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Ecological systems, nonlinear boundary conditions, and Σ-shaped bifurcation curves

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We examine a one-dimensional reaction diffusion model with a weak Allee growth rate that appears in population dynamics. We combine grazing with a certain nonlinear boundary condition that models negative density dependent dispersal on the boundary and analyze the effects on the steady states. In particular, we study the bifurcation curve of positive steady states as the grazing parameter is varied. Our results are acquired through the adaptation of a quadrature method and Mathematica computations. Specifically, we computationally ascertain the existence of Σ-shaped bifurcation curves with several positive steady states for a certain range of the grazing parameter.

1. Introduction

Within population dynamics, the most accepted exemplar for modeling a designated population is the logistic equation

$$f(u) = u(a - bu), \tag{1-1}$$

which illustrates the inference that as a population burgeons, the per capita growth rate

$$\tilde{f}(u) = a - bu \tag{1-2}$$

of that population declines linearly. Yet empirically several authors have witnessed that at lower population densities, the per capita growth rate initially increases (see [Allee 1938; Dennis 1989; Lewis and Kareiva 1993; Shi and Shivaji 2006]). This phenomenon is known in the literature as the Allee effect [1938]. Since the logistic growth model does not compensate for the initial increase, a model of the Allee effect must be implemented to account for this phenomenon.

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The Allee effect can be either strong, in which the per capita growth rate is initially negative, or weak, in which the per capita growth rate is initially positive. The Allee effect is generally modeled in the literature via quadratic per capita growth rate functions of the population density. In this case, the analysis is more difficult since the per capita growth rate is not linear or even nonincreasing. As a contrast with (1-1), a weak Allee effect has been modeled as

$$f(u) = u(u+1)(b-u), \tag{1-3}$$

where $b > 1$.

By analyzing additional factors that can influence a population, such as grazing or harvesting, a better understanding can be had of the dynamics of the population. Therefore, through the inclusion of an extra term to account for these factors, specifically grazing, a more precise model can be obtained. Grazing can be considered as a category of natural predation, for example, when an owl preys upon the surrounding rodent population. The term $cu^2/(1+u^2)$ is commonly employed to model grazing of a population (see [Causey et al. 2010; Lee et al. 2011; Poole et al. 2012; van Nes and Scheffer 2005]).

Density dependent dispersal, or more specifically density dependent emigration, describes a situation in which the dispersal/emigration of individuals living within a patch is based on the population density, in our case, on the habitat border. A positive density dependent emigration characterizes a case where individuals have a tendency to leave if the population density is large and a tendency to stay if the population density is small. On the contrary, a negative density dependent emigration represents a case where individuals have a tendency to stay if the population density is large and a tendency to leave if the population density is small.

Initially and intuitively it was believed that the majority of animals exhibit positive density dependent dispersal. However, recent studies of several animals, including the bighorn sheep, roe deer, house mouse, prairie vole, European badger, and the Glanville fritillary butterfly *Melitaea cinxia*, have proven otherwise (see [Kuussaari et al. 1996; Matthysen 2005]). In the literature, several factors have been suggested as a cause of negative density dependent dispersal, including: niche breadth, increased predator abundance, and, in particular, conspecific attraction (see [Kuussaari et al. 1996; Matthysen 2005]). Conspecific attraction most simply means that there is a predisposition of individuals within a population to become enticed to areas where there are more conspecifics.

Cantrel and Cosner proposed the following nonlinear boundary condition to model conspecific attraction on the boundary of a patch (see [Cantrell and Cosner 2003; 2007; Goddard et al. 2010a; 2010b; 2011a; 2011b; 2012]):

$$d(\nabla u \cdot \eta)\alpha(x,u) + [1-\alpha(x,u)]u = 0; \quad \partial\Omega, \tag{1-4}$$

where $\alpha : \bar{\Omega} \times [0, \infty) \to [0, 1]$ is C^1 and nondecreasing, $d > 0$ is the diffusion parameter, $\nabla u \cdot \eta$ is the outward normal derivative, and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain. The $\alpha(x, u)$'s of biological importance are of the form

$$\alpha(x, u) = \alpha(u) = \frac{u}{u + g(u)}, \tag{1-5}$$

where $g : [0, \infty) \to [\delta, \infty)$ is a C^1 function, $\delta > 0$, and $g(u)/u \to 0$ as $u \to \infty$. Here, $\alpha(u)$ represents the fraction of the population that stays on the boundary when reached. Notice that if $\alpha(u) \equiv 0$ then (1-4) becomes the Dirichlet boundary condition $(u = 0;\ \partial\Omega)$, and if $\alpha(u) \equiv 1$ then (1-4) becomes the Neumann boundary condition $(\nabla u \cdot \eta = 0;\ \partial\Omega)$. In terms of this paper, we consider the case when $g(u) \equiv d$, where $d > 0$ is the diffusion parameter.

Our purpose is to analyze the effects of grazing in combination with a weak Allee effect and the nonlinear boundary conditions (1-4) on the steady state solutions of a reaction diffusion model. In particular, we study the one-dimensional case when $n = 1$ and $\Omega = (0, 1)$:

$$u_t = \frac{1}{\lambda} u_{xx} + u \tilde{f}(u) - \frac{cu^2}{1 + u^2}; \quad (0, 1), \tag{1-6}$$

with nonlinear boundary conditions, namely

$$\begin{aligned} -u'' = \lambda \left[u \tilde{f}(u) - \frac{cu^2}{1 + u^2} \right] &= \lambda f(u); \quad (0, 1), \\ u(0) \left[-\frac{1}{\lambda} u'(0) + \frac{1}{\lambda} \right] &= 0, \\ u(1) \left[\frac{1}{\lambda} u'(1) + \frac{1}{\lambda} \right] &= 0, \end{aligned} \tag{1-7}$$

where u represents the population density, $\tilde{f}(u)$ represents the per capita growth rate, $\lambda = 1/d$ and $d > 0$ represents the diffusion coefficient, and $c \geq 0$ represents the maximum grazing rate. Notice that the boundary conditions found in (1-7) can be separated into the following four cases:

$$-u'' = \lambda f(u);\ (0, 1), \quad u(0) = 0, \quad u(1) = 0, \tag{1-8}$$

$$-u'' = \lambda f(u);\ (0, 1), \quad u(0) = 0, \quad u'(1) = -1, \tag{1-9}$$

$$-u'' = \lambda f(u);\ (0, 1), \quad u'(0) = 1, \quad u(1) = 0, \tag{1-10}$$

$$-u'' = \lambda f(u);\ (0, 1), \quad u'(0) = 1, \quad u'(1) = -1. \tag{1-11}$$

Thus, the positive solutions of (1-8)–(1-11) are the positive solutions of (1-7). Further, it is clear that if $u(x)$ is a positive solution of (1-9), then $v(x) = u(1 - x)$ also satisfies (1-10). Thus, it suffices to only consider (1-8), (1-9), and (1-11).

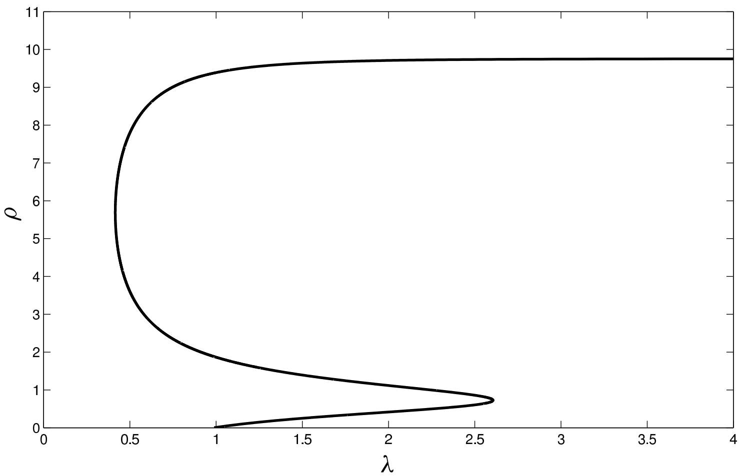

Figure 1. S-shaped bifurcation curve.

Prior studies have gathered information and analyzed the positive solutions to both strong and weak Allee problems. Additionally, the analysis of the positive solutions to the combination of grazing and the Allee effect has also been made; however, to the best of our understanding no analysis has been made in regards to the Allee effect with grazing and nonlinear boundary conditions. In the case when $\alpha(u) \equiv 0$, (1-8) has a rich history. For the logistic case with Dirichlet boundary conditions, Lee, Sasi, and Shivaji proved the existence of an S-shaped bifurcation curve in one dimension, as well as higher dimensions for a certain range of the grazing parameter [Lee et al. 2011]. Regarding the one-dimensional weak Allee effect model with Dirichlet boundary conditions, Poole, Roberson, and Stephenson showed the existence of an S-shaped bifurcation curve, resembling Figure 1, both computationally and analytically for certain parameter ranges [Poole et al. 2012]. In particular, our focus is to further examine the structure of positive solutions of (1-7) when the nonlinear boundary conditions (1-4) are satisfied for the range of the parameters where Poole et al. [2012] showed the existence of an S-shaped bifurcation curve of positive solutions. Computationally, we show the existence of Σ-shaped bifurcation curves as exemplified in Figure 2.

We employ and adapt the quadrature method first developed by Laetsch [1970] to study the structure of positive solutions of (1-7). First, some important preliminaries will be presented in Section 2, followed by a discussion of applying and adapting the quadrature method for the specific cases (1-8), (1-9), and (1-11). In Section 6, we provide the complete evolution of the bifurcation curve of positive solutions of (1-7), followed by analytical results confirming some of our observations in Section 7.

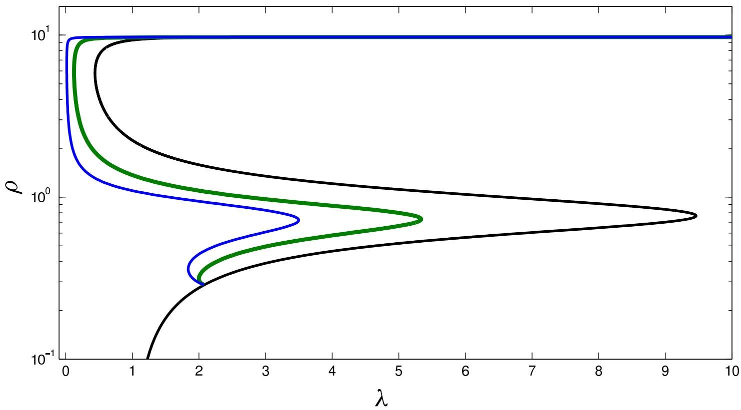

Figure 2. Σ-shaped bifurcation curves.

2. Preliminaries

We examine the combination of the weak Allee effect and grazing in the subsequent reaction term:

$$\begin{aligned} f(u) &= u(u+1)(b-u) - \frac{cu^2}{1+u^2} \quad \text{for } b>1,\ c\geq 0 \\ &= \frac{u(u+1)(b-u)(1+u^2)-cu^2}{1+u^2}. \end{aligned}$$

Through observation it is apparent that the numerator of $f(u)$ can be written as a fifth-degree polynomial. Regardless of any specific values for b and c, by analyzing the roots of $f(u)$ the existence of three roots — a negative root, a positive root, and a root at $u=0$ — can be determined. As c is varied the remaining three roots alternate between imaginary and real values. For the purpose of this paper, denote $\sigma=\sigma(b,c)$ as the smallest positive root of $f(u)$. Also, allow $\sigma_0=\sigma_0(b,c)$ and $\sigma_1=\sigma_1(b,c)$ to represent the remaining roots. Regardless of the value of $c>0$, for certain values of b, specifically $b\in(1,b_0)$ (some $b_0>0$), there exists only one positive real root of $f(u)$ represented by σ.

Remark 1. Through calculation and the use of Mathematica, it is estimated that $b_0\approx 2.852$.

Specifically, when $b\in(b_0,\infty)$, it has been determined that the shape of $f(u)$ changes when c is varied. Note when $c\in[0,c_0)$ (some $c_0=c_0(b)>0$), there exists exactly one positive real root denoted by $\sigma(b,c)$. Figure 3 depicts this case. The shape of $f(u)$ is modified as c becomes larger. Specifically, when $c\in[c_0,c_1)$ (some $c_1=c_1(b)\in(c_0,\infty)$), $f(u)$ has 3 positive real roots, namely $\sigma(b,c)$, $\sigma_0(b,c)$, and $\sigma_1(b,c)$, as depicted in Figure 4. For $c>c_1$, $f(u)$ is shifted downward resulting in

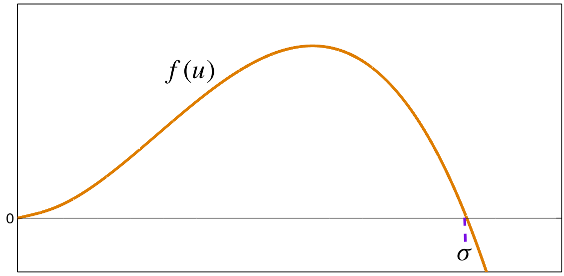

Figure 3. Graph of $f(u)$ for $b > b_0$ and $c \in [0, c_0)$.

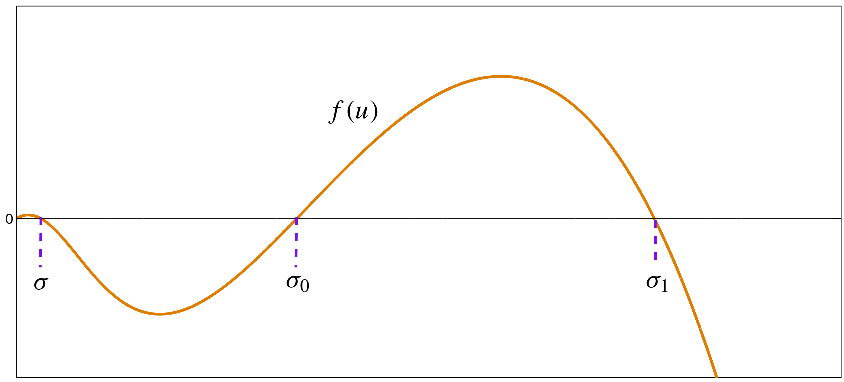

Figure 4. Graph of $f(u)$ for $b > b_0$ and $c \in [c_0, c_1)$.

exactly one positive real root $\sigma(b, c)$, meaning $\sigma_0(b, c)$ and $\sigma_1(b, c)$ are imaginary roots. This particular case is illustrated in Figure 5.

In the preceding cases the structure of the positive solutions of (1-7) varies. As our primary interest is the structure of positive solutions for the range of parameters where Poole et al. [2012] showed the existence of S-shaped bifurcation curves, we focus on the case when $c \in [0, c_0)$.

3. Quadrature method for (1-8)

For completeness, we reestablish the results obtained through the quadrature method actualized by Laetsch [1970] and Brown, Ibrahim, and Shivaji [Brown et al. 1981]. Additionally we recapitulate the subsequent boundary value problem analyzed by Poole et al. [2012] for positive solutions:

$$-u''(x) = \lambda f(u(x)); x \in (0, 1), \quad u(0) = 0, \quad u(1) = 0, \tag{3-1}$$

where $f : [0, \infty) \to (0, \infty)$ is a C^1 function. Clearly, a positive solution of (3-1) must resemble Figure 6.

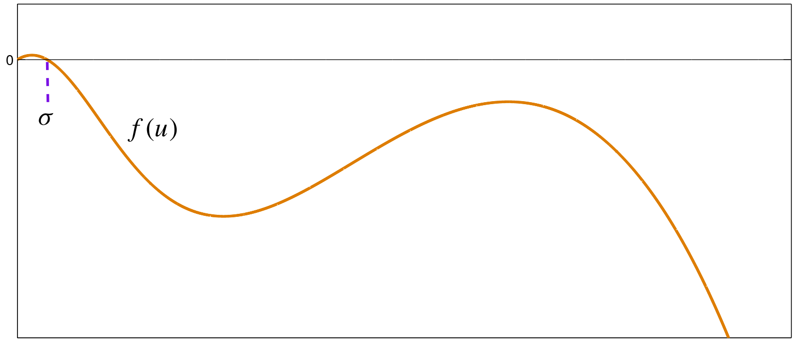

Figure 5. Graph of $f(u)$ for $b > b_0$ and $c > c_1$.

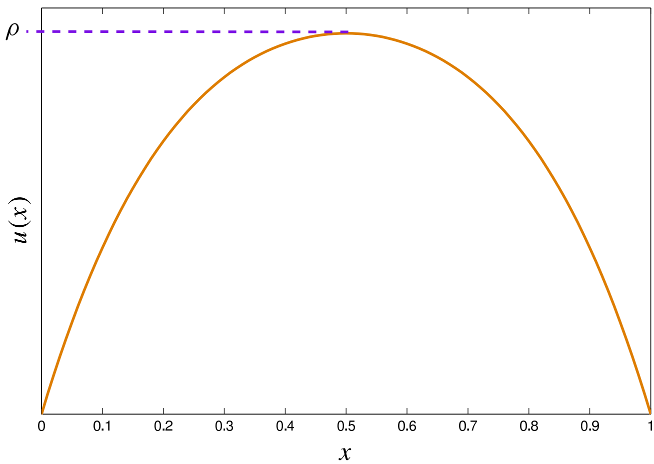

Figure 6. Graph of a typical positive solution of (3-1).

Theorem 3.1 [Brown et al. 1981; Laetsch 1970]. *Suppose $u(x)$ is a positive solution to (3-1) with $\|u\|_\infty = \rho = u(\frac{1}{2})$, where $\rho > 0$. Such a solution to (3-1) exists if and only if*

$$G_1(\rho) = \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} = \sqrt{\lambda}, \tag{3-2}$$

where $F(x) = \int_0^x f(s)\,ds$.

Proof. ($\Rightarrow$) Recognizing that (3-1) is an autonomous differential equation, we see that if u is a positive solution to (3-1) with $u'(x_0) = 0$ for a particular $x_0 \in (0, 1)$, then $m(x) = u(x_0 + x)$ and $n(x) = u(x_0 - x)$ both satiate the initial value problem

$$-k''(x) = \lambda f(k(x)), \quad k(0) = u(x_0), \quad k'(0) = 0, \tag{3-3}$$

where $x \in [0, l)$ and $l = \min\{x_0, 1 - x_0\}$. Using Picard's existence and uniqueness theorem, we have that $u(x_0 + x) \equiv u(x_0 - x)$ for all $x \in [0, l)$ and thus $u(x)$ is symmetric about $x = \frac{1}{2}$, which is notedly where $u(x)$ achieves its maximum.

By multiplying the differential equation in (3-1) by $u'(x)$, we have

$$-\left[\frac{[u'(x)]^2}{2}\right]' = \left[\lambda F(u(x))\right]'. \tag{3-4}$$

Integration of both sides of (3-4) gives

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2\lambda}; \quad x \in [0, \tfrac{1}{2}). \tag{3-5}$$

By integrating a second time and using the fact that $u(0) = 0$, we have

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{2\lambda}x; \quad x \in [0, \tfrac{1}{2}]. \tag{3-6}$$

Substituting $x = \frac{1}{2}$ and utilizing $u(\frac{1}{2}) = \rho$, (3-6) can be written as

$$G_1(\rho) = \sqrt{2}\int_0^{\rho} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{\lambda}. \tag{3-7}$$

Therefore, if $u(x)$ is a positive solution to (3-1) where $\|u\|_\infty = \rho$, then ρ must fulfill $G_1(\rho) = \sqrt{\lambda}$.

($\Leftarrow$) Assume $G_1(\rho) = \sqrt{\lambda}$ for $\rho > 0$. Now define a function $u : [0, \frac{1}{2}] \to [0, \infty)$ by

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{2\lambda}x; \quad x \in [0, \tfrac{1}{2}]. \tag{3-8}$$

We now show that $u(x)$ satisfies (3-1). Notice that $u(x)$ is well defined and via the implicit function theorem also twice differentiable. Hence, differentiating (3-8) yields

$$u'(x) = \sqrt{2\lambda\left[F(\rho) - F(u(x))\right]}.$$

By differentiating a second time we obtain

$$-u''(x) = \lambda f(u(x)).$$

In addition, it is clear that $u(0) = 0$. By defining $u(x)$ as a symmetric solution on [0, 1], it is apparent that $u(x)$ is a positive solution to (3-1) with $\|u\|_\infty = \rho$. □

It is important to discern that $G_1(\rho)$ is well defined and the improper integral is convergent. To that end, we state an important remark.

Remark 2. The improper integral in (3-7) is both well defined and convergent for ρ-values that fulfill:

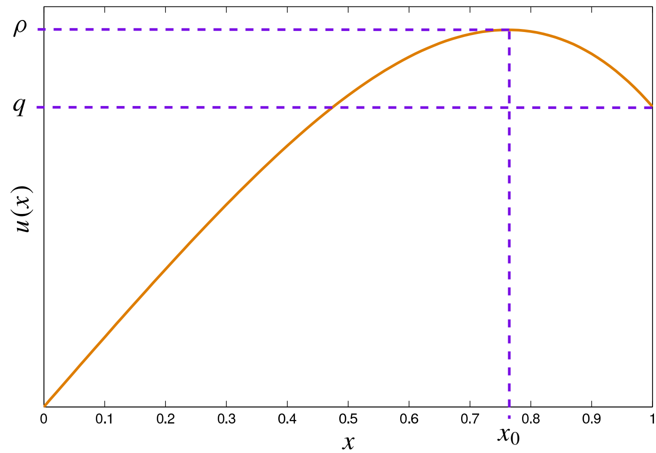

Figure 7. Graph of a typical positive solution of (4-1).

(1) $f(\rho) > 0$;

(2) $F(\rho) > F(s)$ for all $s \in [0, \rho)$.

Notice that from Figure 3 if $c \in [0, c_0)$, then both (1) and (2) will be satisfied for all $\rho \in (0, \sigma(b, c))$. We close this section by recalling an important result from Brown et al.

Theorem 3.2 [Brown et al. 1981]. *$G_1(\rho)$ is both differentiable and continuous on the defined set $T = \{\rho > 0 \mid f(\rho) > 0$ and $F(\rho) - F(s) > 0$ for all $s \in [0, \rho)\}$ where*

$$G_1'(\rho) = \sqrt{2} \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{3/2}}\, dv,$$

in which

$$H(s) = F(s) - \frac{s}{2} f(s).$$

4. Quadrature method for (1-9)

In this section, we adapt the quadrature method to analyze the structure of positive solutions of (1-9):

$$-u'' = \lambda \left[u(u+1)(b-u) - \frac{cu^2}{(u^2+1)} \right]; \ (0, 1), \quad u(0) = 0, \quad u'(1) = -1. \tag{4-1}$$

Define

$$f(u) = \left[u(u+1)(b-u) - \frac{cu^2}{(u^2+1)} \right] \quad \text{and} \quad F(x) = \int_0^x f(s)\, ds.$$

It is apparent that a positive solution of (4-1) must resemble Figure 7.

Assume $u(x)$ is a positive solution to (4-1) with $\|u\|_\infty = \rho$ and $u(1) = q$ for $q \in [0, \rho)$. By multiplying the differential equation in (4-1) by $u'(x)$ we obtain

$$-\left[\frac{[u'(x)]^2}{2}\right]' = \left[\lambda F(u(x))\right]'. \tag{4-2}$$

Integrating both sides of (4-2) yields

$$\frac{-(u'(x))^2}{2} = \lambda F(u(x)) + C. \tag{4-3}$$

Recalling that $u(x_0) = \rho$ and $u'(x_0) = 0$, (4-3) becomes

$$C = -\lambda F(\rho). \tag{4-4}$$

Similarly, using $u(1) = q$ and $u'(1) = -1$, (4-3) is utilized to determine a second value for C,

$$C = -\tfrac{1}{2} - \lambda F(q). \tag{4-5}$$

Combining (4-4) and (4-5) gives

$$\sqrt{2\lambda} = \frac{1}{\sqrt{F(\rho) - F(q)}}. \tag{4-6}$$

In utilizing the C-value from (4-4) while solving for $u'(x)$, (4-3) becomes

$$u'(x) = \sqrt{2\lambda\left[F(\rho) - F(u(x))\right]}; \qquad x \in [0, x_0], \tag{4-7}$$

$$u'(x) = -\sqrt{2\lambda\left[F(\rho) - F(u(x))\right]}; \quad x \in [x_0, 1]. \tag{4-8}$$

Rearranging (4-7) and (4-8) gives

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2\lambda}; \qquad x \in [0, x_0), \tag{4-9}$$

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = -\sqrt{2\lambda}; \quad x \in (x_0, 1]. \tag{4-10}$$

Integration of (4-9) from 0 to x and (4-10) from x_0 to x yields

$$\int_0^x \frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \int_0^x \sqrt{2\lambda}; \qquad x \in [0, x_0), \tag{4-11}$$

$$\int_{x_0}^x \frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \int_{x_0}^x -\sqrt{2\lambda}; \quad x \in (x_0, 1]. \tag{4-12}$$

Using a change of variables and recalling $u(0) = 0$ and $u(x_0) = \rho$ we obtain

$$\int_0^{u(x)} \frac{dw}{\sqrt{F(\rho) - F(w)}} = \sqrt{2\lambda}x; \qquad x \in [0, x_0], \tag{4-13}$$

$$\int_\rho^{u(x)} \frac{dw}{\sqrt{F(\rho) - F(w)}} = -\sqrt{2\lambda}(x - x_0); \quad x \in [x_0, 1]. \tag{4-14}$$

By substituting $x = x_0$ into (4-13) and $x = 1$ into (4-14) we obtain

$$\int_0^{\rho} \frac{dw}{\sqrt{F(\rho) - F(w)}} = \sqrt{2\lambda}x_o, \tag{4-15}$$

$$\int_\rho^{q} \frac{dw}{\sqrt{F(\rho) - F(w)}} = -\sqrt{2\lambda}(1 - x_0). \tag{4-16}$$

Subtracting (4-16) from (4-15) we have

$$\sqrt{2}\int_0^{\rho} \frac{dw}{\sqrt{F(\rho) - F(w)}} - \frac{1}{\sqrt{2}}\int_0^{q} \frac{dw}{\sqrt{F(\rho) - F(w)}} = \sqrt{\lambda}. \tag{4-17}$$

By synthesizing (4-6) with (4-17) we denote

$$\tilde{G}_2(\rho, q) = \sqrt{2}\int_0^{\rho} \frac{dw}{\sqrt{F(\rho)-F(w)}} - \frac{1}{\sqrt{2}}\int_0^{q} \frac{dw}{\sqrt{F(\rho)-F(w)}} - \frac{1}{\sqrt{2}\sqrt{F(\rho)-F(q)}}. \tag{4-18}$$

By Remark 2, the improper integral in $\tilde{G}_2(\rho, q)$ exists and is convergent for ρ in $(0, \sigma(b, c))$. Also, for a given $\rho \in (0, \sigma(b, c))$ Picard's existence and uniqueness theorem guarantees that the corresponding $q = u(1) \in [0, \rho)$ must be unique. If for each $\rho \in (0, \sigma(b, c))$ there exists a unique $q(\rho) \in [0, \rho)$ where $\tilde{G}_2(\rho, q(\rho)) = 0$, then there exists a unique $\lambda \in (0, \infty)$ such that

$$\sqrt{2}\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{\sqrt{2}}\int_0^{q(\rho)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \frac{1}{\sqrt{2}\sqrt{F(\rho) - F(q(\rho))}} = \sqrt{\lambda} \tag{4-19}$$

will be satisfied. Therefore it is imperative to examine the existence and uniqueness of such a $q = q(\rho)$. Hence, we recall and prove Lemma 1, adapted from [Goddard et al. 2010a], which outlines necessary properties of $\tilde{G}_2(\rho, q)$.

Lemma 1 [Goddard et al. 2010a]. *If* $\rho \in (0, \sigma(b, c))$ *then*:

(1) $\tilde{G}_2(\rho, q) \to -\infty$ *as* $q \to \rho^-$ *for fixed* $\rho \in (0, \sigma(b, c))$.

(2) $[\tilde{G}_2]_q < 0$ *for every* $q \in [0, \rho)$ *and fixed* $\rho \in (0, \sigma(b, c))$.

(3) $\tilde{G}_2(\rho, 0) \to \infty$ *when* $\rho \to \sigma(b, c)^-$.

(4) $\tilde{G}_2(\rho, 0) \to -\infty$ *when* $\rho \to 0^+$.

Proof. (1) Accomplished via the mean value theorem and the fact that $F(u)$ is an increasing function on $(0, \sigma(b, c))$.

(2) Let $\rho \in (0, \sigma(b, c))$. Thus

$$[\tilde{G}_2(\rho, q)]_q = -\frac{1}{\sqrt{2}\sqrt{F(\rho) - F(q)}} - \frac{f(q)}{2\sqrt{2}[F(\rho) - F(q)]^{\frac{3}{2}}} < 0$$

for all $q \in [0, \rho)$, since $f(s) > 0$ for $s \in (0, \sigma(b, c))$.

(3) For every $\rho \in (0, \sigma(b, c))$, we have

$$\tilde{G}_2(\rho, 0) = \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{\sqrt{2}\sqrt{F(\rho)}} = G_1(\rho) - \frac{1}{\sqrt{2}\sqrt{F(\rho)}}. \quad (4\text{-}20)$$

Laetsch [1970] showed that $G_1(\rho) \to \infty$ as $\rho \to \sigma(b, c)^-$. This implies that $\tilde{G}_2(\rho, 0) \to \infty$ when $\rho \to \sigma(b, c)^-$.

(4) Ascertained via the mean value theorem and the monotonicity of $F(u)$ on $(0, \sigma(b, c))$. □

According to Lemma 1, $\tilde{G}_2(\rho, q)$ must resemble Figure 8, whereas Figures 9 and 10 illustrate $\tilde{G}_2(\rho, 0)$. Noteworthy from Lemma 1, if $\tilde{G}_2(\rho, 0) \geq 0$ then there exists a unique $q(\rho) \in [0, \rho)$ wherefore $\tilde{G}_2(\rho, q(\rho)) = 0$. We conjecture as a result of our computations that there is a unique $\rho^* = \rho^*(b, c) > 0$ wherefore if $\rho \geq \rho^*$, then $\tilde{G}_2(\rho, 0) \geq 0$. Also if $\rho < \rho^*$ then $\tilde{G}_2(\rho, 0) < 0$. So, for all $\rho \in [\rho^*, \infty)$ there exists a unique $q = q(\rho) \in [0, \rho)$ where $\tilde{G}_2(\rho, q(\rho)) = 0$. In this case, we have

$$G_2(\rho, q(\rho)) = \frac{1}{\sqrt{2}\sqrt{F(\rho) - F(q)}} = \sqrt{\lambda}. \quad (4\text{-}21)$$

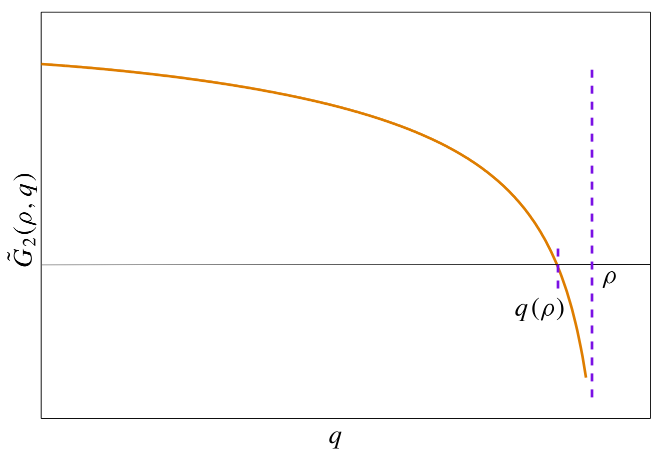

Figure 8. Graph of $\tilde{G}_2(\rho, q)$.

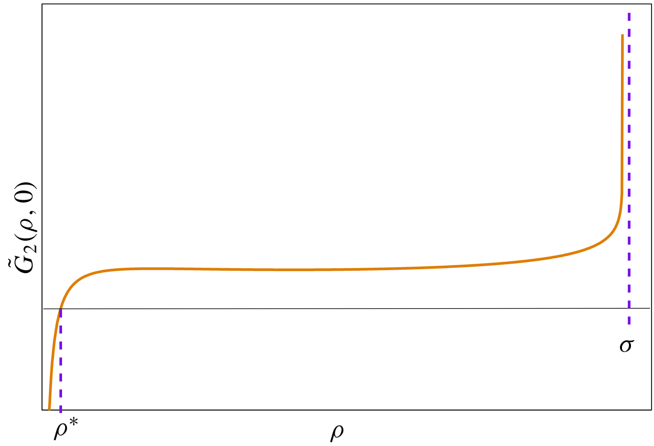

Figure 9. Graph of $\tilde{G}_2(\rho,0)$ when $b=10$ and $c=0$.

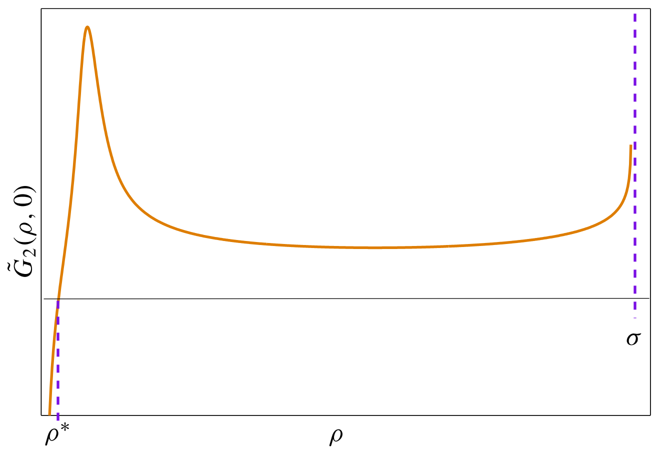

Figure 10. Graph of $\tilde{G}_2(\rho,0)$ when $b=10$ and $c=33$.

We now state and prove the main theorem of the section.

Theorem 4.1. *The function $u(x)$ is a positive solution to (4-1) with*

$$\|u\|_\infty = \rho \in S(b,c) := \big[\rho^*(b,c), \sigma(b,c)\big)$$

if and only if

$$G_2(\rho, q(\rho)) = \frac{1}{\sqrt{2}\sqrt{F(\rho)-F(q)}} = \sqrt{\lambda}$$

for a positive λ for which $q = q(\rho) \in [0,\rho)$ is the unique solution of

$$\tilde{G}_2(\rho, q(\rho)) = \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}} - \frac{1}{\sqrt{2}}\int_0^q \frac{dw}{\sqrt{F(\rho)-F(w)}} - \frac{1}{\sqrt{2}\sqrt{F(\rho)-F(q)}} = 0.$$

Proof. ($\Rightarrow$) Accomplished in the above analysis.

($\Leftarrow$) Assume that there exist $\lambda \in (0, \infty)$ and $\rho \in S(b, c)$ wherefore $G_2(\rho, q(\rho)) = \sqrt{\lambda}$ in which the unique solution of $\tilde{G}_2(\rho, q(\rho)) = 0$ is $q(\rho) \in [0, \rho)$. Define

$$u(x) : [0, 1] \to \mathbb{R}$$

via

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x; \qquad x \in [0, x_0], \tag{4-22}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2\lambda}(x - x_0); \quad x \in [x_0, 1]. \tag{4-23}$$

Now, we will exhibit $u(x)$ as a positive solution to (4-1). Note that $u(x)$ has a turning point at x_0 denoted by

$$x_0 = \frac{1}{\sqrt{2\lambda}} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{4-24}$$

For the given $\lambda > 0$, it is apparent that

$$\frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \tag{4-25}$$

is both a differentiable function of u and an increasing function ranging from 0 to x_0 when u takes on the values from 0 to ρ. Therefore, for each $x \in [0, x_0]$ there is a unique $u(x)$ wherefore

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x. \tag{4-26}$$

The implicit function theorem gives that $u(x)$ is a twice-differentiable function with respect to x. Differentiating (4-26) with respect to x gives

$$u'(x) = \sqrt{2\lambda\big[F(\rho) - F(u(x))\big]}; \quad x \in [0, x_0]. \tag{4-27}$$

Through a similar argument,

$$u'(x) = -\sqrt{2\lambda\big[F(\rho) - F(u(x))\big]}; \quad x \in [x_0, 1]. \tag{4-28}$$

By utilizing (4-27) and (4-28) we obtain

$$\frac{[u'(x)]^2}{2} = \lambda\big[F(\rho) - F(u(x))\big]; \quad x \in [0, 1]. \tag{4-29}$$

Through differentiation of (4-29) we have

$$-u''u' = \lambda f(u)u'; \quad x \in (0, 1), \tag{4-30}$$

which can be rewritten as

$$-u'' = \lambda f(u); \quad x \in (0, 1). \tag{4-31}$$

Thus, we have proved that $u(x)$ satisfies the differential equation in (4-1). Now, we show that $u(x)$ satisfies the boundary value conditions in (4-1); however, it is apparent that $u(0) = 0$. Additionally, using $G_2(\rho, q(\rho)) = \sqrt{\lambda}$, we ascertain

$$\sqrt{F(\rho) - F(q)} = \frac{1}{\sqrt{2\lambda}}. \tag{4-32}$$

Substitution of $x = 1$ in (4-28) yields

$$u'(1) = -\sqrt{2\lambda}\sqrt{F(\rho) - F(q)}. \tag{4-33}$$

When (4-32) and (4-33) are synthesized we obtain

$$u'(1) = -1. \tag{4-34}$$

Therefore, the boundary conditions in (4-1) are satisfied by $u(x)$. □

5. Quadrature method for (1-11)

Further extension of the quadrature method is performed in this section to analyze the structure of positive solutions of (1-11):

$$-u'' = \lambda\left[u(u+1)(b-u) - \frac{cu^2}{(u^2+1)}\right]; (0, 1), \quad u'(0) = 1, \quad u'(1) = -1. \tag{5-1}$$

Define

$$f(u) = \left[u(u+1)(b-u) - \frac{cu^2}{(u^2+1)}\right] \quad \text{and} \quad F(x) = \int_0^x f(s)\,ds.$$

Clearly, a positive solution of (5-1) must resemble Figure 11, where $\|u\|_\infty = \rho$, $\rho \in (0, \infty)$, $q = u(0) = u(1)$, and $q \in [0, \rho)$. Through a similar argument as in Section 4, we articulate the main theorem of this section.

Theorem 5.1. *The function $u(x)$ is a positive solution of (5-1) with*

$$\|u\|_\infty = \rho \in S(b, c) = \big[\rho^*(b, c), \sigma(b, c)\big)$$

if and only if

$$G_3(\rho, q(\rho)) = \frac{1}{\sqrt{2}\sqrt{F(\rho) - F(q)}} = \sqrt{\lambda}, \tag{5-2}$$

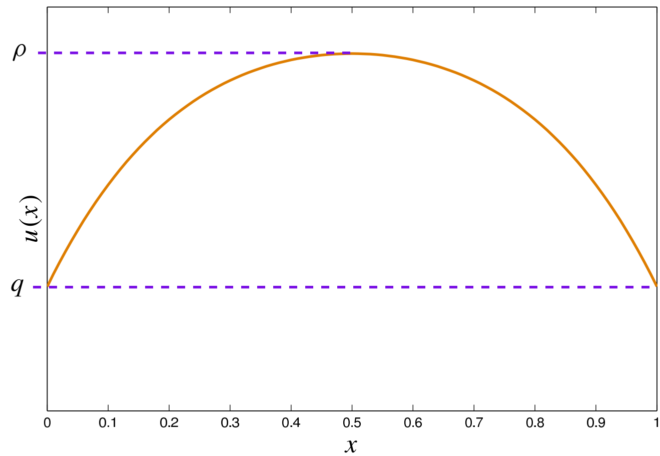

Figure 11. Graph of a typical positive solution of (5-1).

for which $q = q(\rho) \in [0, \rho)$ *is the unique solution of*

$$\tilde{G}_3(\rho, q(\rho)) = \sqrt{2}\int_0^{\rho} \frac{dw}{\sqrt{F(\rho)-F(w)}} - \sqrt{2}\int_0^{q} \frac{dw}{\sqrt{F(\rho)-F(w)}} - \frac{1}{\sqrt{2}\sqrt{F(\rho)-F(q)}} = 0.$$

6. Computational results

Within this section we exhibit the complete evolution of the bifurcation curve of positive solutions of (1-7) for $c \in [0, c_0(b))$. The results for (1-8) are reestablished via Mathematica computations and by recalling Theorem 3.1. For (1-9) and (1-11), we recall Theorems 4.1 and 5.1 and utilize a standard root-finding algorithm to find the unique $\rho^*(b, c) > 0$. Then for $\rho \in [\rho^*(b, c), \sigma(b, c))$ we employ a root-finding algorithm to find the corresponding unique $q(\rho)$, which is delineated in Theorems 4.1 and 5.1. These diagrams were acquired via Mathematica for a single b-value as c-values are varied. If $b \in (b_0, \infty)$ then there exist

$$0 < c_0^* < c_1^* < c_2^* < c_3^* < c_4^* < c_5^* < c_6^* < c_7^* < c_0(b)$$

such that we have the following cases. In the subsequent figures, (1-8) is represented in black, cases (1-9) and (1-10) in red, and (1-11) in blue.

Case 1. If $c \in [0, c_0^*)$ then there exist $\lambda_i > 0$ for $i = 1, 2, 3, 4$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in [\lambda_0, \infty)$, then (1-7) has exactly 4 positive solutions;

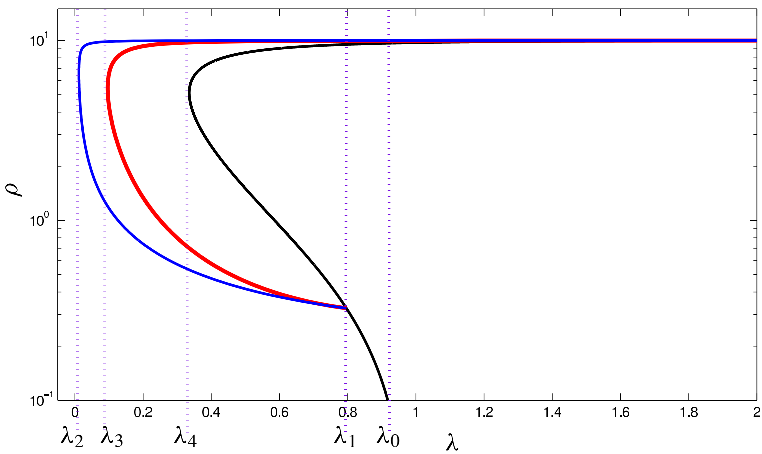

Figure 12. ρ versus λ when $b = 10$ and $c = 0$ (Case 1).

- $\lambda \in (\lambda_1, \lambda_0)$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_1]$, then (1-7) has exactly 8 positive solutions.

Figure 12 illustrates Case 1.

Case 2. If $c \in [c_0, c_1)$ (some $c_1(b) > 0$) then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 6$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in [\lambda_0, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda \in (\lambda_1, \lambda_5)$ and $\lambda \in (\lambda_6, \lambda_0)$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, $\lambda = \lambda_5$, and $\lambda = \lambda_6$, then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$ and $\lambda \in (\lambda_5, \lambda_6)$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_1]$, then (1-7) has exactly 8 positive solutions.

Figure 13 illustrates Case 2.

Case 3. If $c = c_1$ then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 5$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;

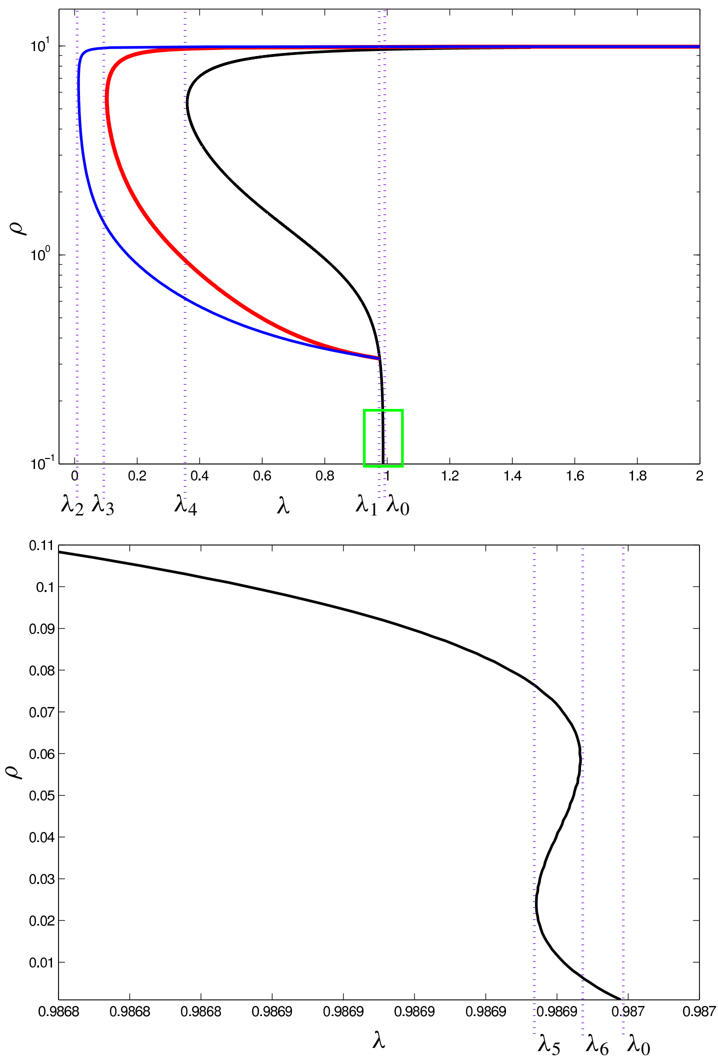

Figure 13. ρ versus λ (top) and cross-section (bottom) for $b = 10$ and $c = 8.97$ (Case 2).

- $\lambda = \lambda_3$ and $\lambda \in [\lambda_0, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda \in (\lambda_1, \lambda_5)$ and $\lambda = \lambda_0$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, $\lambda = \lambda_5$, then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$ and $\lambda \in (\lambda_5, \lambda_0)$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_1]$, then (1-7) has exactly 8 positive solutions.

Figure 14 illustrates Case 3.

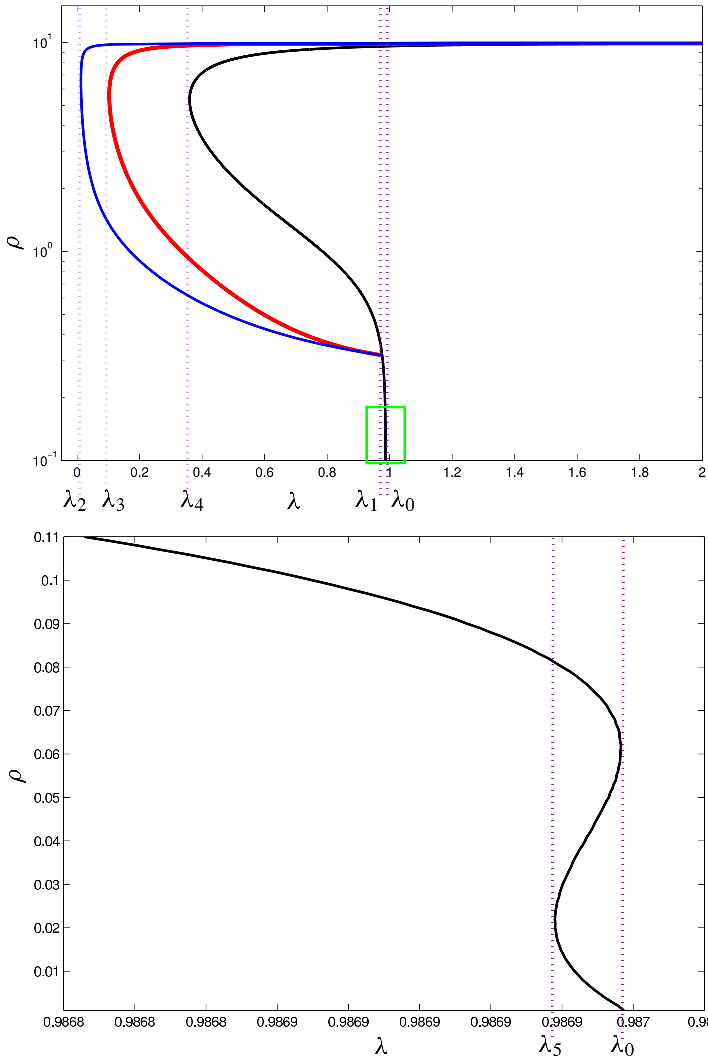

Figure 14. ρ versus λ (top) and cross-section (bottom) for $b = 10$ and $c = 8.972$ (Case 3).

Case 4. If $c \in (c_1, c_2)$ (some $c_2(b) > 0$) then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 6$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_6, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda \in (\lambda_1, \lambda_5)$ and $\lambda = \lambda_6$, then (1-7) has exactly 5 positive solutions;

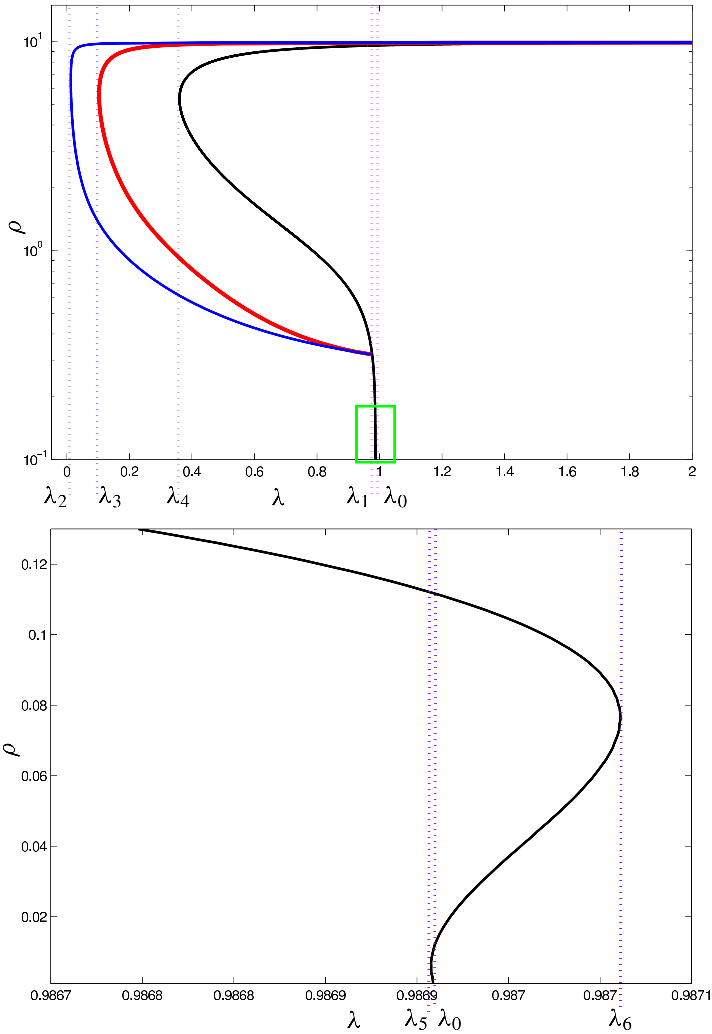

Figure 15. ρ versus λ (top) and cross-section (bottom) for $b = 10$ and $c = 8.99$ (Case 4).

- $\lambda \in (\lambda_3, \lambda_4)$, $\lambda \in [\lambda_0, \lambda_6)$, and $\lambda = \lambda_5$, then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$ and $\lambda \in (\lambda_5, \lambda_0)$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_1]$, then (1-7) has exactly 8 positive solutions.

Figure 15 illustrates Case 4.

Case 5. If $c = c_2$ then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 5$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;

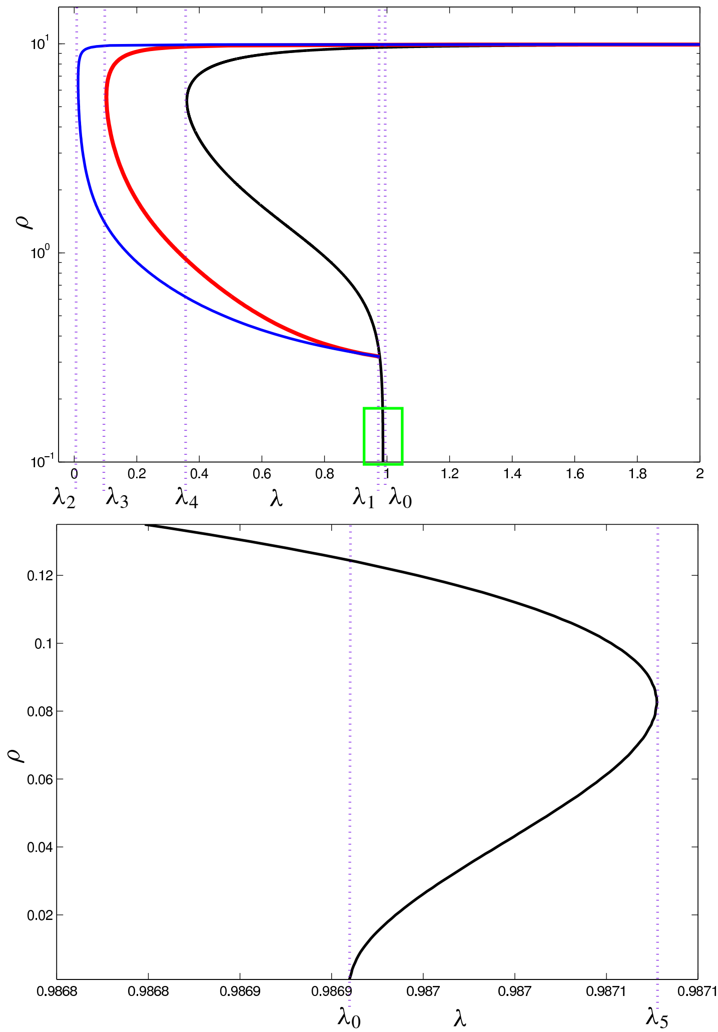

Figure 16. ρ versus λ (top) and cross-section (bottom) for $b = 10$ and $c = 9$ (Case 5).

- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_5, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda \in (\lambda_1, \lambda_0]$ and $\lambda = \lambda_5$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$ and $\lambda \in (\lambda_0, \lambda_5)$, then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_1]$, then (1-7) has exactly 8 positive solutions.

Figure 16 illustrates Case 5.

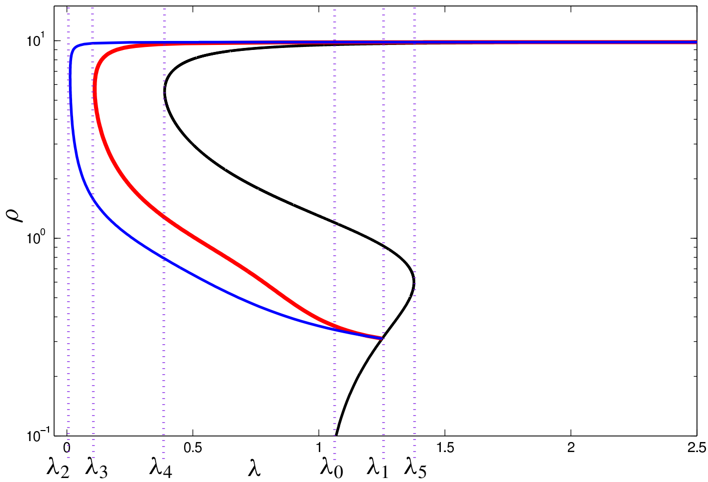

Figure 17. ρ versus λ when $b = 10$ and $c = 18$ (Case 6).

Case 6. If $c \in (c_2, c_3]$ (some $c_3(b) > 0$) then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 5$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_5, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_5$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_1, λ_5), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_1]$, then (1-7) has exactly 9 positive solutions.

Figure 17 illustrates Case 6.

Case 7. If $c \in (c_3, c_4)$ (some $c_4 > 0$) then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 7$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_7, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_7$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_1, λ_7), then (1-7) has exactly 6 positive solutions;

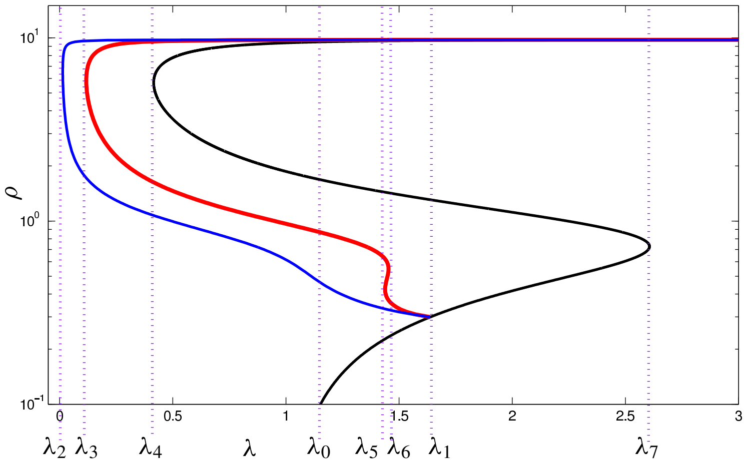

Figure 18. ρ versus λ when $b = 10$ and $c = 26$ (Case 7).

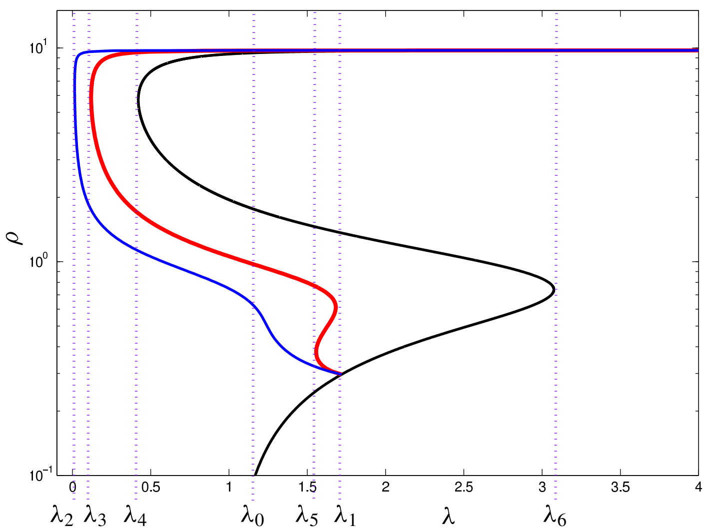

Figure 19. ρ versus λ when $b = 10$ and $c = 27.3$ (Case 8).

- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, $(\lambda_6, \lambda_1]$, then (1-7) has exactly 9 positive solutions;
- $\lambda = \lambda_5$, $\lambda = \lambda_6$, then (1-7) has exactly 11 positive solutions;
- $\lambda \in (\lambda_5, \lambda_6)$, then (1-7) has exactly 13 positive solutions.

Figure 18 illustrates Case 7. Notice that the red curve has become Σ-shaped and this shape persists through $c \leq c_0(b)$.

Case 8. If $c = c_4$ then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 6$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_6, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_6$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_1, λ_6), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, then (1-7) has exactly 9 positive solutions;
- $\lambda = \lambda_1$, $\lambda = \lambda_5$, then (1-7) has exactly 11 positive solutions;
- $\lambda \in (\lambda_5, \lambda_1)$, then (1-7) has exactly 13 positive solutions.

Figure 19 illustrates Case 8.

Case 9. If $c \in (c_4, c_5]$ (some $c_5(b) > 0$), then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 7$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_7, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_7$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_6, λ_7), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, $\lambda = \lambda_6$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, then (1-7) has exactly 9 positive solutions;
- $\lambda \in (\lambda_1, \lambda_6)$, then (1-7) has exactly 10 positive solutions;
- $\lambda = \lambda_5$, then (1-7) has exactly 11 positive solutions;
- $\lambda \in (\lambda_5, \lambda_1]$, then (1-7) has exactly 13 positive solutions.

Figure 20 illustrates Case 9.

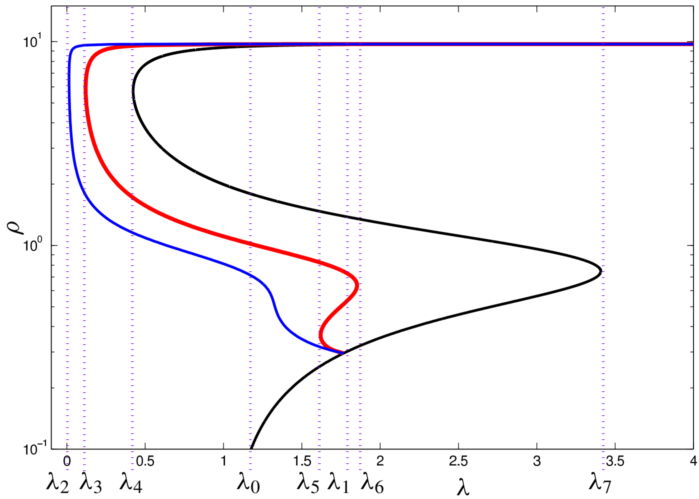

Figure 20. ρ versus λ when $b = 10$ and $c = 28$ (Case 9).

Case 10. If $c \in (c_5, c_6)$ (some $c_6(b) > 0$) then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 9$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_9, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_9$, then (1-7) has exactly 5 positive solutions;

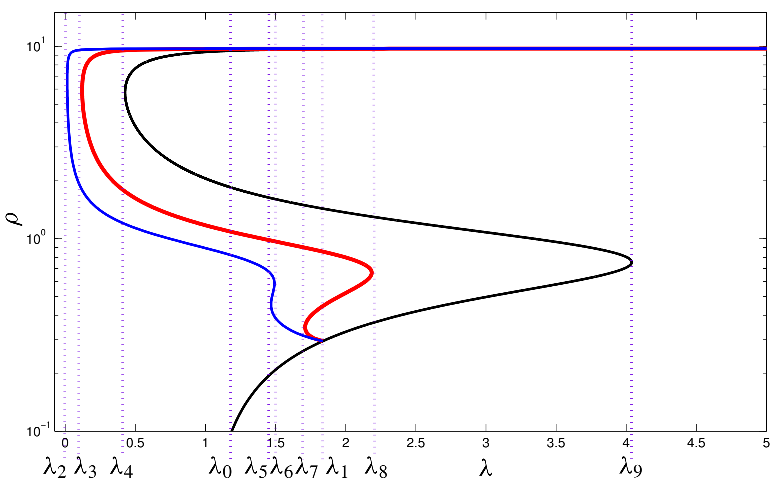

Figure 21. ρ versus λ when $b = 10$ and $c = 29$ (Case 10).

- $\lambda \in (\lambda_3, \lambda_4)$, (λ_8, λ_9), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, $\lambda = \lambda_8$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, (λ_6, λ_7), then (1-7) has exactly 9 positive solutions;
- $\lambda = \lambda_5$, $\lambda = \lambda_6$, $\lambda \in (\lambda_1, \lambda_8)$, then (1-7) has exactly 10 positive solutions;
- $\lambda \in (\lambda_5, \lambda_6)$, $\lambda = \lambda_7$, then (1-7) has exactly 11 positive solutions;
- $\lambda \in (\lambda_7, \lambda_1]$, then (1-7) has exactly 13 positive solutions.

Figure 21 illustrates Case 10. Notice that the blue curve has now also become Σ-shaped and its shape persists through $c \le c_0(b)$.

Case 11. If $c = c_6$ then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 8$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_8, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_8$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_7, λ_8), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, $\lambda = \lambda_7$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, then (1-7) has exactly 9 positive solutions;

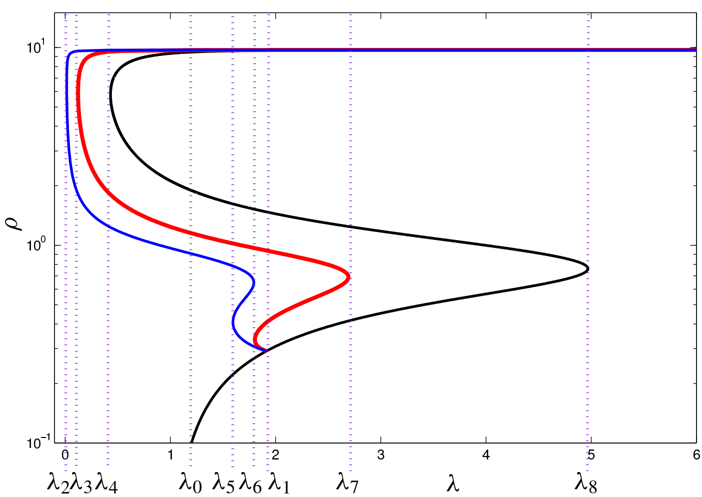

Figure 22. ρ versus λ when $b = 10$ and $c = 30$ (Case 11).

- $\lambda = \lambda_5$, $\lambda \in (\lambda_1, \lambda_7)$, then (1-7) has exactly 10 positive solutions;
- $\lambda \in (\lambda_5, \lambda_6)$, then (1-7) has exactly 11 positive solutions;
- $\lambda = \lambda_6$, then (1-7) has exactly 12 positive solutions;
- $\lambda \in (\lambda_6, \lambda_1]$, then (1-7) has exactly 13 positive solutions.

Figure 22 illustrates Case 11.

Case 12. If $c \in (c_6, c_7)$ (some $c_7(b) > 0$) then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 9$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_9, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_9$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_8, λ_9), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, $\lambda = \lambda_8$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, then (1-7) has exactly 9 positive solutions;
- $\lambda = \lambda_5$, $\lambda \in (\lambda_1, \lambda_8)$, then (1-7) has exactly 10 positive solutions;
- $\lambda \in (\lambda_5, \lambda_6)$, then (1-7) has exactly 11 positive solutions;
- $\lambda = \lambda_6$, $\lambda \in (\lambda_7, \lambda_1]$, then (1-7) has exactly 13 positive solutions;

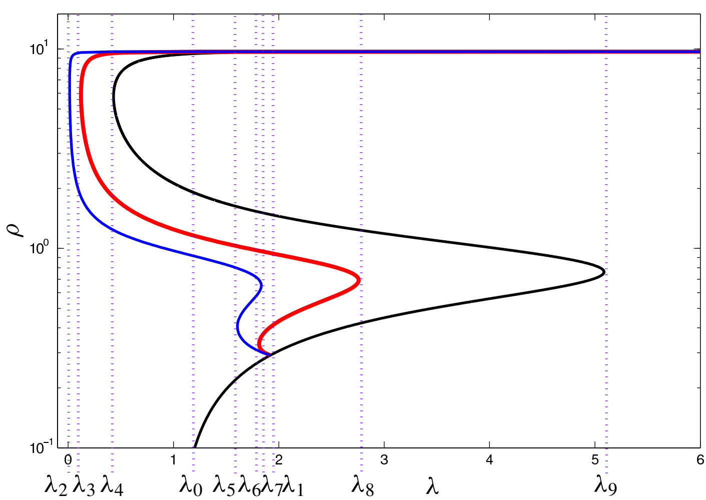

Figure 23. ρ versus λ when $b = 10$ and $c = 30.1$ (Case 12).

- $\lambda = \lambda_7$, then (1-7) has exactly 14 positive solutions;
- $\lambda \in (\lambda_6, \lambda_7)$, then (1-7) has exactly 15 positive solutions.

Figure 23 illustrates Case 12.

Case 13. If $c = c_7$ then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 8$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_8, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_8$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_7, λ_8), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, $\lambda = \lambda_7$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, then (1-7) has exactly 9 positive solutions;
- $\lambda = \lambda_5$, $\lambda \in (\lambda_1, \lambda_7)$, then (1-7) has exactly 10 positive solutions;
- $\lambda \in (\lambda_5, \lambda_6)$, then (1-7) has exactly 11 positive solutions;
- $\lambda = \lambda_6$, then (1-7) has exactly 13 positive solutions;
- $\lambda = \lambda_1$, then (1-7) has exactly 14 positive solutions;
- $\lambda \in (\lambda_6, \lambda_1)$, then (1-7) has exactly 15 positive solutions.

Figure 24 illustrates Case 13.

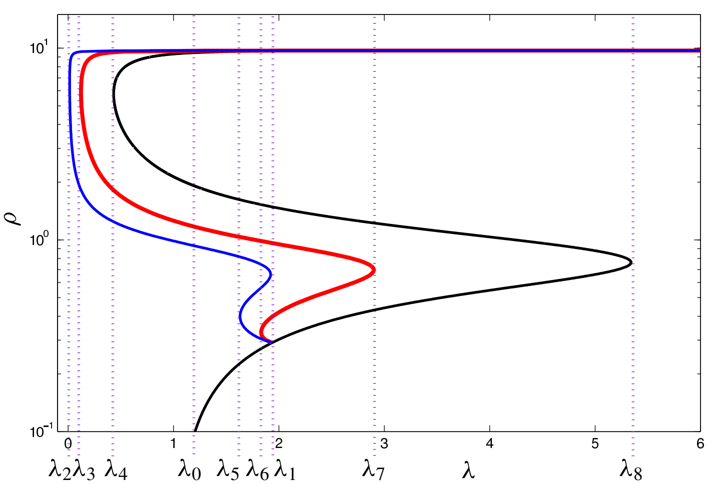

Figure 24. ρ versus λ when $b = 10$ and $c = 30.3$ (Case 13).

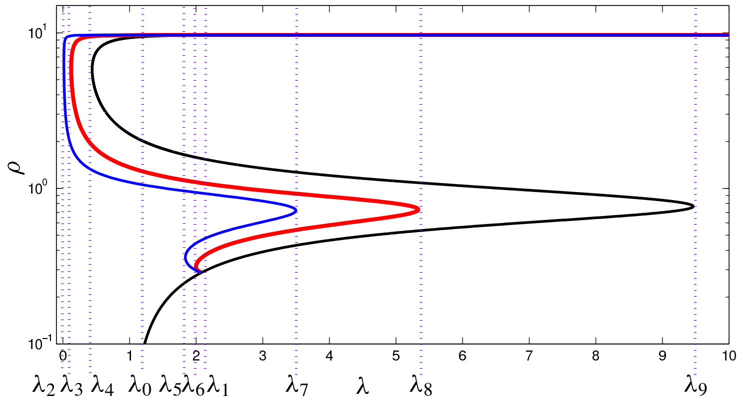

Figure 25. ρ versus λ when $b = 10$ and $c = 32$ (Case 14).

Case 14. If $c \in (c_7, c_0(b))$ then there exist $\lambda_i > 0$ for $i = 1, 2, \ldots, 9$ such that if

- $\lambda \in [0, \lambda_2)$, then (1-7) has no positive solution;
- $\lambda = \lambda_2$, then (1-7) has a unique positive solution;
- $\lambda \in (\lambda_2, \lambda_3)$, then (1-7) has exactly 2 positive solutions;
- $\lambda = \lambda_3$ and $\lambda \in (\lambda_9, \infty)$, then (1-7) has exactly 4 positive solutions;
- $\lambda = \lambda_9$, then (1-7) has exactly 5 positive solutions;
- $\lambda \in (\lambda_3, \lambda_4)$, (λ_8, λ_9), then (1-7) has exactly 6 positive solutions;
- $\lambda = \lambda_4$, then (1-7) has exactly 7 positive solutions;
- $\lambda \in (\lambda_4, \lambda_0]$, $\lambda = \lambda_8$, then (1-7) has exactly 8 positive solutions;
- $\lambda \in (\lambda_0, \lambda_5)$, then (1-7) has exactly 9 positive solutions;
- $\lambda = \lambda_5$, $\lambda \in (\lambda_7, \lambda_8)$, then (1-7) has exactly 10 positive solutions;
- $\lambda \in (\lambda_5, \lambda_6)$, $\lambda = \lambda_7$, then (1-7) has exactly 11 positive solutions;
- $\lambda \in (\lambda_1, \lambda_7)$, then (1-7) has exactly 12 positive solutions;
- $\lambda = \lambda_6$, then (1-7) has exactly 13 positive solutions;
- $\lambda \in (\lambda_6, \lambda_1]$, then (1-7) has exactly 15 positive solutions;

Figure 25 illustrates Case 14.

7. Analytical results

In order to bolster our computational results as well as elaborate on the behavior of the bifurcation curves, we procure some analytical results. First, we recall some results from [Laetsch 1970] detailing the behavior of $G_1(\rho)$ when $\rho \to \sigma(b, c)^-$

and when $\rho \to 0^+$ in the following lemmas, where $\sigma(b, c)$ represents the smallest positive root of $f(u)$.

Lemma 2 [Laetsch 1970]. $\lim_{\rho\to\sigma(b,c)^-} G_1(\rho) = \infty$.

Lemma 3 [Laetsch 1970]. $\lim_{\rho\to 0^+} G_1(\rho) = \pi/(2\sqrt{b})$.

Our main goal for this section is to establish the following analytical results for (1-9) and (1-11). Recall that $\lambda = [G_2(\rho, q)]^2$ and $\lambda = [G_3(\rho, q)]^2$ from Theorems 4.1 and 5.1, respectively. Thus, we can obtain some global behavior of the ρ versus λ bifurcation curve via study of $G_2(\rho, q)$ and $G_3(\rho, q)$.

Theorem 7.1.

(1) $\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}} \le G_2(\rho, q) \le \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}}$;

(2) $$G_3(\rho, q) \le \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}}.$$

Proof. To prove (1), recall

$$\begin{aligned} G_2(\rho, q) &= \frac{1}{\sqrt{2}\sqrt{F(\rho)-F(q)}} \\ &= \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}} - \frac{1}{\sqrt{2}}\int_0^q \frac{dw}{\sqrt{F(\rho)-F(w)}}. \end{aligned} \tag{7-1}$$

We ascertain an upper bound by substituting $q = 0$ into (7-1) yielding

$$G_2(\rho, q) \le \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}}.$$

Also, recalling $q \in [0, \rho)$ and allowing $q \to \rho^-$ in (7-1) we obtain

$$G_2(\rho, q) \ge \left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}}$$

as the lower bound. Hence,

$$\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}} \le G_2(\rho, q) \le \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}}.$$

Now to prove (2). Recall

$$G_3(\rho, q) = \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}} - \sqrt{2}\int_0^q \frac{dw}{\sqrt{F(\rho)-F(w)}}. \tag{7-2}$$

Similarly, by substituting $q = 0$ into (7-2) we obtain

$$G_3(\rho, q) \le \sqrt{2}\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}}$$

as the upper bound. □

Theorem 7.2. $$\lim_{\rho\to\sigma(b,c)^-} G_2(\rho,q)=\infty.$$

Proof. By Theorem 7.1, we have

$$G_2(\rho,q)\geq\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\int_0^\rho \frac{dw}{\sqrt{F(\rho)-F(w)}}. \tag{7-3}$$

From Lemma 2, it is clear that the right side of (7-3) approaches infinity as

$$\rho\to\sigma(b,c)^-.$$

Therefore, it is apparent that

$$\lim_{\rho\to\sigma(b,c)^-} G_2(\rho,q)=\infty. \qquad \square$$

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