w' is continuous and nonvanishing on D. Therefore, $1/w'$ is also continuous on D and thus bounded above by some $M < \infty$. We then have $\int_D g^2 = \int_D (f \circ w)^2 = \int f^2/w' \le M \|f\|_2^2 < \infty$, and so $g \in L^2[D]$.

Thus, we can write $g(x)$ as the convergent series $\sum_i c_i \psi_i(x)$, where $c_i = \langle g, \psi_i \rangle$. Remembering that $g = f \circ w = \sum_j a_j (\psi_j \circ w)$, we have

$$\begin{aligned} c_i &= \langle g, \psi_i \rangle = \langle f \circ w, \psi_i \rangle \\ &= \left\langle \sum_j a_j (\psi_j \circ w), \psi_i \right\rangle = \sum_j a_j \langle \psi_j \circ w, \psi_i \rangle = \sum_j a_j \alpha_{ij}. \end{aligned}$$ □

Note that the coefficients $\{\alpha_{ij}\}$ can be computed independently of f. Given a deformation w, these may be computed and reused for multiple choices of f. Alas, such a clean theorem requires dues to be paid elsewhere. We will see below the challenges of computing the $\{\alpha_{ij}\}$ coefficients in specific cases.

3. Explicit relationships: linear deformations

3.1. ***Linear deformations and the Legendre basis for $L^2[-1, 1]$.*** We first examine deformations of the form $w(x) = \beta x$, with $0 < \beta < 1$, for $D = [-1, 1]$. We are cheating slightly here, as $h = w^{-1}$ maps D to a larger interval $D \subset h(D)$, and so the setting of this first example does not match with Theorem 1. Nonetheless, the results for linear w will be helpful in understanding the results for polynomial w in Section 4, and so we persevere. We start with a simple fact from calculus:

Fact. *For $A = [-a, a]$ and t odd, $\int_A x^t\,dx = 0$.* □

Theorem 2. *Following Theorem 1, we take $D = [-1, 1]$, $\{\psi_i(x)\}$ as the Legendre basis, and $w(x) = \beta x$, $\beta > 0$. Then*

$$\alpha_{ij} = \begin{cases} 2\displaystyle\sum_{n,m=0}^{N,M} \frac{K_{in}K_{jm}\beta^{i-2n}}{(i-2n)+(j-2m)+1} & \text{if } i+j \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Expanding a function f in the Legendre basis, we can write $f(x)$ as $\sum_i a_i \sum_{n=0}^{N} K_{in}x^{i-2n}$, where $N = i/2$ when i is even and $N = (i-1)/2$ when i is odd. We are concerned with $g(x) = f(w(x)) = \sum_i a_i\psi_i(w(x))$, where

$$\psi_i(w(x)) = \psi_i(\beta x) = \sum_{n=0}^{N} K_{in}(\beta x)^{i-2n} = \sum_{n=0}^{N} K_{in}x^{i-2n}\beta^{i-2n}.$$

Therefore,

$$g(x) = \sum_i a_i \sum_{n=0}^{N} K_{in}x^{i-2n}\beta^{i-2n}.$$

Substituting in βx, we obtain the following formula for $\{\alpha_{ij}\}$:

$$\begin{aligned}
\alpha_{ij} &= \langle \psi_i(\beta x), \psi_j(x)\rangle \\
&= \int_{-1}^{1} \Bigl(\sum_{n=0}^{N} K_{in}x^{i-2n}\beta^{i-2n}\Bigr)\Bigl(\sum_{m=0}^{M} K_{jm}x^{j-2m}\Bigr)\,dx \\
&= \int_{-1}^{1} \sum_{n,m=0}^{N,M} (K_{in}x^{i-2n}\beta^{i-2n})(K_{jm}x^{j-2m})\,dx \\
&= \sum_{n,m=0}^{N,M} \int_{-1}^{1} (K_{in}x^{i-2n}\beta^{i-2n})(K_{jm}x^{j-2m})\,dx. \\
&= \sum_{n,m=0}^{N,M} \int_{-1}^{1} K_{in}K_{jm}x^{i-2n+j-2m}\beta^{i-2n}\,dx
\end{aligned}$$

In view of the Fact quoted above, if $i+j$ is odd, the integral is zero. Otherwise,

$$\alpha_{ij}=\sum_{n,m=0}^{N,M}\int_{-1}^{1}K_{in}K_{jm}x^{i-2n+j-2m}\beta^{i-2n}\,dx=2\sum_{n,m=0}^{N,M}\frac{K_{in}K_{jm}\beta^{i-2n}}{(i-2n)+(j-2m)+1}. \quad \square$$

3.2. ***Linear deformations and the Haar basis for $L^2[0,1]$.*** We again examine linear deformations of the form $w(x)=\beta x$, now with $\beta>0$ and $D=[0,1]$. Note that for the Haar wavelet basis, each basis element has two indices: one for scale and one for location. Hence, the $\{\alpha_{ij}\}$ coefficients defined in Theorem 1 become $\{\alpha_{ijkl}\}=\langle\psi_{ij}\circ w,\psi_{kl}\rangle$.

As before, we must compute

$$\psi_{ij}(w(x))=\psi_{ij}(\beta x)=\begin{cases}2^{i/2} & \text{for } \frac{j}{2^i}\le\beta x<\frac{j+1/2}{2^i}\\ -2^{i/2} & \text{for } \frac{j+1/2}{2^i}\le\beta x<\frac{j+1}{2^i}\\ 0 & \text{otherwise.}\end{cases}$$
$$=\begin{cases}2^{i/2} & \text{for } \frac{j}{\beta 2^i}\le x<\frac{j+1/2}{\beta 2^i}\\ -2^{i/2} & \text{for } \frac{j+1/2}{\beta 2^i}\le x<\frac{j+1}{\beta 2^i}\\ 0 & \text{otherwise.}\end{cases}$$

Let

$$I_{ij}^{+}=\Big[\frac{j}{\beta 2^i},\frac{j+1/2}{\beta 2^i}\Big)\quad\text{and}\quad I_{ij}^{-}=\Big[\frac{j+1/2}{\beta 2^i},\frac{j+1}{\beta 2^i}\Big),$$

the regions where $\psi_{ij}(w(x))>0$ and $\psi_{ij}(w(x))<0$, respectively. Similarly, let

$$I_{kl}^{+}=\Big[\frac{l}{2^k},\frac{l+1/2}{2^k}\Big)\quad\text{and}\quad I_{kl}^{-}=\Big[\frac{l+1/2}{2^k},\frac{l+1}{2^k}\Big).$$

Note that a particular α_{ijkl} will be nonzero only if (a) $(I_{ij}^{+}\cup I_{ij}^{-})\cap I_{kl}^{+}\neq\varnothing$ and $(I_{ij}^{+}\cup I_{ij}^{-})\cap I_{kl}^{-}\neq\varnothing$ and (vice versa) (b) $(I_{kl}^{+}\cup I_{kl}^{-})\cap I_{ij}^{+}\neq\varnothing$ and $(I_{kl}^{+}\cup I_{kl}^{-})\cap I_{ij}^{-}\neq\varnothing$. Otherwise, α_{ijkl} will vanish; either the supports will be disjoint, or the support of one will be contained entirely in the positive or negative domain of the other. Analyzing the possibilities for nonzero values of α_{ijkl} produces the following theorem.

Theorem 3. *Following Theorem 1, we take $D=[0,1]$, $\{\psi_{ij}(x)\}$ as the Haar wavelet basis, and $w(x)=\beta x$, $\beta>0$. Then nonzero values for α_{ijkl} are of the form*

$$\alpha_{ijkl}=\sum_{m=1}^{3}\xi_m 2^{m-k-2}+\tilde{\xi}_m 2^{m-i-2},$$

where $\xi_1=1\pm\beta$, $\xi_2\in\{-\beta,\pm l\beta,(1+l)\beta,\pm(3l+1)\beta\}$, $\xi_3\in\{\pm l\beta,(1+l)\beta\}$, $\tilde{\xi}_1=0$, $\tilde{\xi}_2\in\{\pm 1,\pm j,\pm(1+j),3j,-(3j+1)\}$, and $\tilde{\xi}_3\in\{1,\pm j,(1+j)\}$.

Proof. Expanding some f in the Haar basis, we can write $f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{2^i-1} a_{ij}\psi_{ij}(x)$. Therefore,

$$g(x) = f(w(x)) = \sum_{i=0}^{\infty} \sum_{j=0}^{2^i-1} a_{ij}\psi_{ij}(w(x)) = \sum_{i=0}^{\infty} \sum_{j=0}^{2^i-1} a_{ij}\psi_{ij}(\beta x).$$

The formula for α_{ijkl} is then

$$\begin{aligned}
\alpha_{ijkl} &= \langle \psi_{ij}(\beta x), \psi_{kl}(x) \rangle \\
&= \int_{(I_{ij}^+ \cap I_{kl}^+) \cup (I_{ij}^- \cap I_{kl}^-)} 2^{i/2} 2^{k/2}\, dx - \int_{(I_{ij}^+ \cap I_{kl}^-) \cup (I_{ij}^- \cap I_{kl}^+)} 2^{i/2} 2^{k/2}\, dx \\
&= 2^{(i+k)/2} \left(\int_{(I_{ij}^+ \cap I_{kl}^+) \cup (I_{ij}^- \cap I_{kl}^-)} dx - \int_{(I_{ij}^+ \cap I_{kl}^-) \cup (I_{ij}^- \cap I_{kl}^+)} dx \right) \\
&= 2^{(i+k)/2} \big[\mu\big((I_{ij}^+ \cap I_{kl}^+) \cup (I_{ij}^- \cap I_{kl}^-) \big) - \mu\big((I_{ij}^+ \cap I_{kl}^-) \cup (I_{ij}^- \cap I_{kl}^+) \big) \big],
\end{aligned}$$

where μ is the standard Lebesgue measure. Hence, to compute α_{ijkl}, we must compute $M = \mu\big((I_{ij}^+ \cap I_{kl}^+) \cup (I_{ij}^- \cap I_{kl}^-)\big) - \mu\big((I_{ij}^+ \cap I_{kl}^-) \cup (I_{ij}^- \cap I_{kl}^+)\big)$. From the 14 possible arrangements of the values $\left\{ \frac{l}{2^k}, \frac{l+1/2}{2^k}, \frac{l+1}{2^k}, \frac{j}{\beta 2^i}, \frac{j+1/2}{\beta 2^i}, \frac{j+1}{\beta 2^i} \right\}$ satisfying (a) and (b) as in the discussion preceding Theorem 3, we find possible values for M as follows. Given positive integers $\{ijkl\}$ corresponding to α_{ijkl}, the value of $\beta 2^{i+k+2} M \neq 0$ is one of

- $(1+j)2^{k+3} - l\beta 2^{i+3} - \beta 2^{i+2}$
- $(1+j)2^{k+2} - l\beta 2^{i+2}$
- $-j2^{k+3} + l\beta 2^{i+3} + (1+\beta 2^{i+1}\}$
- $-(3j+1)2^{k+2} + (3l+1)\beta 2^{i+2} + (1+\beta)2^{i+1}$
- $-j2^{k+2} + l\beta 2^{i+2} + (1-\beta)2^{i+1}$
- $\pm j2^{k+2} \mp (1+l)\beta 2^{i+2}$
- $-(1+j)2^{k+2} + (1+l)\beta 2^{i+2}$
- $-j2^{k+3} - 2^{k+2} + (1+l)\beta 2^{i+3}$
- $j2^{k+3} + 2^{k+2} - l\beta 2^{i+3}$
- $2^{k+3} + 3j2^{k+2} - (3l+1)\beta 2^{i+2}$
- $j2^{k+2} - l\beta 2^{i+2}$
- $-(1+j)2^{k+2} + l\beta 2^{i+2}$
- $(1+j)2^{k+2} - (1+l)\beta 2^{i+2}$.

Substituting these values for M into the formula for α_{ijkl} gives the desired result. □

4. Explicit relationships: polynomial deformations and the Legendre basis

Because the Legendre basis is a basis of polynomials, it is less challenging to compute values for $\{\alpha_{ij}\}$ when w is a polynomial than it would be for a nonpolynomial basis such as the Haar basis. We now consider deformations $w(x) = \sum_{s=0}^{v} \beta_s x^s$, where the $\{\beta_s\}$ are chosen so that $w(x)$ maps $[-1, 1]$ onto itself diffeomorphically and $dw/dx > 0$. This increase in complexity of the deformations requires careful accounting, as we shall see below.

As before, we compute

$$\begin{aligned}
\psi_i(w(x)) = \psi_i\left(\sum_{s=0}^{v} \beta_s x^s\right) &= \sum_{n=0}^{N} K_{in}\left(\sum_{s=0}^{v} \beta_s x^s\right)^{i-2n} \\
&= \sum_{n=0}^{N} K_{in}\left(\sum_{p_0+p_1+\cdots+p_v=i-2n} \binom{i-2n}{p_0, p_1, \ldots, p_v} (\beta_0 x^0)^{p_0} (\beta_1 x^1)^{p_1} \cdots (\beta_v x^v)^{p_v}\right) \\
&= \sum_{n=0}^{N} K_{in}\left(\sum_{p_0+p_1+\cdots+p_v=i-2n} \binom{i-2n}{p_0, p_1, \ldots, p_v} \left(\prod_{s=0}^{v} {\beta_s}^{p_s}\right) x^{\sum_{s=0}^{v} s p_s}\right) \\
&= \sum_{n=0}^{N} \sum_{P} K_{in} \binom{i-2n}{p_0, p_1, \ldots, p_v} \left(\prod_{s=0}^{v} {\beta_s}^{p_s}\right) x^{\sum_{s=0}^{v} s p_s}
\end{aligned}$$

using the multinomial theorem, where $P = p_0 + p_1 + \cdots + p_v$ is the collective sum of partitions of $i - 2n$. Therefore,

$$g(x) = f(w(x)) = \sum_{i} a_i \sum_{n=0}^{N} \sum_{P} K_{in} \binom{i-2n}{p_0, p_1, \ldots, p_v} \left(\prod_{s=0}^{v} {\beta_s}^{p_s}\right) x^{\sum_{s=0}^{v} s p_s}.$$

In order to apply the Fact to compute $\langle \psi_j(w(x)), \psi_i(x) \rangle$, we must identify which of the powers of x, given by $\sum_{s=0}^{v} s p_s$, are even and which are odd. Certainly, when s is even, $s p_s$ will be even. We rewrite

$$\sum_{s=0}^{v} s p_s = \sum_{t=0}^{\lfloor v/2 \rfloor} \bigl(2t p_{2t} + (2t+1) p_{2t+1}\bigr) = \sum_{t=0}^{\lfloor v/2 \rfloor} 2t p_{2t} + \sum_{t=0}^{\lfloor v/2 \rfloor} (2t+1) p_{2t+1}.$$

Analyzing the sum over odd $s = 2t+1$, we see that if p_{2t+1} is even for a given t, the product $(2t+1) p_{2t+1}$ will be even. In other words, the parity of the total exponent $\sum_{s=0}^{v} s p_s$ is determined entirely by the parity of the number of odd-indexed elements of the partition that are themselves odd. More precisely, let N_P be the number of odd-valued elements in the set $\{p_{2t+1}\}$. If N_P is odd, then $\sum_{t=0}^{\lfloor v/2 \rfloor} (2t+1) p_{2t+1}$ will sum an odd number of odd elements, and will therefore be odd. If N_P is even, $\sum_{t=0}^{\lfloor v/2 \rfloor} (2t+1) p_{2t+1}$ will sum an even number of odd elements, and will therefore be even. We have proved the following lemma.

Lemma. *Let $P : p_1 + \cdots + p_v = i - 2n$ be a particular choice of partition. Then the value of $\sum_{s=0}^{v} s p_s$ will be even if N_P, the number of odd-indexed, odd-valued elements of P, is even, or odd if N_P is odd.*

We now state the result for polynomial deformations.

Theorem 4. *Following Theorem 1, we take $D = [-1, 1]$, $\{\psi_i(x)\}$ as the Legendre basis, and $w(x) = \sum_{s=0}^{v} \beta_s x^s$ to be monotone increasing on D. Then*

$$\alpha_{ij} = 2 \sum_{n,m=0}^{N,M} \sum_{P, 2|j+N_P} \binom{i-2n}{p_0, p_1, \ldots, p_v} \frac{K_{in} K_{jm} \left(\prod_{s=0}^{v} {\beta_s}^{p_s}\right)}{j - 2m + 1 + \sum_{s=0}^{v} s p_s}.$$

Proof. Calculating α_{ij}, we find

$$\begin{aligned}
\alpha_{ij} &= \langle \psi_i(w(x)), \psi_j(x) \rangle \\
&= \int_{-1}^{1} \left[\sum_{n=0}^{N} \sum_{P}^{i-2n} K_{in} \binom{i-2n}{p_0, p_1, \ldots, p_v} \left(\prod_{s=0}^{v} {\beta_s}^{p_s} \right) x^{\sum_{s=0}^{v} s p_s} \right] \left[\sum_{m=0}^{M} K_{jm} x^{j-2m} \right] dx \\
&= \sum_{n,m=0}^{N,M} \sum_{P}^{i-2n} K_{in} K_{jm} \binom{i-2n}{p_0, p_1, \ldots, p_v} \left(\prod_{s=0}^{v} {\beta_s}^{p_s} \right) \int_{-1}^{1} x^{j-2m+\sum_{s=0}^{v} s p_s} \, dx.
\end{aligned}$$

Each integral term of the sum will vanish or not depending on the parity of $j - 2m + \sum_{s=0}^{v} s p_s$. Because $2m$ is always even, we focus on the parity of $j + \sum_{s=0}^{v} s p_s$. For each α_{ij}, j is fixed along with its parity. From the discussion leading up to Theorem 4, we know that N_P determines the parity of $\sum_{s=0}^{v} s p_s$. Putting this together, we see that the exponent $j - 2m + \sum_{s=0}^{v} s p_s$ will be odd (and so will have vanishing integral) when $j + N_P$ is odd. When $j + N_P$ is even, however, the exponent will be even and the integral nonzero. □

5. Conclusion and future work

Based on the computational challenges apparent in the few simple examples given in this paper, we believe there are very few cases where the coefficients $\{\alpha_{ij}\}$ that capture the relationship between the deformed and undeformed function can be computed explicitly. Nonetheless, we would like to be able to say something in other situations. Currently, we are exploring distributions of coefficients of periodic functions after deformation by randomly generated b-splines with between 5 and 25 knots. We hope to make conjectures based on those empirical results about what we can realistically say mathematically. Because of the highly structured nature of periodic functions, we expect meaningful results. For example, since the oscillations of a periodic function cannot change in number or amplitude after composition with a deformation, there should be a formulation for a wavelet basis that relates scale and location of periodic behavior with the local energy of a deformation.

The motivation for this project comes from a similar problem in two dimensions related to modeling textures in images [Liu et al. 2004a; 2004b; Park et al. 2009]. When a periodic texture such as a wallpaper pattern appears in an image, it is often not periodic within the image. That is, geometric distortions arising from lighting, occlusion, or projection of a three-dimensional object onto the two-dimensional image plane, create a near-periodic texture in the image. To recognize the periodic structures in these distorted textures requires solving this problem: given a deformed near-periodic function, what is the underlying periodic function and the associated deformation? This inverse problem is ill-posed, but our work gives insight into a similar problem in one dimension. Future work will focus on examining that inverse problem in the one-dimensional setting and deriving similar results to the ones in this paper for functions on $\mathbb{R}^2$.

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