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For an integral domain D, the *irreducible divisor graph* $G_D(x)$ of a nonunit $x \in D$ gives a visual representation of the factorizations of x. Here we consider a higher-dimensional generalization of this notion, the *irreducible divisor simplicial complex* $S_D(x)$. We show how this new structure is a true generalization of $G_D(x)$, and show that it often carries more information about the element x and the domain D than its two-dimensional counterpart.

1. Introduction and preliminaries

The concept of an irreducible divisor graph of an element x in an integral domain D was introduced in [Coykendall and Maney 2007]. The vertices of this graph are a prechosen set of irreducible divisors of x, and any pair of vertices are connected by an edge if and only if the corresponding irreducible divisors appear in the same factorization of x. The relevance of the irreducible divisor graph was illustrated in the same paper and in [Axtell et al. 2011]: an integral domain D is a unique factorization domain if and only if each irreducible divisor graph is complete if and only if each irreducible divisor graph is connected.

Since their introduction, irreducible divisor graphs have been studied in the context of integral domains [Axtell et al. 2011; Maney 2008] and in more general contexts [Axtell and Stickles 2008; Bachman et al. 2012; Smallwood and Swartz 2009; 2008]. Despite the appealing result mentioned above, it is difficult to pick out the factorizations of an element given its irreducible divisor graph. In short, irreducible divisor graphs fail to give us all the information we might wish to glean about an element's factorizations.

Our main goal is to introduce the concept of an *irreducible divisor simplicial complex*, effectively a generalization of the irreducible divisor graph to higher dimensions. As we shall see, irreducible divisor simplicial complexes often convey more information about the factorization of an element than its two-dimensional

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counterpart. Maney [2008] uses homologies to study irreducible divisor graphs, linking irreducible divisor graphs to certain zeroth and first homologies. Moreover, higher homologies are considered which, although not explicitly mentioned in [Maney 2008], are related to irreducible divisor simplicial complexes. This gives yet another motivation for studying this new construct.

We now provide a brief overview of the ring- and graph-theoretic terminology that will be required in the sequel. Throughout, D will denote an integral domain, D^* the nonzero elements of D, and $U(D)$ the units of D. It will often be convenient for us to speak of the set of nonzero nonunits of D which will be denoted by $D^* \setminus U(D)$. An element $x \in D^* \setminus U(D)$ is *irreducible* if whenever $x = yz$ with $y, z \in D$, then either $y \in U(D)$ or $z \in U(D)$. We say $x \in D^* \setminus U(D)$ is *prime* if whenever $x \mid yz$ with $y, z \in D$, then either $x \mid y$ or $x \mid z$. An element $x \in D$ is *square-free* if it is not divisible by any perfect square $z^2 \in D^* \setminus U(D)$, that is, if z^2 divides x for $z \in D$, then $z \in U(D)$. Two elements a and b of D are called *associates* if $a = ub$ where $u \in U(D)$. The relation $a \sim b$ on elements of D is an equivalence relation that partitions D into *associate classes*. We denote the set of irreducibles in D as $\operatorname{Irr}(D) = \{x : x \text{ is irreducible}\}$ and define $\overline{\operatorname{Irr}}(D)$ to be a (prechosen) set of associate class representatives, one from each class of nonzero associates. We denote the irreducible divisors of a particular element $x \in D$ as $\operatorname{Irr}(x)$ and set $\overline{\operatorname{Irr}}(x) = \operatorname{Irr}(x) \cap \overline{\operatorname{Irr}}(D)$. Note that by considering only $\overline{\operatorname{Irr}}(D)$, we do not distinguish elements in D from their associates and we are implicitly working in the *reduced multiplicative monoid* $D^{\bullet}_{\text{red}}$ (see [Geroldinger and Halter-Koch 2006, Chapter 1]), which is the multiplicative monoid whose elements are associate classes and whose identity is the set of units.

As we will be studying the factorization of elements of D as products of irreducibles, it will be useful to restrict our study to only *atomic domains* where each element $x \in D^* \setminus U(D)$ can be factored into a finite product of irreducible elements. Clearly every prime in an integral domain is irreducible. If D is an atomic domain, then D is a *unique factorization domain* (UFD) if and only if all irreducibles in D are prime [Geroldinger and Halter-Koch 2006, Theorem 1.1.10.2]. We now give a brief introduction of some special types of atomic domains and related terminology. We say D is a *finite factorization domain* (FFD) if every nonzero nonunit in D has only finitely many distinct nonassociate irreducible divisors. If D is a finite factorization domain, the *set of lengths* (of factorizations) of $x \in D^* \setminus U(D)$ is $L(x) = \{t : x = a_1 a_2 \cdots a_t \text{ where each } a_i \text{ is irreducible}\}$. A FFD D is a *bounded factorization domain* (BFD) if there is a bound on the length of factorization into products of irreducible elements for each nonzero nonunit element in D. If $|L(x)| = 1$ for all $x \in D^* \setminus U(D)$, we say that D is a *half-factorial domain* (HFD). The *elasticity* $\rho(D)$ of D gives a measure of how far D is from being a HFD; it is defined as the supremum of the *elasticity* $\rho(x) := \max L(x)/\min L(x)$ of each element $x \in D^* \setminus U(D)$.

A *graph* is an ordered pair of sets (V, E), where V is called the *vertex set*, and E is the *edge set*, whose elements are subsets of V of cardinality 2. We denote an edge between vertices a and b as $\{a, b\}$ and note that the edge $\{a, b\}$ is the same as the edge $\{b, a\}$. The edge $\{a, b\}$ is said to be *incident* with both vertices a and b. We denote the set of vertices of a graph G as $V(G)$ and the set of edges of G as $E(G)$. In addition, we define a *loop* to be an edge between a and itself. We now define a higher-dimensional analog of graphs. A *simplicial complex* S is an ordered pair (V, F) where V is a set of vertices and the *set of faces* F is a collection of subsets of V satisfying: (1) $\{v\} \in F$ for all $v \in V$ (vertices are faces) and (2) if $\sigma \in F$ and $\tau \subseteq \sigma$, then $\tau \in F$ (subsets of faces are faces). As with graphs, we denote the set of vertices of S as $V(S)$, and the set of faces of S as $F(S)$. The *dimension* of a face β of finite cardinality in a simplicial complex S is one less than its cardinality and is denoted as $\dim(\beta) = |\beta| - 1$. Faces with maximal cardinality (with respect to inclusion) are referred to as *facets*. For a nonnegative integer k, the *k-skeleton* $K_k(S)$ of a simplicial complex S is the subsimplex of S consisting of all the faces of S whose dimension is at most k. We note that $K_1(S)$ is a graph.

2. Irreducible divisor graphs

In this section, we introduce the irreducible divisor graph of an element in an atomic domain and summarize results from [Axtell et al. 2011; Coykendall and Maney 2007].

Definition 2.1. Let D be an atomic domain and let $x \in D^* \setminus U(D)$. The *irreducible divisor graph* of x, denoted $G_D(x)$, is given by (V, E) where the vertex set $V = \{\overline{\mathrm{Irr}}(x) : x \in D\}$, and given $y_1, y_2 \in V$, there is an edge $\{y_1, y_2\} \in E$ between vertices y_1 and y_2 if and only if $y_1 y_2 \mid x$.

When it is clear from context, we will drop the subscript D from $G_D(x)$ and write $G(x)$. If the same element $a \in \overline{\mathrm{Irr}}(D)$ appears multiple times in a particular factorization of $x \in D^* \setminus U(D)$, then we add one or more loops to the vertex a in $G(x)$. We place n loops on vertex a provided $a^{n+1} \mid x$ and $a^{n+2} \nmid x$. When a vertex has more than one loop, we will denote the number of loops in the graph with a superscript over the loop.

Example 2.2. Let $D = \mathbb{Z}[\sqrt{-5}]$ and consider the irreducible divisor graph $G(18)$. Recall that 18 factors as

$$18 = 2 \cdot 3^2 = 3(1 + \sqrt{-5})(1 - \sqrt{-5}) = 2(2 + \sqrt{-5})(2 - \sqrt{-5}).$$

To simplify notation, we set $\alpha = (1 + \sqrt{-5})$ and $\beta = (2 + \sqrt{-5})$, with $\bar{\alpha}$ and $\bar{\beta}$ denoting their complex conjugates. Using the rules provided in Definition 2.1, we construct the irreducible divisor graph shown in Figure 1. For example, $\{2, \beta\}$ is an

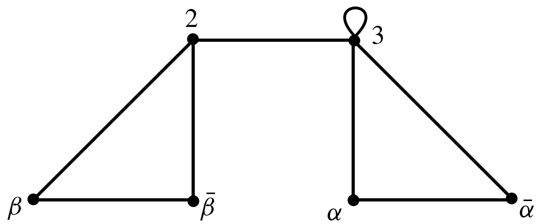

Figure 1. $G(18)$ in $\mathbb{Z}[\sqrt{-5}]$.

edge in $G(18)$ since $2\beta \mid 18$. Since $3^2 \mid 18$ but $3^3 \nmid 18$, we place a single loop on vertex 3. We note $G(18)$ is connected but not complete.

One of the goals in studying the irreducible divisor graph of an element x in an integral domain D is to be able to draw conclusions about the factorization of the element in question and of other elements of D. When we look closely at $G(18)$ and briefly try to forget what the factorizations of 18 look like, we can see that there will be some factorization that will include β, $\bar{\beta}$, and 2. Since β and $\bar{\beta}$ are connected by an edge in $G(18)$, $18 = \beta\bar{\beta}x$ for some $x \in D^* \setminus U(D)$. Similarly, $18 = 2\beta y$ and $18 = 2\bar{\beta}z$ for some $y, z \in D^* \setminus U(D)$. Since all irreducible factors of x appear together with β and $\bar{\beta}$ in a factorization of 18 and since none of 2, β, or $\bar{\beta}$ are looped in $G(18)$, it must be the case that $x = 2$. Similarly, $y = \bar{\beta}$ and $z = \beta$. Thus 18 factors as $18 = 2\beta\bar{\beta}$, and this factorization corresponds to the complete subgraph with vertex set $\{2, \beta, \bar{\beta}\}$. Note that the maximal complete subgraphs $\{2, 3\}$ and $\{3, \alpha, \bar{\alpha}\}$ also correspond to factorizations of 18. However, this correspondence requires a priori knowledge of the factorizations of 18 in $\mathbb{Z}[\sqrt{-5}]$ and we cannot see simply by looking at the graph $G(18)$ what the remaining factorizations of 18 are. This problem occurs because of the loop on the vertex 3. When we look at the graph, we really have no way of assigning the element 3^2 to any one factorization. As irreducible divisor graphs get more complicated with more irreducible divisors, we will have much difficulty in deciphering what the factorization of a particular element is by simply looking at its irreducible divisor graph.

In most situations, factorizations do not correspond to complete subgraphs. Conversely, complete subgraphs need not correspond to factorizations. We now consider another example where this is certainly the case.

Example 2.3. Let $D = \mathbb{Z}[\sqrt{-5}]$ and consider $G(108)$. By considering norms, we see that 108 factors only as

$$\begin{aligned} 108 &= 2^2 3^3 = 2 \cdot 3^2(1+\sqrt{-5})(1-\sqrt{-5}) \\ &= 2^2 \cdot 3(2+\sqrt{-5})(2-\sqrt{-5}) = 3(1+\sqrt{-5})^2(1-\sqrt{-5})^2 \\ &= 2(1+\sqrt{-5})(1-\sqrt{-5})(2+\sqrt{-5})(2-\sqrt{-5}). \end{aligned}$$

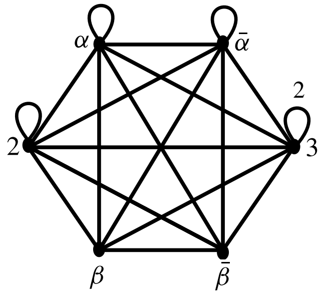

Figure 2. $G(108)$ in $D = \mathbb{Z}[\sqrt{-5}]$.

As before, let $\alpha = (1+\sqrt{-5})$ and $\beta = (2+\sqrt{-5})$ with $\bar{\alpha}$ and $\bar{\beta}$ denoting their complex conjugates. The irreducible divisor graph is given in Figure 2. This graph is complete, even though D is not a UFD. Certainly not all complete subgraphs correspond to factorizations of 108, thus making it hard to glean factorization-theoretic information from the irreducible divisor graph. We will return to this example later in Example 3.4.

For variety, we now give an example of the irreducible divisor graph of an element in a nonhalf-factorial domain.

Example 2.4. Let k be a field and let $D = k[x^{10}, x^{12}, x^{18}, x^{33}]$ denote the subring of the polynomial ring $k[x]$. Then $x^{66} \in D$ and the only irreducible divisors of x^{66} in D are x^{10}, x^{12}, x^{18}, and x^{33}. Moreover, x^{66} factors only as

$$x^{66} = (x^{12})(x^{18})^3 = (x^{12})^4(x^{18}) = (x^{10})^3(x^{18})^2 = (x^{10})^3(x^{12})^3 = (x^{33})^2.$$

Therefore, the irreducible divisor graph $G_D(x^{66})$, shown in Figure 3, consists of a complete graph on three vertices (x^{10}, x^{12}, and x^{18}) with 2, 3, and 2 loops on these respective vertices, along with a single vertex (x^{33}) having a single loop.

We now turn to several important results that can be found in [Axtell et al. 2011; Coykendall and Maney 2007]. The first result gives necessary and sufficient

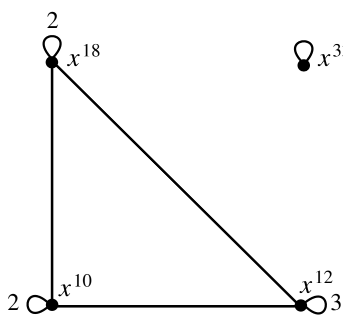

Figure 3. $G(x^{66})$ in $D = k[x^{10}, x^{12}, x^{18}, x^{33}]$.

conditions for an atomic domain to be a UFD. The second result gives a bound on the elasticity of an element given by its irreducible divisor graph. We will prove generalizations of these results in Section 3.

Theorem 2.5 [Axtell et al. 2011, Theorem 2.1]. *Let D be an atomic domain. The following statements are equivalent.*

(1) *D is a UFD.*

(2) *$G(x)$ is complete for all $x \in D^* \setminus U(D)$.*

(3) *$G(x)$ is connected for all $x \in D^* \setminus U(D)$.*

Proposition 2.6 [Axtell et al. 2011, Proposition 4.1]. *Let x be an element which is not irreducible of a BFD D. Then $\rho(x)$ does not exceed*

$$\tfrac{1}{2}\max\{t+l : G(x) \textit{ contains a complete subgraph with } t \textit{ vertices and } l \textit{ loops}\}.$$

The proof of this result makes note of the fact that if $x = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}$, where $a_1, a_2, \ldots, a_n \in \overline{\mathrm{Irr}}(x)$, then $G(x)$ contains a complete subgraph with a vertex corresponding to each a_i. In the special case that x is square-free we produce a more accurate result.

Corollary 2.7 [Axtell et al. 2011, Corollary 4.4]. *Let x be a square-free nonirreducible element of a domain D. Then*

$$\rho(x) \leq \tfrac{1}{2}\max\{t : G(x) \textit{ contains a complete subgraph with } t \textit{ vertices}\}.$$

We note that the bounds given in Proposition 2.6 and Corollary 2.7 are, in general, not tight. There are three reasons: First, it is often the case that not all vertices belonging to a complete subgraph of $G_D(x)$ are involved in a single factorization of x. Second, the minimal length of a factorization of x is often larger than 2. Finally, when counting loops, it is impossible to know how many come from a given factorization of x. As was done in Corollary 2.7, assuming that x is square-free eliminates the third problem. We will consider these other two issues in Section 3.

3. Irreducible divisor simplicial complexes

We now extend the definition of irreducible divisor graphs given in Section 2 to higher dimensions. We do this in the hopes that this extension will yield more information about the factorization of elements in an atomic domain. After giving a couple of examples, we generalize the results given in Section 2, but in terms of irreducible divisor simplicial complexes.

Definition 3.1. Let D be an atomic domain and let $x \in D^* \setminus U(D)$. The *irreducible divisor simplicial complex* of x, denoted $S_D(x)$, is given by (V, F) with vertex set V given by $V = \{\overline{\mathrm{Irr}}(x) : x \in D\}$ and with $\{y_1, y_2, \ldots, y_n\} \in F$ a face if and only if $y_1 y_2 \cdots y_n \mid x$. In addition, to satisfy convention, we also put $\varnothing \in F$.

Whenever the context is clear we will drop the subscript D from $S_D(x)$ giving $S(x)$.

Remark 3.2. Let $S(x) = (V, F)$ be an irreducible simplicial complex. Clearly F is a collection of subsets of V. If $y \in V = \overline{\operatorname{Irr}}(x)$, then $\{y\} \in F$ since $y \mid x$ and hence vertices are faces. Second, suppose that $\sigma \in F$ and $\tau \subseteq \sigma$. Since $\sigma \in F$, we know $\sigma = \{y_1, \ldots, y_n\}$ where $y_1 \cdots y_n \mid x$. Hence $\tau = \{y_{i_1}, \ldots, y_{i_j}\}$, some subcollection of the y_i, and clearly $y_{i_1} \cdots y_{i_j} \mid x$. Thus $\tau \in F$, and hence subsets of faces are faces. Therefore irreducible simplicial complexes are indeed simplicial complexes.

We graphically represent irreducible divisor simplicial complexes and irreducible divisor graphs in similar ways. Points represent vertices and edges represent faces of dimension 1. If we have some element x which factors into irreducibles as $x_1^{m_1} \cdots x_n^{m_n}$ with distinct irreducible x_i and $m_i \geq 1$ for all i, then the vertex representing x_i will be drawn with $m_i - 1$ loops. Graphically, we illustrate two-dimensional faces by shaded triangles and three-dimensional faces by solid tetrahedra. We have no effective way to graphically depict higher-dimensional faces, so readers are on their own.

Example 3.3. Recall the irreducible divisor graph $G(18)$ in Figure 1. We now show the corresponding irreducible divisor simplicial complex $S(18)$:

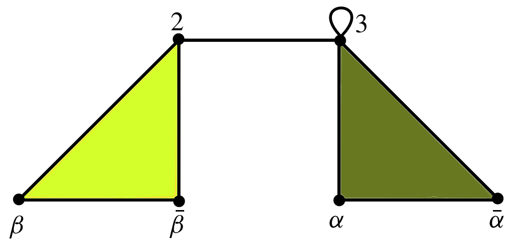

Figure 4. $S(18)$ in $\mathbb{Z}[\sqrt{-5}]$.

Here we have the same general structure as $G(18)$, but we now have two-dimensional facets $\{2, \beta, \bar{\beta}\}$ and $\{3, \alpha, \bar{\alpha}\}$ which are represented graphically as shaded faces. In this higher-dimensional structure, we avoid the difficulty in determining factorizations as in Example 2.2. Indeed, the facets $\{2, 3\}$, $\{2, \beta, \bar{\beta}\}$, and $\{3, \alpha, \bar{\alpha}\}$ correspond directly to the factorizations of 18. We will make this idea more precise in Propositions 3.7 and 3.8.

Example 3.4. We now consider the irreducible divisor simplicial complex $S(108)$ in $D = \mathbb{Z}[\sqrt{-5}]$. Recall that 108 factors as

$$108 = 2^2 3^3 = 2 \cdot 3^2 \alpha\bar{\alpha} = 2^2 3 \beta\bar{\beta} = 3\alpha^2\bar{\alpha}^2 = 2\alpha\bar{\alpha}\beta\bar{\beta}.$$

If we investigate Figure 2, we can see the difficulty in extracting a particular factorization by simply analyzing the graph. However, this becomes much easier if we consider the irreducible divisor simplicial complex $S(108)$. We have that

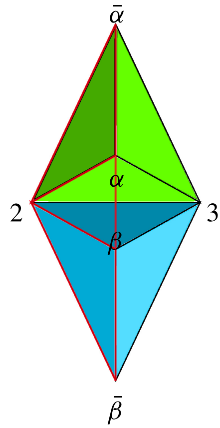

Figure 5. $S(108)$ in $\mathbb{Z}[\sqrt{-5}]$.

$S(108) = (V, F)$, with $V = \{2, 3, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}$ and $F = \{\varnothing\} \cup F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4$, where F_i denotes the set of faces of $S(108)$ with dimension i:

$$\begin{aligned}
F_0 &= \{\{v\} : v \in V\},\\
F_1 &= \{S \subseteq V : |S| = 2\},\\
F_2 &= \{S \subseteq V : |S| = 3\} - \{\{3, \alpha, \beta\}, \{3, \alpha, \overline{\beta}\}, \{3, \overline{\alpha}, \beta\}, \{3, \overline{\alpha}, \overline{\beta}\}\},\\
F_3 &= \big\{\{2, 3, \beta, \overline{\beta}\}, \{2, 3, \alpha, \overline{\alpha}\}, \{\alpha, \overline{\alpha}, \beta, \overline{\beta}\}, \{2, \overline{\alpha}, \beta, \overline{\beta}\},\\
&\qquad \{2, \alpha, \beta, \overline{\beta}\}, \{2, \alpha, \overline{\alpha}, \overline{\beta}\}, \{2, \alpha, \overline{\alpha}, \beta\}\big\},\\
F_4 &= \{2, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}.
\end{aligned}$$

The maximal faces (facets) of $S(x)$ are

$$\{2, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}, \quad \{2, 3, \beta, \overline{\beta}\}, \quad \{2, 3, \alpha, \overline{\alpha}\}.$$

Note that in Figure 5, the red-colored outline illustrates the 4-dimensional facet $\{2, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}$. Unlike in $G(108)$, we can actually see that there are factorizations of 108 that contain $2, \alpha, \overline{\alpha}, \beta$, and $\overline{\beta}$, since they form a face of $S(108)$. We can also conclude that there is a factorization of x that only contains $2, 3, \alpha$, and $\overline{\alpha}$, a fact that is not immediately apparent when examining $G(108)$. If we consider only $G(108)$ and consider the set $A = \{2, 3, \alpha, \overline{\alpha}\}$, we see no clear way of proving that a factorization of 108 given by $2 \cdot 3 \cdot \alpha\overline{\alpha}$ will not include β or $\overline{\beta}$. After all, there are edges connecting β or $\overline{\beta}$ to each element of A. In other words, the graph $G(108)$ does not seem to provide enough information to support the conclusion that $108 = 2^i 3^j \alpha^k \overline{\alpha}^l$ for $i, j, k, l \geq 1$. In contrast, $S(108)$ contains far more information, as we will see in the results that follow.

Example 3.5. Recall the element $x^{66} \in D = k[x^{10}, x^{12}, x^{18}, x^{33}]$ from Example 2.4. Since no three distinct irreducible divisors of x^{66} occur together in a factorization of x^{66}, the irreducible divisor simplicial complex contains no faces of dimension higher than 1 and $S_D(x^{66}) = G_D(x^{66})$ as shown in Figure 3. Even though these two constructions give identical objects in this case, the simplicial complex carries more information. In particular, only by looking at $S_D(x^{66})$ can we see that there are no factorizations involving more than two distinct irreducible factors.

We now generalize and extend the results from Section 2. First we note that the irreducible divisor simplicial complex $S_D(x)$ properly contains as a subsimplex the irreducible divisor graph $G_D(x)$.

Proposition 3.6. *For D an atomic domain and $x \in D^* \setminus U(D)$, we have*

$$K_1(S(x)) = G(x).$$

Proof. Let $G(x) = (V, E)$ denote the irreducible divisor graph of x, and let $S(x) = (V', F)$ denote the irreducible divisor simplicial complex of x. By definition,

$$V' = V = \overline{\mathrm{Irr}}(x).$$

Furthermore, $E \subseteq F$ since if $\{a, b\} \in E$, then $ab \mid x$ and hence $\{a, b\} \in F$. Moreover, if $\{a, b\}$ is a one-dimensional face of F, then $ab \mid x$ and hence $\{a, b\} \in E$. That is, the one-dimensional faces of $S(x)$ are precisely the edges of $G(x)$. $\square$

The following results give a means for finding factorizations of an element x by considering $S_D(x)$.

Proposition 3.7. *For D an atomic domain and $x \in D^* \setminus U(D)$, let $A = \{a_1, \dots, a_n\}$ be a facet of the irreducible divisor simplicial complex $S(x)$. Then there exists a factorization of x given by $x = a_1^{m_1} \cdots a_n^{m_n}$, where $m_i \geq 1$ for each i.*

Proof. Since A is a face of $S(x)$, we know that $a_1 \cdots a_n \mid x$. In fact, since $x/(a_1 a_2 \cdots a_n)$ also has a factorization, there is a factorization of x that involves each a_i. Suppose, by way of contradiction, that there exists some factorization of x of the form $a_1^{m_1} \cdots a_n^{m_n} b_1 \cdots b_k$, where each b_j is irreducible and b_j is not an associate of a_i for all i, j. Then by the definition of $S(x)$, $\{a_1, \dots, a_n, b_1\}$ is a face of $S(x)$ properly containing A, contradicting the fact that A is a facet of $S(x)$. $\square$

The converse to Proposition 3.7 does not hold in general as seen in Example 3.4. Indeed, $108 = 2^2 3^3 = 3\alpha^2\overline{\alpha}^2$ and yet neither $\{2, 3\}$ nor $\{3, \alpha, \overline{\alpha}\}$ is a facet since they are properly contained in the facet $\{2, 3, \alpha, \overline{\alpha}\}$. However, if we apply an additional restriction we find a partial converse.

Proposition 3.8. *Let D be an atomic domain and suppose $x \in D^* \setminus U(D)$ is square-free. Then every factorization of x corresponds to a facet of $S(x)$.*

Proof. By way of contradiction, suppose there exists a factorization $x = a_1a_2\cdots a_n$, with each a_i irreducible, corresponding to the face $A = \{a_1, a_2, \ldots, a_n\}$ of $S(x)$ that is not a facet. That is, $A \subsetneq B$ for some facet $B = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$ of $S(x)$, where no b_i is associate to any a_i. Applying Proposition 3.7 and the fact that x is square-free, the facet B corresponds to the factorization $x = a_1a_2\cdots a_nb_1b_2\cdots a_m$. Setting these two factorizations equal, we have

$$x = a_1a_2\cdots a_n = a_1a_2\cdots a_nb_1b_2\cdots b_m.$$

As D is an integral domain, we may repeatedly apply left-cancellation to find that $1 = b_1\cdots b_m$. This is a contradiction, since each of the b_i is a nonunit irreducible of D. Hence, $A = B$ is a facet of $S(x)$. $\square$

We now produce a result analogous to Theorem 2.5 providing another necessary and sufficient condition for an integral domain D to be a UFD. Recall that if X is a set, then $\mathcal{P}(X)$ denotes the power set of X consisting of all subsets of X. We abuse notation and write $\mathcal{P}(X)$ to denote the simplicial complex $(X, \mathcal{P}(X))$ with vertex set X and face set $\mathcal{P}(X)$. Recall that $V(\mathcal{P}(X)) = X$ and $F(\mathcal{P}(X)) = \mathcal{P}(X)$. Recall from Definition 3.1 that if the singleton $\{y\}$ is a face of $S(x)$, then y is an irreducible divisor of x. In the next theorem, we may safely ignore all loops in both $S(x)$ and $G(x)$.

Theorem 3.9. *Let D be an atomic domain. The following are equivalent.*

(1) *For every $x \in D^* \setminus U(D)$, $S(x) = \mathcal{P}(A)$ for some $A \subseteq \overline{\mathrm{Irr}}(x)$.*

(2) *D is a UFD.*

Proof. Assume (1) and let $x \in D^* \setminus U(D)$. Then $S(x) = \mathcal{P}(A)$ for some $A \subseteq \overline{\mathrm{Irr}}(x)$. Since $G(x) = K_1(S(x))$ by Proposition 3.6, and since $K_1(\mathcal{P}(A))$ is a complete graph, $G(x)$ is complete. Since this holds for all $x \in D^* \setminus U(D)$, D is a UFD by Theorem 2.5.

If D is a UFD, any x factors uniquely as $x = a_1^{m_1}\cdots a_n^{m_n}$, $m_i \geq 1$. Then $a_{i_1}\cdots a_{i_t} \mid x$ for any subset $\{a_{i_1}, \ldots, a_{i_t}\} \subseteq \{a_1, \ldots, a_n\}$, and hence $F(S(x)) = \mathcal{P}(\{a_i, \ldots, a_n\})$. That is, $S(x) = \mathcal{P}(\overline{\mathrm{Irr}}(x))$. $\square$

We now examine another necessary and sufficient condition for an integral domain D to be a UFD. First, we require a definition and two lemmas. Recall that for two simplicial complexes $S = (V, F)$ and $T = (W, G)$, their *join* $S * T$ is the simplicial complex with vertex set $V \cup W$ and with face set $\{A \cup B : A \in F, B \in G\}$.

Lemma 3.10. *Let A and B be two sets. As simplicial complexes, $\mathcal{P}(A \cup B) = \mathcal{P}(A) * \mathcal{P}(B)$.*

Proof. First we show that the vertex sets are equal. Suppose $a \in V(\mathcal{P}(A \cup B))$. Then $a \in A \cup B$, which by definition means $a \in V(\mathcal{P}(A) * \mathcal{P}(B))$. For the other containment, suppose $b \in V(\mathcal{P}(A) * \mathcal{P}(B))$. By definition, $b \in A \cup B$ and hence $b \in V(\mathcal{P}(A \cup B))$.

Now we show that $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A) * \mathcal{P}(B)$ have the same face set. Let $\alpha \in F(\mathcal{P}(A \cup B))$, that is, $\alpha \subseteq A \cup B$. Set $\alpha_A := \alpha \cap A \subseteq A$ and $\alpha_B := \alpha \setminus \alpha_A \subseteq B$. Clearly $\alpha = \alpha_A \cup \alpha_B$, and hence $\alpha \in F(\mathcal{P}(A) * \mathcal{P}(B))$. To show the other containment, select $\alpha \in F(\mathcal{P}(A) * \mathcal{P}(B))$ and write $\alpha = \alpha_A \cup \alpha_B$ for some $\alpha_A \subseteq A$, $\alpha_B \subseteq B$. Then $\alpha \subseteq A \cup B$, and thus $\alpha \in F(\mathcal{P}(A \cup B))$. Since $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A) * \mathcal{P}(B)$ have the same vertex and face sets, they are equal as simplicial complexes. □

Lemma 3.11. *Let $a, b \in D^* \setminus U(D)$. Then $V(S(b)) \cup V(S(a)) \subseteq V(S(ab))$. Moreover, if D is a UFD, then equality holds.*

Proof. Suppose $x \in V(S(a)) \cup V(S(b))$. If $x \in V(S(a))$, then $x \mid a$. If $x \in V(S(b))$, then $x \mid b$. In either case, $x \mid ab$ and hence $x \in V(S(ab))$.

Now suppose that D is a UFD and let $x \in V(S(ab))$. Then $x \mid ab$, with x irreducible and hence prime. If $x \mid a$, then $x \in V(S(a))$. If $x \nmid a$, then $x \mid b$ and hence $x \in V(S(b))$. Thus $x \in V(S(a)) \cup V(S(b))$. □

Theorem 3.12. *Let D be an atomic domain. The following are equivalent.*

(1) *$S(a) * S(b) = S(ab)$ for all $a, b \in D^* \setminus U(D)$.*

(2) *D is a UFD.*

Proof. Suppose D is not a UFD. Then there exists an irreducible $z \in D$ that is not prime. That is, there exists $a, b \in D$ where $z \mid ab$, but $z \nmid a$ and $z \nmid b$. Since $z \mid ab$, we have $z \in V(S(ab))$. We now consider $S(a) * S(b)$. By definition, $z \notin V(S(a))$ and $z \notin V(S(b))$, and hence $z \notin V(S(a)) \cup V(S(b))$. But then $z \notin V(S(a) * S(b))$, since $V(S(a) * S(b)) = V(S(a)) \cup V(S(b))$. Therefore $S(a) * S(b) \neq S(ab)$.

Now let D be a UFD and let $a, b \in D^* \setminus U(D)$. We want to show that $S(a) * S(b) = S(ab)$. Since D is a UFD, we know from Theorem 3.9 that $S(x) = \mathcal{P}(V(S(x)))$ for any $x \in D^* \setminus U(D)$. From Lemma 3.11, we have $V(S(ab)) = V(S(a)) \cup V(S(b))$. Also, using Lemma 3.10, we have

$$\begin{aligned} S(ab) &= \mathcal{P}\big(V(S(ab))\big) = \mathcal{P}\big(V(S(a)) \cup V(S(b))\big) \\ &= \mathcal{P}\big(V(S(a))\big) * \mathcal{P}\big(V(S(b))\big) = S(a) * S(b). \end{aligned}$$

Thus $S(ab) = S(a) * S(b)$ for all $a, b \in D^* \setminus U(D)$. □

We now provide improvements to the elasticity results of Section 2.

Theorem 3.13. *Let D be a BFD. For $x \in D^* \setminus U(D)$ a non irreducible element, let $A(x)$ and $B(x)$ be sets of positive integers defined as:*

$A(x) = \{v + l : S(x)$ *contains a facet with v vertices and l loops*$\}$,

$B(x) = \{v + l : G(x)$ *contains a complete subgraph with v vertices and l loops*$\}$.

Then

$$\max L(x) \leq \max A(x) \leq \max B(x).$$

Moreover,

$$\rho(x) \leq \tfrac{1}{2}\max A(x) \leq \tfrac{1}{2}\max B(x).$$

Note that $\frac{1}{2}B(x)$ is precisely the bound given in Proposition 2.6.

Proof. Let $\{a_1, \ldots, a_v\}$ be a facet of $S(x)$ with a total of l loops on these vertices. Then $a_1 \cdots a_v \mid x$, and thus $\{a_1, \ldots, a_v\}$ is the vertex set of a complete subgraph of $G(x)$. Loops are preserved when moving from $S(x)$ to $G(x)$. Therefore if $n \in A(x)$, then $n \in B(x)$. Thus $\max A(x) \leq \max B(x)$. If $M = \max L(x)$, then we can write $x = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}$, where the a_i are distinct irreducibles and $\sum_{i=1}^{t} n_i = M$. The set $\{a_1, a_2, \ldots, a_t\}$ is a face in $S(x)$ which is contained in some facet of $S(x)$. Also, for each i with $1 \leq i \leq t$, there are $n_i - 1$ loops drawn on the vertex a_i. Thus for any factorization of x of length M we can find a facet of $S(x)$ that contains at least M vertices/loops, and hence $\max L(x) \leq \max X(x)$. Finally, since x is not irreducible, $\min(L(x)) \geq 2$ and thus $\rho(x) \leq \frac{1}{2}\max(A(x)) \leq \frac{1}{2}\max(B(x))$. □

We now consider the sharpness of these bounds by looking at two examples.

Example 3.14. Consider $G(108)$ in Figure 2. The graph $G(108)$ is complete and thus to find the bound on elasticity using Corollary 2.7 we count all vertices and all loops giving us $\rho(108) \leq \frac{1}{2}(6+5) = \frac{11}{2}$. Though not explicitly mentioned in Corollary 2.7, we also see that $\max L(x) \leq 11$. Now consider $S(108)$ in Example 3.4. In order to maximize the total of vertices of and loops in a facet of $S(108)$, we select the facet $\{2, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}$. By Theorem 3.13, $\max L(x) \leq 5$ and $\rho(108) \leq \frac{1}{2}(5+3) = 4$. Here we see that the bound on $\max L(x)$ achieved by Theorem 3.13 is sharp, while the bound on $\max L(x)$ from Corollary 2.7 is not. Since $\mathbb{Z}[\sqrt{-5}]$ is half-factorial, $\rho(108) = 1$ and neither of the bounds on elasticity are sharp.

Example 3.15. Consider $x^{66} \in D = k[x^{10}, x^{12}, x^{18}, x^{33}]$ from Examples 2.4 and 3.5. Since we know precisely the factorizations of x^{66}, we see that $\max L(x^{66}) = 6$, $\min L(x^{66}) = 2$, and $\rho(x^{66}) = 3$. The bounds given by Corollary 2.7 are $\max L(x^{66}) \leq 13$ and $\rho(x^{66}) \leq \frac{13}{2}$. The bounds from Theorem 3.13 are much sharper, with $\max L(x^{66}) \leq 7$ and $\rho(x^{66}) \leq \frac{7}{2}$.

In the special case where x is square-free, we determine in Theorem 3.16 both the minimum and maximum of $L(x)$ as well as the elasticity precisely when using irreducible divisor simplicial complexes, which is a vast improvement over the bound given in Corollary 2.7.

Theorem 3.16. *Let D be a BFD and let $x \in D^* \setminus U(D)$ be square-free. Choose facets β and α such that β has maximal cardinality and α has minimal cardinality among the set of all facets of $S(x)$. Then*

$$\max L(x) = \dim(\beta) + 1, \quad \min L(x) = \dim(\alpha) + 1, \quad \rho(x) = \frac{\dim(\beta)+1}{\dim(\alpha)+1}.$$

Proof. By Proposition 3.8, each factorization of x corresponds to a facet of $S(x)$. Therefore

$$\max L(x) = \max\{|\beta| : \beta \in F(S(x))\},$$
$$\min L(x) = \min\{|\alpha| : \alpha \in F(S(x))\}.$$

By definition,

$$\rho(x) = \frac{\max L(x)}{\min L(x)},$$

and thus

$$\rho(x) = \frac{\dim(\beta)+1}{\dim(\alpha)+1}.$$ □

Example 3.17. Let k be a field and let

$$D = k\big[xy^2w,\, xz,\, y^2,\, z^3w,\, x^2y^2,\, y^2z^2,\, z^2w^2\big]$$

be a subring of the polynomial ring $k[x, y, z, w]$. Then the element $x^2y^4z^4w^2$ factors in D only as

$$x^2y^4z^4w^2 = (xy^2w)(xz)(y^2)(z^3w) = (x^2y^2)(y^2z^2)(z^2w^2).$$

Thus $L(x^2y^4z^4w^2) = \{3, 4\}$ and $\rho(x^2y^4z^4w^2) = \frac{4}{3}$. The irreducible divisor graph $G_D(x^2y^4z^4w^2)$, shown at the top of Figure 6, consists of two disjoint components, a 4-clique and a 3-clique, with no looped vertices. The irreducible divisor simplicial complex, shown at the bottom of Figure 6, consists of two disjoint facets, one of dimension 3, the other of dimension 2. Again, no vertices are looped. The bounds

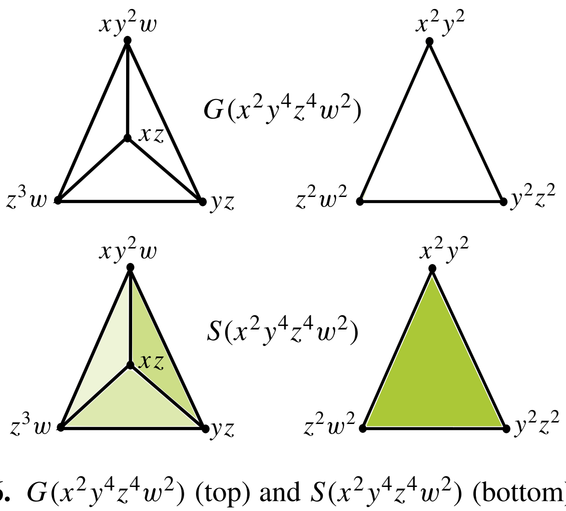

Figure 6. $G(x^2y^4z^4w^2)$ (top) and $S(x^2y^4z^4w^2)$ (bottom) in $D = k[xy^2w, xz, y^2, z^3w, x^2y^2, y^2z^2, z^2w^2]$.

from Corollary 2.7 are

$$\max L(x^2y^4z^4w^2) \le 4 \quad \text{and} \quad \rho(x^2y^4z^4w^2) \le 2.$$

The values from Theorem 3.13 are precise, with

$$\max L(x^2y^4z^4w^2) = 4, \quad \min L(x^2y^4z^4w^2) = 3, \quad \rho(x^2y^4z^4w^2) = \tfrac{4}{3}.$$

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