

involve

a journal of mathematics

Hyperbolic construction of Cantor sets

Zair Ibragimov and John Simanyi



Hyperbolic construction of Cantor sets

Zair Ibragimov and John Simanyi

(Communicated by Kenneth S. Berenhaut)

In this paper we present a new construction of the ternary Cantor set within the context of Gromov hyperbolic geometry. Unlike the standard construction, where one proceeds by removing middle-third intervals, our construction uses the collection of the removed intervals. More precisely, we first hyperbolize (in the sense of Gromov) the collection of the removed middle-third open intervals, then we define a visual metric on its boundary at infinity and then we show that the resulting metric space is isometric to the Cantor set.

1. The ternary Cantor set

The ternary Cantor set \mathcal{C} is one of the most familiar fractals in mathematics. Recall its standard construction, which is based on the Euclidean notion of length. Begin with the closed unit interval $C_0 = [0, 1] \subseteq \mathbb{R}$, then remove the open middle-third interval, constructing $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. We then remove the middle-third of each resulting closed interval again, finding

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

Continuing in this manner, we construct \mathcal{C} by taking the intersection of all C'_k s,

$$\mathcal{C} = \bigcap_{k=1}^{\infty} C_k.$$

Graphically, C_0 through C_6 are shown in Figure 1. The ternary Cantor set has many interesting properties. As the intersection of closed intervals in $(\mathbb{R}, | \cdot |)$, it is compact. It is also perfect (i.e., it contains no isolated points), uncountable and totally disconnected. The complement of the ternary Cantor set in $[0, 1]$, \mathcal{C}^c , is called the *Cantor string*. It consists of the countable union of the removed open middle-third intervals. Cantor strings are subjects of study in fractal geometry [Lapidus and van Frankenhuijsen 2006].

MSC2010: primary 30C65; secondary 05C25.

Keywords: Cantor set, Gromov hyperbolic spaces.

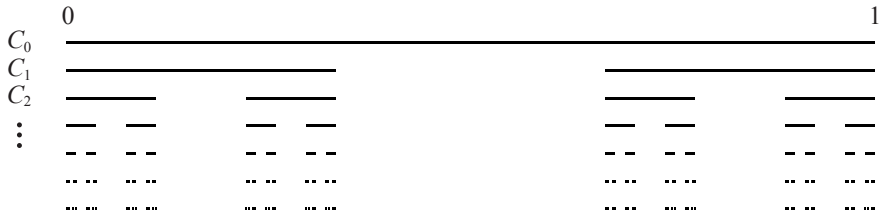


Figure 1. Sets C_0 through C_6 .

2. Hyperbolic construction

We begin with a brief discussion of Gromov hyperbolic spaces. A metric space (X, d) is called *Gromov δ -hyperbolic* (or *δ -hyperbolic*) if there exists a $\delta \geq 0$ such that for all $x, y, z, w \in X$,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta. \tag{2-1}$$

For $x, y, z \in X$, the *Gromov product* of x and y with respect to z is defined by

$$(x|y)_z = \frac{1}{2}[d(x, z) + d(y, z) - d(x, y)]. \tag{2-2}$$

Alternatively, the space (X, d) is δ -hyperbolic if

$$(x|y)_v \geq \min\{(x|z)_v, (z|y)_v\} - \delta,$$

for all $x, y, z, v \in X$ (see, for example, [Väisälä 2005]). A bounded metric space X is always δ -hyperbolic with $\delta \leq \text{diam } X$, so only unbounded metric spaces may have more interesting characteristics.

To each Gromov hyperbolic space X , we associate a boundary at infinity, ∂X (also called the *Gromov boundary*). Fix a base point $v \in X$. A sequence $\{a_i\}$ in X is said to *converge at infinity* if $(a_i|a_j)_v \rightarrow \infty$ as $i, j \rightarrow \infty$. Two such sequences $\{a_i\}$ and $\{b_i\}$ are *equivalent* if $(a_i|b_i)_v \rightarrow \infty$ as $i \rightarrow \infty$. The boundary at infinity is defined to be the equivalence classes of sequences converging at infinity. The boundary at infinity supports a family of so-called *visual metrics*. A metric d on ∂X is called a visual metric if there exists a $v \in X$, $C \geq 1$ and $\epsilon > 0$ such that for all $x, y \in \partial X$,

$$\frac{1}{C} \rho_{\epsilon, v}(x, y) \leq d(x, y) \leq C \rho_{\epsilon, v}(x, y), \quad \text{where } \rho_{\epsilon, v}(x, y) = e^{-\epsilon(x|y)_v}.$$

Here $(x|y)_v$ is the Gromov product on ∂X , defined by

$$(x|y)_v = \inf\{\liminf_{i \rightarrow \infty} (a_i|b_i)_v : a_i \in x, b_i \in y\}$$

and we set $e^{-\infty} = 0$. The boundary at infinity of any Gromov hyperbolic space endowed with a visual metric is bounded and complete [Bonk and Schramm 2000].

Our goal is to produce the ternary Cantor set within the framework of Gromov hyperbolic spaces. As mentioned above, we do this by hyperbolizing the collection of the removed middle-third intervals. Let X be the collection of all such intervals. Hence, X contains intervals such as $(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})$ and so on. Note that

$$\mathcal{C} = [0, 1] \setminus \bigcup_{I \in X} I.$$

We now proceed to construct a metric h on X so that the space (X, h) is Gromov hyperbolic. Let u_H be a distance function defined on the set of all nonempty subsets of $[0, 1]$, defined by

$$u_H(A, B) = \sup\{|x - y| : x \in A \text{ and } y \in B\}.$$

This distance function is called the *upper Hausdorff* distance (see, for instance, [Hausdorff 1957; Ibragimov 2011a]). If $I, J \in X$ with $I = (a, b), J = (c, d)$ and $b < c$, then $u_H = |a - d|$. Note also that for each $I, J \in X$, we have

$$u_H(I, J) \geq l(I) \vee l(J) \geq \sqrt{l(I) \cdot l(J)}, \tag{2-3}$$

where the first equality holds only if $I = J$ and the second equality holds only if $l(I) = l(J)$. Here, and in what follows, $l(I)$ denotes the Euclidean length of $I \in X$ and $a \vee b = \max\{a, b\}$ for positive numbers $a, b \in \mathbb{R}$.

Observe that since X consists of a disjoint collection of open intervals, it has a natural order \preceq induced by the usual order \leq on \mathbb{R} . Namely, we say that $I \preceq J$ if I is to the left of J or if $I = J$. Observe also that if $I \preceq J \preceq K$, then

$$u_H(I, K) \geq u_H(I, J). \tag{2-4}$$

Now we define a distance function h on X . Given $I, J \in X$, let

$$h(I, J) = 2 \log \frac{u_H(I, J)}{\sqrt{l(I) \cdot l(J)}}.$$

It is an immediate consequence of (2-3) that h is nonnegative, symmetric and $h(I, J) = 0$ if and only if $I = J$. To show that h also satisfies the triangle inequality, let I, J and K be arbitrary elements of X . Then the triangle inequality $h(I, J) \leq h(I, K) + h(K, J)$ is equivalent to

$$\begin{aligned} 2 \log \frac{u_H(I, J)}{\sqrt{l(I) \cdot l(J)}} &\leq 2 \log \frac{u_H(I, K)}{\sqrt{l(I) \cdot l(K)}} + 2 \log \frac{u_H(K, J)}{\sqrt{l(K) \cdot l(J)}} \\ &= 2 \log \frac{u_H(I, K) \cdot u_H(K, J)}{l(K) \sqrt{l(I) \cdot l(J)}}. \end{aligned}$$

This is true if and only if

$$l(K) \cdot u_H(I, J) \leq u_H(I, K) \cdot u_H(K, J). \quad (2-5)$$

It is also a consequence of (2-3) and (2-4) that inequality (2-5) holds if either $K \preceq I$ and $K \preceq J$ or $I \preceq K$ and $J \preceq K$. Therefore, due to symmetry, it is enough to verify the validity of (2-5) when $I \preceq K \preceq J$. In this case, since

$$u_H(I, J) = u_H(I, K) + u_H(J, K) - l(K),$$

inequality (2-5) is equivalent to $(u_H(I, K) - l(K))(u_H(J, K) - l(K)) \geq 0$, whose validity follows from (2-3). Thus, h is a metric on X .

Next, we will show that h satisfies the Gromov hyperbolicity condition (2-1) with $\delta = \log 2$. We will need the following lemma.

Lemma 2.6. For all $I, J, K, L \in X$, we have

$$u_H(I, J) \cdot u_H(K, L) \leq u_H(I, K) \cdot u_H(J, L) + u_H(I, L) \cdot u_H(J, K).$$

Proof. Without loss of generality we can assume that $I \preceq J \preceq K \preceq L$. Then inequality (2-4) implies that

$$u_H(I, K) \cdot u_H(J, L) \geq u_H(I, J) \cdot u_H(K, L).$$

It also implies that

$$(u_H(I, K) - u_H(J, K))(u_H(J, L) - u_H(J, K)) \geq 0,$$

which is equivalent to

$$u_H(I, K) \cdot u_H(J, L) \geq u_H(J, K)(u_H(I, K) + u_H(J, L) - u_H(J, K)).$$

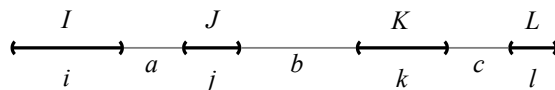
Since $u_H(I, L) = u_H(I, K) + u_H(J, L) - u_H(J, K)$, we obtain that

$$u_H(I, K) \cdot u_H(J, L) \geq u_H(J, K) \cdot u_H(I, L).$$

Therefore, to prove the lemma it is enough to show that

$$u_H(I, K) \cdot u_H(J, L) \leq u_H(I, L) \cdot u_H(J, K) + u_H(I, K) \cdot u_H(J, L).$$

Let i, j, k, l denote the lengths of I, J, K, L and let a, b, c denote the distances between I and J , J and K , K and L , respectively:



Then

$$\begin{aligned}
 u_H(I, K) \cdot u_H(J, L) &= (i+a+j+b+k)(j+b+k+c+l) \\
 &= (i+a+j+b+k)(j+b+k) + (i+a+j+b+k)(c+l) \\
 &= (i+a+j+b+k)(j+b+k) + (c+l)(j+b+k) + (i+a)(c+l) \\
 &< (i+a+k+b+j+c+l)(j+b+k) + (i+a+j)(k+c+l) \\
 &= u_H(I, L) \cdot u_H(J, K) + u_H(I, J) \cdot u_H(K, L),
 \end{aligned}$$

completing the proof. □

Theorem 2.7. The metric space (X, h) is Gromov δ -hyperbolic with $\delta \leq \log 2$.

Proof. Let $I, J, K, L \in X$ be arbitrary. Lemma 2.6 implies that

$$u_H(I, J) \cdot u_H(K, L) \leq 2[u_H(I, K) \cdot u_H(J, L) \vee u_H(I, L) \cdot u_H(J, K)].$$

Hence

$$\begin{aligned}
 h(I, J) + h(K, L) &= 2 \log \frac{u_H(I, J)}{\sqrt{l(I) \cdot l(J)}} + 2 \log \frac{u_H(K, L)}{\sqrt{l(K) \cdot l(L)}} \\
 &= 2 \log \frac{u_H(I, J) \cdot u_H(K, L)}{\sqrt{l(I) \cdot l(J) \cdot l(K) \cdot l(L)}} \\
 &\leq 2 \log \frac{2([u_H(I, K) \cdot u_H(J, L)] \vee [u_H(I, L) \cdot u_H(J, K)])}{\sqrt{l(I) \cdot l(J) \cdot l(K) \cdot l(L)}} \\
 &= [h(I, K) + h(J, L)] \vee [h(I, L) + h(J, K)] + 2 \log 2,
 \end{aligned}$$

as required. □

3. The boundary at infinity

We now discuss the boundary at infinity ∂X of the Gromov hyperbolic space (X, h) . Our goal is to construct a visual metric d on ∂X so that the space $(\partial X, d)$ is isometric to the Cantor set \mathcal{C} equipped with the standard Euclidean metric of the real line. Denote the distance between real numbers x and y by $|x - y|$. Recall that ∂X is the collection of equivalence classes of sequences in X converging at infinity. Fix $V = (\frac{1}{3}, \frac{2}{3}) \in X$ to be the base point. Observe that if the sequence $\{I_n\}$ converges at infinity, then $\lim_{j,k \rightarrow \infty} (I_j | I_k)_V = \infty$.

Lemma 3.1. Given $a \in \partial X$, there exists unique $x_a \in \mathcal{C}$ with the property that

$$\lim_{n \rightarrow \infty} u_H(I_n, \{x_a\}) = 0 \quad \text{for each } I_n \in a.$$

Conversely, for each $x \in \mathcal{C}$ there exists $a \in \partial X$ such that

$$\lim_{n \rightarrow \infty} u_H(J_n, \{x\}) = 0 \quad \text{for each } J_n \in a.$$

Proof. Given $\{I_n\} \in a$, we have

$$\begin{aligned} (I_j|I_k)_V &= \frac{1}{2}(h(I_j, V) + h(I_k, V) - h(I_j, I_k)) = \log \frac{u_H(I_j, V) \cdot u_H(I_k, V)}{l(V) \cdot u_H(I_j, I_k)} \\ &\leq \log \frac{\frac{2}{3} \cdot \frac{2}{3}}{\frac{1}{3} \cdot u_H(I_j, I_k)} = \log \frac{\frac{4}{3}}{u_H(I_j, I_k)}. \end{aligned}$$

Since $\lim_{j,k \rightarrow \infty} (I_j|I_k)_V = \infty$, we obtain $\lim_{j,k \rightarrow \infty} u_H(I_j, I_k) = 0$. For each n choose some point $x_n \in I_n$.

Next, given $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$|x_j - x_k| \leq u_H(I_j, I_k) < \epsilon \quad \text{whenever } j, k \geq n_0.$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence in $[0, 1]$. Since $[0, 1]$ is complete, it converges to some point in $[0, 1]$, call it x_a . Now if we choose a different sequence $\{y_n\}$, where $y_n \in I_n$, then

$$|y_n - x_a| \leq |y_n - x_n| + |x_n - x_a| \leq u_H(I_n, I_n) + |x_n - x_a|,$$

which implies that $\{y_n\}$ also converges to x_a . Therefore, the point x_a is well defined. Finally, since

$$u_H(I_n, \{x_a\}) \leq u_H(I_n, \{x_n\}) + u_H(\{x_n\}, \{x_a\}) \leq u_H(I_n, I_n) + |x_n - x_a|,$$

we obtain that $\lim_{n \rightarrow \infty} u_H(I_n, \{x_a\}) = 0$, as required.

Now let $\{K_n\}$ be another sequence converging at infinity and equivalent to $\{I_n\}$, i.e., $\{K_n\} \in a$. Then we need to show that $\lim_{n \rightarrow \infty} u_H(K_n, \{x_a\}) = 0$. Recall that the equivalence of the two sequences $\{I_n\}$ and $\{K_n\}$ means that $\lim_{n \rightarrow \infty} (I_n|K_n)_V = \infty$. The latter implies, by the same argument as above, that $\lim_{n \rightarrow \infty} u_H(I_n, K_n) = 0$. Since

$$u_H(K_n, \{x_a\}) \leq u_H(K_n, I_n) + u_H(I_n, \{x_a\}),$$

we obtain that $\lim_{n \rightarrow \infty} u_H(K_n, \{x_a\}) = 0$. Thus, we have shown the existence and uniqueness of x_a .

It remains to show that $x_a \in \mathcal{C}$. Assume by contrary that $x \in [0, 1] \setminus \mathcal{C}$. Then $x_a \in I$ for some $I \in X$. Since

$$0 < \frac{l(I)}{2} \leq u_H(I_n, \{x_a\}) \quad \text{and} \quad \lim_{n \rightarrow \infty} u_H(I_n, \{x_a\}) = 0,$$

we obtain the required contradiction. Thus, $x_a \in \mathcal{C}$, completing the proof of the first part.

To prove the second part, we first show that there exists a sequence $\{J_n\}$ in X converging at infinity and such that $\lim_{n \rightarrow \infty} u_H(J_n, \{x\}) = 0$. To construct such a sequence, index X as follows: let $J_{i,j} \in X$, where 3^{-i} is the length of the interval $J_{i,j}$ and j represents each interval in X of length 3^{-i} . Here

$$i = 1, 2, 3, \dots \quad \text{and} \quad j = 1, 2, \dots, 2^{i-1}.$$

Note that $J_{1,1} = V$ and that $u_H(\{x\}, J_{1,1}) \leq 2/3$ while $l(J_{1,1}) = 1/3$. We can then find J_{2,j_2} such that $u_H(\{x\}, J_{2,j_2}) \leq \frac{2}{9}$ and $l(J_{2,j_2}) = \frac{1}{9}$. Continuing in this manner, for each $n \in \mathbb{N}$, there exists j_n such that

$$u_H(\{x\}, J_{n,j_n}) \leq \frac{2}{3^n} \quad \text{and} \quad l(J_{n,j_n}) = 3^{-n}.$$

Put $J_n = J_{n,j_n}$. Then $\lim_{n \rightarrow \infty} u_H(J_n, \{x\}) = 0$, as required. Observe that since $u_H(J_j, J_k) \leq u_H(J_j, \{x\}) + u_H(J_k, \{x\})$, we have $\lim_{j,k \rightarrow \infty} u_H(J_j, J_k) = 0$. Also, since

$$(J_j|J_k)_V = \log \frac{u_H(J_j, V) \cdot u_H(J_k, V)}{l(V) \cdot u_H(J_j, J_k)}$$

and $u_H(J_j, V) \cdot u_H(J_k, V) \leq \frac{4}{9}$, we obtain that the sequence $\{J_n\}$ converges at infinity.

Finally, we let $a \in \partial X$ to be the equivalence class of sequences converging at infinity and equivalent to $\{J_n\}$. Then it follows from the first part that

$$\lim_{n \rightarrow \infty} u_H(J_n, \{x\}) = 0 \quad \text{for each } J_n \in a,$$

completing the proof of the lemma. □

Lemma 3.1 implies that the map $f: \partial X \rightarrow \mathcal{C}$, given by $f(a) = x_a$, is a well defined, bijective map. Now we define a metric d on ∂X by setting $d(a, b) = |x_a - x_b|$.

Lemma 3.2. The metric d is a visual metric. More precisely,

$$\frac{1}{3}e^{-(a|b)_V} \leq d(a, b) \leq 3e^{-(a|b)_V} \quad \text{for all } a, b \in \partial X.$$

Proof. Recall that $V = (\frac{1}{3}, \frac{2}{3})$ and

$$(a|b)_V = \inf \{ \liminf_{n \rightarrow \infty} (I_n|J_n)_V : I_n \in a, J_n \in b \}.$$

Given $a, b \in \partial X$, let $I_n \in a$ and $J_n \in b$ be arbitrary sequences. Then

$$(I_n|J_n)_V = \log \frac{u_H(I_n, V) \cdot u_H(I_n, V)}{l(V) \cdot u_H(I_n, J_n)}.$$

Lemma 3.1 implies that

$$\lim_{n \rightarrow \infty} u_H(I_n, \{x_a\}) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} u_H(J_n, \{x_b\}) = 0.$$

In particular, since

$$|u_H(I_n, J_n) - |x_a - x_b|| \leq u_H(I_n, \{x_a\}) + u_H(J_n, \{x_b\}),$$

we have

$$\lim_{n \rightarrow \infty} u_H(I_n, J_n) = |x_a - x_b| = d(a, b).$$

Also, since

$$|u_H(V, I_n) - u_H(V, \{x_a\})| \leq u_H(I_n, \{x_a\}),$$

we have

$$\lim_{n \rightarrow \infty} u_H(V, I_n) = u_H(V, \{x_a\}) \quad \text{and} \quad \lim_{n \rightarrow \infty} u_H(V, J_n) = u_H(V, \{x_b\}).$$

Therefore, as the sequences $\{I_n\} \in a$ and $\{J_n\} \in b$ were arbitrary, we obtain

$$(a|b)_V = \log \frac{u_H(V, \{x_a\}) \cdot u_H(V, \{x_b\})}{l(V) \cdot d(a, b)}.$$

Finally, since $l(V) = \frac{1}{3}$ and since $\frac{1}{3} \leq u_H(V, \{x\}) \leq \frac{2}{3}$ for all $x \in [0, 1]$, we have

$$\frac{1}{3}d(a, b) \leq \frac{3}{4}d(a, b) = \frac{\frac{1}{3}}{\frac{2}{3} \cdot \frac{2}{3}}d(a, b) \leq e^{-(a|b)_V} \leq \frac{\frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{3}}d(a, b) = 3d(a, b).$$

Equivalently,

$$\frac{1}{3}e^{-(a|b)_V} \leq d(a, b) \leq 3e^{-(a|b)_V},$$

completing the proof. □

As an immediate consequence of Lemma 3.2 we obtain our main result.

Theorem 3.1. The spaces $(\partial X, d)$ and $(\mathcal{C}, | - |)$ are isometric.

4. Further remarks

Although this particular geometric approach was successful, there is no guarantee that *any* such construction will produce the desired results. Consider, for example, the following seemingly natural distance function \hat{h} , defined for any $I, J \in X$ by

$$\hat{h}(I, J) = 2 \log \frac{l(I \cup J)}{\sqrt{l(I) \cdot l(J)}}.$$

Since the distinct intervals in X are disjoint, $l(I \cup J) = l(I) + l(J)$ whenever $I \neq J$, from which it follows that $\hat{h}(I, J) \leq \hat{h}(I, K) + \hat{h}(K, J)$, for all $I, J, K \in X$. Hence the space (X, \hat{h}) is a metric space. In fact, it is Gromov hyperbolic. Indeed, by setting $\mu(I, J) = l(I \cup J)$, we find that $\mu(I, I) > 0$, $\mu(I, J) = \mu(J, I)$

and $\mu(I, J) \leq \mu(I, K) + \mu(K, J)$, for all $I, J, K \in X$. By [Ibragimov 2011a, Lemma 3.7], we have

$$\mu(I, J) \cdot \mu(K, L) \leq 4[(\mu(I, K) \cdot \mu(J, L)) \vee (\mu(I, L) \cdot \mu(J, K))],$$

for all $I, J, K, L \in X$. Hence the space (X, \hat{h}) is δ -hyperbolic with $\delta \leq \log 4$ (see, for example, the proof of [Ibragimov 2011b, Theorem 2.1(2)]).

Next, we investigate the boundary at infinity of (X, \hat{h}) . Observe that

$$m(I, J) \leq \hat{h}(I, J) \leq m(I, J) + \log 4 \quad \text{for all } I, J \in X,$$

where

$$m(I, J) = \log \frac{\max\{l(I), l(J)\}}{\min\{l(I), l(J)\}}.$$

Fix $V = (\frac{1}{3}, \frac{2}{3}) \in X$ to be the base point. Then we have the following estimates for the Gromov products in (X, \hat{h}) with respect to V :

$$(I|J)_V = \frac{1}{2}[\hat{h}(I, V) + \hat{h}(J, V) - \hat{h}(I, J)] \leq \frac{1}{2}[m(I, V) + m(J, V) - m(I, J)] + \log 4,$$

for all $I, J \in X$ and, similarly

$$(I|J)_V = \frac{1}{2}[\hat{h}(I, V) + \hat{h}(J, V) - \hat{h}(I, J)] \geq \frac{1}{2}[m(I, V) + m(J, V) - m(I, J)] - \log 2.$$

Since

$$\frac{1}{2}[m(I, V) + m(J, V) - m(I, J)] = \log \frac{1}{l(I) \vee l(J)} - \log 3,$$

we find

$$\log \frac{1}{l(I) \vee l(J)} - \log 6 \leq (I|J)_V \leq \log \frac{1}{l(I) \vee l(J)} + \log \frac{4}{3}.$$

Hence a sequence $\{I_n\}$ in (X, \hat{h}) converges at infinity if and only if

$$\max\{l(I_n), l(I_k)\} \rightarrow 0 \quad \text{as } n, k \rightarrow \infty.$$

But all such sequences are equivalent and, consequently, we obtain that the boundary at infinity of (X, \hat{h}) consists of a single point.

We would like to also point out that this geometric construction differs from a topological approach. Topologically, the Cantor set can be viewed as the end space of the infinite binary tree, known as the Cantor tree (Figure 2), when the latter is endowed with a path metric. The end space of such a tree is the collection of all possible infinite branches emanating from its root, and is an ultrametric space when equipped with a visual metric (see [Hughes 2004] for details). As the end space is an ultrametric space, it can not be isometric to the Cantor set, although it is homeomorphic to it.

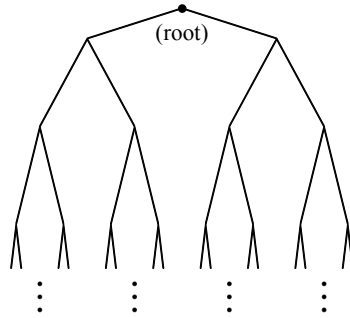


Figure 2. The Cantor tree.

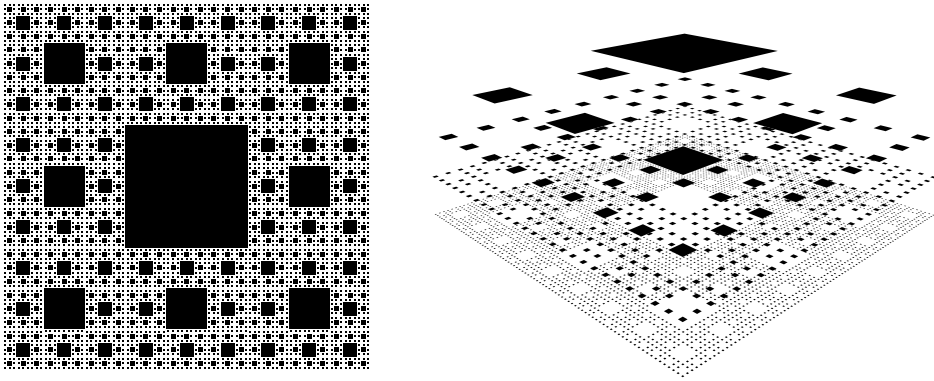


Figure 3. The standard Sierpiński carpet and some of its removed squares.

Finally, although we will not pursue it in this paper, many other fractals, such as Sierpiński carpets, can also be isometrically identified with the boundary at infinity of a similarly hyperbolized collection of removed squares (Figure 3).

References

- [Bonk and Schramm 2000] M. Bonk and O. Schramm, “Embeddings of Gromov hyperbolic spaces”, *Geom. Funct. Anal.* **10**:2 (2000), 266–306. MR 2001g:53077 Zbl 0972.53021
- [Hausdorff 1957] F. Hausdorff, *Set theory*, Chelsea, New York, 1957. MR 19,111a Zbl 0081.04601
- [Hughes 2004] B. Hughes, “Trees and ultrametric spaces: A categorical equivalence”, *Adv. Math.* **189**:1 (2004), 148–191. MR 2005m:57001 Zbl 1061.57021
- [Ibragimov 2011a] Z. Ibragimov, “Hyperbolizing hyperspaces”, *Michigan Math. J.* **60**:1 (2011), 215–239. MR 2012g:53063 Zbl 1220.30075
- [Ibragimov 2011b] Z. Ibragimov, “Hyperbolizing metric spaces”, *Proc. Amer. Math. Soc.* **139**:12 (2011), 4401–4407. MR 2012f:30084 Zbl 1244.30066

[Lapidus and van Frankenhuijsen 2006] M. L. Lapidus and M. van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions: Geometry and spectra of fractal strings*, Springer, New York, 2006. MR 2007j:11001 Zbl 1119.28005

[Väisälä 2005] J. Väisälä, “Gromov hyperbolic spaces”, *Expo. Math.* **23**:3 (2005), 187–231. MR 2006j:53055 Zbl 1087.53039

Received: 2012-07-06 Revised: 2012-12-21 Accepted: 2013-01-06

zibragimov@fullerton.edu *Department of Mathematics, California State University,
Fullerton, McCarthy Hall 154, Fullerton, CA 92831, United States*

jsimanyi@csu.fullerton.edu *Department of Mathematics, California State University,
Fullerton, McCarthy Hall 154, Fullerton, CA 92831, United States*

involve

msp.org/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsgdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2013 is US \$105/year for the electronic version, and \$145/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

involve

2013

vol. 6

no. 3

Potentially eventually exponentially positive sign patterns MARIE ARCHER, MINERVA CATRAL, CRAIG ERICKSON, RANA HABER, LESLIE HOGBEN, XAVIER MARTINEZ-RIVERA AND ANTONIO OCHOA	261
The surgery unknotting number of Legendrian links BIANCA BORANDA, LISA TRAYNOR AND SHUNING YAN	273
On duals of p -frames ALI AKBAR AREFIJAMAAL AND LEILI MOHAMMADKHANI	301
Shock profile for gas dynamics in thermal nonequilibrium WANG XIE	311
Expected conflicts in pairs of rooted binary trees TIMOTHY CHU AND SEAN CLEARY	323
Hyperbolic construction of Cantor sets ZAIR IBRAGIMOV AND JOHN SIMANYI	333
Extensions of the Euler–Satake characteristic for nonorientable 3-orbifolds and indistinguishable examples RYAN CARROLL AND CHRISTOPHER SEATON	345
Rank numbers of graphs that are combinations of paths and cycles BRIANNA BLAKE, ELIZABETH FIELD AND JOBBY JACOB	369



1944-4176(2013)6:3;1-8