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Fermat hypersurface

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This paper studies the existence of free and very free curves on the degree 5 Fermat hypersurface in \mathbb{P}^5 over an algebraically closed field of characteristic 2. We explicitly compute a free curve in degree 8, and a very free curve in degree 9. We also prove that free and very free curves cannot exist in lower degrees.

1. Introduction

Any smooth projective Fano variety in characteristic zero is rationally connected and hence contains a very free rational curve. In positive characteristic a smooth projective Fano variety is rationally chain-connected. However, it is not known whether such varieties are separably rationally connected, or equivalently, whether they have a very free rational curve. This is an open question even for nonsingular Fano hypersurfaces. See [Kollár 1996], as well as [Debarre 2001].

Following [Shen 2012], we consider the degree 5 Fermat hypersurface

$$X : X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 = 0$$

in \mathbb{P}^5 over an algebraically closed field k of characteristic 2. This is a nonsingular projective Fano variety.

Theorem 1.1. *Any free rational curve $\varphi : \mathbb{P}^1 \rightarrow X$ has degree ≥ 8 , and there exists a free rational curve of degree 8. Any very free rational curve $\varphi : \mathbb{P}^1 \rightarrow X$ has degree ≥ 9 , and there exists a very free rational curve of degree 9.*

This result, although perhaps expected, is interesting for several reasons. First, it is known that X is unirational; see [Debarre 2001, p. 52] (the corresponding rational map $\mathbb{P}^4 \dashrightarrow X$ is inseparable). Second, in [Beauville 1990], it is shown that

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every nonsingular hyperplane section of X is isomorphic to a Fermat hypersurface of dimension 3, and this property characterizes Fermat hypersurfaces among all hypersurfaces of degree 5 in characteristic 2. We believe that these facts single out the Fermat as a likely candidate for a counterexample to the conjecture below; instead, our theorem shows that they are evidence for it.

Conjecture 1.2. Nonsingular Fano hypersurfaces have very free rational curves.

Zhu [2011] discusses this question more broadly. Let us discuss a little bit about the method of proof. In Section 2, we translate the geometric question into an algebraic question which is computationally more accessible. In Sections 3, 4, and 5, we exclude low-degree solutions by theoretical methods. Finally, in Sections 6 and 7, we explicitly describe some curves which are free and very free in degrees 8 and 9, respectively.

2. The overall setup

In the rest of this paper, k will be an algebraically closed field of characteristic 2 and X will be the Fermat hypersurface of degree 5 over k . Let $\varphi : \mathbb{P}^1 \rightarrow X$ be a nonconstant morphism. We will repeatedly use that every vector bundle on \mathbb{P}^1 is a direct sum of line bundles; see [Grothendieck 1957]. Thus we can choose a splitting

$$\varphi^*T_X = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(a_4).$$

Recall that φ is said to be a *free curve* on X if $a_i \geq 0$, and φ is said to be *very free* if $a_i > 0$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_X & \xlongequal{\quad} & \mathcal{O}_X & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & E_X & \longrightarrow & \mathcal{O}_X(1)^{\oplus 6} & \longrightarrow & \mathcal{O}_X(5) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbb{P}^5}|_X & \longrightarrow & N_{X/\mathbb{P}^5} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array} \tag{2-1}$$

with exact rows and columns as indicated. We will call E_X the *extended tangent bundle* of X . The left vertical exact sequence determines a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \varphi^*E_X \rightarrow \varphi^*T_X \rightarrow 0.$$

The splitting type of $\varphi^* E_X$ will consistently be denoted $(f_1, f_2, f_3, f_4, f_5)$ in this paper. Since $\text{Hom}_{\mathbb{P}^1}(\mathbb{O}_{\mathbb{P}^1}(f), \mathbb{O}_{\mathbb{P}^1}(a)) = 0$ if $f > a$, we conclude:

- (1) If $f_i \geq 0$ for all i , then φ is free.
- (2) If $f_i > 0$ for all i , then φ is very free.

For the converse, note that the map $\mathbb{O}_{\mathbb{P}^1} \rightarrow \varphi^* E_X$ has image contained in the direct sum of the summands with $f_i \geq 0$. Hence, if $f_i < 0$ for some i , then φ is not free. Finally, suppose that $f_i \geq 0$ for all i . If there are at least two f_i equal to 0, then we see that φ is free but not very free. We conclude:

- (3) If φ is free, then $f_i \geq 0$ for all i .
- (4) If φ is very free, then either
 - (a) $f_i > 0$ for all i , or
 - (b) exactly one f_i vanishes and all others are positive.

We do not know if (4b) occurs.

Translation into algebra. Here we work over the graded k -algebra $R = k[S, T]$. As usual, we let $R(e)$ be the graded free R -module whose underlying module is R with grading given by $R(e)_n = R_{e+n}$. A *graded free R -module* will be any graded R -module isomorphic to a finite direct sum of $R(e)$'s. Such a module M has a *splitting type* which is uniquely defined up to reordering, namely, the sequence of integers u_1, \dots, u_r such that $M \cong R(u_1) \oplus \dots \oplus R(u_r)$.

We will think of a degree d morphism $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^5$ as a 6-tuple (G_0, \dots, G_5) of homogeneous elements in R of degree d with no common factors. Then φ is a morphism into X if and only if $G_0^5 + \dots + G_5^5 = 0$. In this situation we define two graded R -modules. The first is called the *pullback of the cotangent bundle*

$$\Omega_X(\varphi) = \text{Ker}(\tilde{\varphi} : R^{\oplus 6}(-d) \rightarrow R),$$

where the map $\tilde{\varphi}$ is given by $(A_0, \dots, A_5) \mapsto \sum A_i G_i$. The second is called the *pullback of the extended tangent bundle*

$$E_X(\varphi) = \text{Ker}(R^{\oplus 6}(d) \rightarrow R(5d)),$$

where the map is given by $(A_0, \dots, A_5) \mapsto \sum A_i G_i^4$. Since the kernel of a map of graded free R -modules is a graded free R -module, both $\Omega_X(\varphi)$ and $E_X(\varphi)$ are themselves graded free R -modules of rank 5.

Lemma 2.1. *The splitting type of $\varphi^* E_X$ is equal to the splitting type of the R -module $E_X(\varphi)$.*

Proof. Recall that $\mathbb{P}^1 = \text{Proj}(R)$. Thus, a finitely generated graded R -module corresponds to a coherent sheaf on \mathbb{P}^1 ; see [Hartshorne 1977, Proposition 5.11]. Under this correspondence, the module $R(e)$ corresponds to $\mathbb{O}_{\mathbb{P}^1}(e)$. The lemma follows

if we show that φ^*E_X is the coherent sheaf associated to $E_X(\varphi)$. Diagram (2-1) shows that φ^*E_X is the kernel of a map $\mathbb{O}_{\mathbb{P}^1}(d)^{\oplus 6} \rightarrow \mathbb{O}_{\mathbb{P}^1}$ given by substituting (G_0, \dots, G_5) into the partial derivatives of the polynomial defining X . Since the equation is $X_0^5 + \dots + X_5^5$, the derivatives are X_i^4 , and substituting we obtain G_i^4 as desired. \square

3. Relating the splitting types

Observe that $\Omega_X(\varphi)$ is also a graded free module of rank 5 and so has a splitting type, which we denote using e_1, \dots, e_5 . In this section, we relate the splitting type of $\Omega_X(\varphi)$ to the splitting type of $E_X(\varphi)$.

If $(A_0, \dots, A_5) \in \Omega_X(\varphi)$, then $A_0G_0 + \dots + A_5G_5 = 0$ so that

$$A_0^4G_0^4 + \dots + A_5^4G_5^4 = 0$$

by the Frobenius endomorphism in characteristic 2. Let

$$\mathcal{T} = \{(A_0^4, \dots, A_5^4) \mid (A_0, \dots, A_5) \in \Omega_X(\varphi)\}$$

in $E_X(\varphi)$. We denote the R -module generated by \mathcal{T} as $R\langle\mathcal{T}\rangle$.

Lemma 3.1. *In the notation above, $E_X(\varphi) = R\langle\mathcal{T}\rangle$.*

Proof. Let (B_0, \dots, B_5) be an element of $E_X(\varphi)$, where B_i is a homogeneous polynomial of degree b . We consider the case $b \equiv 0 \pmod 4$.

Observe that we can rewrite each monomial term of B_i as $(c^{1/4}S^\ell T^k)^4 S^i T^{4-i}$ or $(c^{1/4}S^\ell T^k)^4$ for some integers ℓ, k , where $c \in k$ and $0 < i < 4$. After collecting terms and applying the Frobenius endomorphism, we obtain

$$B_i = a_{i1}^4 + a_{i2}^4 S^3 T + a_{i3}^4 S^2 T^2 + a_{i4}^4 S T^3,$$

where each a_{ij} is an element of R . Then, since $B_0G_0 + \dots + B_5G_5 = 0$, substituting our expression for the B_i 's and applying Frobenius, we obtain

$$\left(\sum_{i=0}^5 a_{i1}G_i\right)^4 + \left(\sum_{i=0}^5 a_{i2}G_i\right)^4 S^3 T + \left(\sum_{i=0}^5 a_{i3}G_i\right)^4 S^2 T^2 + \left(\sum_{i=0}^5 a_{i4}G_i\right)^4 S T^3 = 0.$$

The sums $\sum_{i=0}^5 a_{ij}G_i$ are each themselves homogeneous polynomials. But since the degree of T in each term above is distinct modulo 4, the equation $\sum_{i=0}^5 a_{ij}G_i = 0$ implies that $(a_{0j}, \dots, a_{5j}) \in \Omega_X(\varphi)$ so that $(a_{0j}^4, \dots, a_{5j}^4) \in \mathcal{T}$ for $1 \leq j \leq 4$.

Hence, every homogeneous element of $E_X(\varphi)$ is contained in the submodule generated by \mathcal{T} . Since the reverse containment is trivial, it follows that $E_X(\varphi) = R\langle\mathcal{T}\rangle$. The cases for $b \equiv 1, 2, 3 \pmod 4$ follow similarly. \square

Proposition 3.2. *If $x_i = (x_{i0}, \dots, x_{i5})$, for $1 \leq i \leq 5$, form a basis for $\Omega_X(\varphi)$, then $y_i = (x_{i0}^4, \dots, x_{i5}^4)$, for $1 \leq i \leq 5$, form a basis for $E_X(\varphi)$.*

Proof. If $x_i \in \Omega_X(\varphi)$, then $y_i \in \mathcal{T}$, and every element of \mathcal{T} is an R -linear combination of the y_i 's. Since $E_X(\varphi) = R\langle \mathcal{T} \rangle$, every element of $E_X(\varphi)$ is also an R -linear combination of the y_i 's so that the y_i 's generate $E_X(\varphi)$. Moreover, $E_X(\varphi)$ is a free module of rank 5 over a domain, so the generators y_i for $E_X(\varphi)$ must also be linearly independent and hence form a basis. \square

Accounting for twist, a simple computation using the results above gives us the following.

Corollary 3.3. *Let φ be a degree d morphism, and e_1, \dots, e_5 be the splitting type of $\Omega_X(\varphi)$. If $f_1 = 4e_1 + 5d, f_2 = 4e_2 + 5d, \dots, f_5 = 4e_5 + 5d$, then f_1, \dots, f_5 is the splitting type of $E_X(\varphi)$.*

4. Numerology

We now utilize some facts about graded free modules in order to give constraints on potential splitting types. Given a graded free module

$$M = R(u_1) \oplus \dots \oplus R(u_r),$$

one can observe that the Hilbert polynomial H_M is given by

$$H_M(m) = rm + u_1 + \dots + u_r + r.$$

Let φ denote a free morphism of degree d into X . Noting that the map

$$\tilde{\varphi} : R(-d)_m^{\oplus n+1} \rightarrow R_m$$

is surjective for $m \gg 0$, we obtain

$$\begin{aligned} H_{\Omega(\varphi)}(m) &= \dim_k(\ker(R(-d)_m^{\oplus n+1} \rightarrow R_m)) \\ &= (n+1)(-d+m+1) - (m+1) \\ &= nm + -d(n+1) + n. \end{aligned}$$

A similar calculation shows that

$$H_{E_X(\varphi)}(m) = nm + d(n+1-5) + n.$$

We continue to refer to the splitting type components of $\Omega(\varphi)$ and $E_X(\varphi)$ as e_i and f_i , respectively. In both cases $n = r = 5$, so combining these two equations with the general form for the Hilbert polynomial of a graded free module, we obtain our first constraints:

$$\begin{aligned} e_1 + e_2 + e_3 + e_4 + e_5 &= -6d, \\ f_1 + f_2 + f_3 + f_4 + f_5 &= d. \end{aligned}$$

Recall from [Section 2](#) that a curve is free or very free if $f_i \geq 0$ or $f_i > 0$, respectively, for each i . Since $f_i = 4e_i + 5d$, it follows that

$$e_i \geq -\frac{5d}{4},$$

where strict inequality implies the curve is very free. With these two bounds, we can quickly observe a few facts about curves of different degrees.

Remarks. (1) There exist no free curves in degrees 1, 2, 3, 6, and 7.

(2) Any free curve of degree not divisible by 4 must be very free.

(3) There are no very free curves in degrees 4 or 8.

(4) The splitting type of $\Omega(\varphi)$ of a free curve of degree 4 must be

$$(-5, -5, -5, -5, -4).$$

(5) The splitting type of $\Omega(\varphi)$ of a very free curve of degree 5 must be

$$(-6, -6, -6, -6, -6).$$

All of these observations follow directly from the two constraints. For example, in degree 6, $e_1 + e_2 + e_3 + e_4 + e_5 = -6d = -36$. However, each $e_i \geq -30/4 = -7.5$. So even if each e_i is at best -7 , the e_i cannot sum to -36 .

The rest of the remarks follow in a similar manner. Note that one can glean even more information about these curves from the constraints, but the remarks listed above are sufficient for our purposes.

5. Degree 4 and 5 morphisms into X

We will now show that there are no free morphisms of degrees 4 or 5 into X . A morphism $\varphi = (G_0, \dots, G_5)$, where each $G_i = \sum_{j=0}^d a_{ij} S^{d-j} T^j$ is a homogeneous polynomial of degree d , gives us a $6 \times (d+1)$ matrix (a_{ij}) . We will denote this matrix as M_φ .

Lemma 5.1. *If φ is a degree 4 or 5 free morphism into X , then M_φ has maximal rank.*

Proof. This follows from [Remarks\(4\)](#) and (5) by observing that for a degree d morphism into X , the transpose of M_φ is the matrix of the k -linear map

$$\tilde{\varphi}_d : (R(-d)^{\oplus 6})_d \rightarrow R_d. \quad \square$$

Lemma 5.2.

(a) *There are no degree 4 free morphisms into X .*

(b) *There are no degree 5 free morphisms into X .*

Proof. (a) Assume a degree 4 free morphism $\varphi = (G_0, \dots, G_5)$ exists. By the previous lemma, the 6×5 matrix $M_\varphi = (a_{ij})$ has maximal rank. Since permuting the G_i 's does not affect the splitting type of $E_X(\varphi)$, we can assume that the first 5 rows of M_φ are linearly independent over k . Then $\det((a_{ij})_{i \leq 4}) \neq 0$. Now consider the matrix $\overline{M}_\varphi = (a_{ij}^4)$. By the Frobenius endomorphism on k ,

$$\det((a_{ij}^4)_{i \leq 4}) = \det((a_{ij})_{i \leq 4})^4 \neq 0,$$

proving that \overline{M}_φ has maximal rank as well.

Since $G_0^5 + \dots + G_5^5 = 0$, computing the coefficients of $G_0^5 + \dots + G_5^5$, we obtain for $0 \leq j \leq 4$

$$\sum_{i=0}^5 a_{ij}^4 a_{i1} = 0 \quad \text{and} \quad \sum_{i=0}^5 a_{ij}^4 a_{i3} = 0. \quad (5-1)$$

The kernel of the map $k^6 \rightarrow k^5$ given by right multiplication by the matrix \overline{M}_φ has dimension 1 because $\text{rank}(\overline{M}_\varphi) = 5$. By (5-1),

$$(a_{01}, a_{11}, \dots, a_{51}), (a_{03}, a_{13}, \dots, a_{53}) \in \ker(k^6 \rightarrow k^5),$$

and since these 6-tuples are columns of M_φ , they are linearly independent over k . Then $\dim_k(\ker(k^6 \rightarrow k^5)) \geq 2$, a contradiction.

(b) Assume $\varphi = (G_0, \dots, G_5)$ is a degree 5 free morphism. By the previous lemma, the matrix $M_\varphi = (a_{ij})$ has maximal rank and is invertible. Thus $\overline{M}_\varphi = (a_{ij}^4)$ is invertible by the same argument above. Since $G_0^5 + \dots + G_5^5 = 0$, computing the coefficients of the polynomial $G_0^5 + \dots + G_5^5$, we get

$$\sum_{i=0}^5 a_{ij}^4 a_{i2} = 0 \quad \text{for } 0 \leq j \leq 5.$$

Thus, the product of the row matrix $(a_{02}, a_{12}, \dots, a_{52})$ and the matrix \overline{M}_φ is 0, which is impossible because $(a_{02}, a_{12}, \dots, a_{52}) \neq 0$ and \overline{M}_φ is invertible. \square

6. Computations for the degree 8 free curve

Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^5$ be a morphism given by the 6-tuple

$$\begin{aligned} G_0 &= S^7 T, & G_1 &= S^4 T^4 + S^3 T^5, \\ G_2 &= S^4 T^4 + S^3 T^5 + T^8, & G_3 &= S^7 T + S^6 T^2 + S^5 T^3 + S^4 T^4 + S^3 T^5, \\ G_4 &= S^8 + S^7 T + S^6 T^2 + S^5 T^3 + S^4 T^4 + S^3 T^5 + T^8, \\ G_5 &= S^8 + S^7 T + S^6 T^2 + S^5 T^3 + S^4 T^4 + S^3 T^5 + S^2 T^6 + S T^7. \end{aligned}$$

One can check by computer or by hand that this curve lies on the Fermat hypersurface $X \subset \mathbb{P}^5$. Due to twisting, the domain of the map $\tilde{\varphi} : R(-8)^{\oplus 6} \rightarrow R$ has its first non-trivial graded piece in dimension 8. The G_i are linearly independent over k , hence the kernel is trivial in dimension 8. The matrix for the map $\tilde{\varphi}_9 : R(-8)_9^{\oplus 6} \rightarrow R_9$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where each direct summand of the domain has a basis $\{(S, 0), (0, T)\}$, of which we take six copies (for total dimension 12), and the range has basis given by the degree 9 monomials in S and T , ordered by increasing T -degree (for total dimension 10). This matrix has rank 10, which means that the map in degree 9 is surjective. By rank-nullity, two dimensions of the kernel live in degree 9; denote the generators by x_1, x_2 . Surjectivity of $\tilde{\varphi}$ in degree 9 implies surjectivity in all higher degrees. A second application of rank-nullity gives $\dim_k \Omega(\varphi)_{10} = 7$. Four of the generators are inherited from the previous degree, taking the forms

$$x_1S, x_2S, x_1T, x_2T.$$

We conclude that there are three additional generators in degree 10. Therefore, the splitting type of $\Omega_X(\varphi)$ is $(e_1, \dots, e_5) = (-10, -10, -10, -9, -9)$, which corresponds to a splitting type for $E_X(\varphi)$ of $(f_1, \dots, f_5) = (0, 0, 0, 4, 4)$, hence the curve is free.

7. A very free rational curve of degree 9

We conclude by giving an example of a degree 9 very free curve lying on X . Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^5$ be a morphism into the Fermat hypersurface given by the 6-tuple

$$\begin{aligned} G_0 &= S^4T^5, & G_1 &= S^9 + S^8T + S^5T^4, \\ G_2 &= S^9 + S^4T^5 + ST^8, & G_3 &= S^9 + S^8T + S^4T^5 + S^3T^6 + S^2T^7 + ST^8, \\ G_4 &= S^9 + S^5T^4 + S^3T^6 + S^2T^7 + ST^8 + T^9, \\ G_5 &= S^7T^2 + S^6T^3 + S^5T^4 + S^3T^6 + S^2T^7 + ST^8 + T^9. \end{aligned}$$

Let e_1, \dots, e_5 again denote the splitting type of $\Omega_X(\varphi)$. As in Section 6, we know that $e_i \leq -9$. Since the G_i are linearly independent over k , $\dim_k(\Omega_X(\varphi)_9) = 0$. Next we claim that $\tilde{\varphi}_{10} : R_1^{\oplus 6} \rightarrow R_{10}$ is surjective. In fact, it can be checked that the $\tilde{\varphi}(b_i)$ span R_{10} , where the b_i are distinct basis elements of $R_1^{\oplus 6}$. It follows that $\tilde{\varphi}_n : R(-9)_n^{\oplus 6} \rightarrow R_n$ is surjective for $n \geq 10$. Hence,

$$\begin{aligned}\dim_k(\Omega_X(\varphi)_{10}) &= \dim_k(R_1^{\oplus 6}) - \dim_k(R_{10}) = 1, \\ \dim_k(\Omega_X(\varphi)_{11}) &= \dim_k(R_2^{\oplus 6}) - \dim_k(R_{11}) = 6.\end{aligned}$$

After reordering, this yields $(e_1, \dots, e_5) = (-11, -11, -11, -11, -10)$, which corresponds to the splitting type $(1, 1, 1, 1, 5)$ of $E_X(\varphi)$, showing that φ is very free. This completes the proof of Theorem 1.1.

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