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An interesting proof of the nonexistence of a continuous bijection between $\mathbb{R}^n$ and $\mathbb{R}^2$ for $n \neq 2$

Hamid Reza Daneshpajouh, Hamed Daneshpajouh and Fereshte Malek

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We show that there is no continuous bijection from $\mathbb{R}^n$ onto $\mathbb{R}^2$ for $n \neq 2$ by an elementary method. This proof is based on showing that for any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of R^n $(n \geq 3)$ into β arcwise connected dense subsets.

1. Introduction

In 1877 Cantor discovered a bijection of $\mathbb{R}$ onto $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Cantor's map was discontinuous, but the discovery of the Peano curve in 1890 showed that there existed continuous (although not injective) maps of $\mathbb{R}$ onto $\mathbb{R}^n$. Between then and 1910, several mathematicians showed that there does not exist a bicontinuous bijection (homeomorphism) from $\mathbb{R}^m$ onto $\mathbb{R}^n$ for the cases $m = 2$ and $m = 3$ and $n > m$. Finally in 1911, Brouwer showed that there does not exist a homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $n \neq m$ (for a modern treatment, see [Munkres 1984, p. 109]). The present paper proves the nonexistence of a continuous bijection from $\mathbb{R}^n$ onto $\mathbb{R}^2$ for $n \neq 2$ by an elementary method.

Rudin [1963] showed that for any countable cardinal $\alpha > 2$, we cannot partition the plane into α arcwise connected dense subsets. In this paper we show that for any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of $\mathbb{R}^n$ $(n \geq 3)$ into β arcwise connected dense subsets; then by using this we show that there is no continuous bijection from $\mathbb{R}^n$ onto $\mathbb{R}^2$ for $n \neq 2$.

Lemma 1. *There is a partition of $\mathbb{R}^+$ into $2^{\aleph_0}$ dense subsets.*

Proof. Consider the additive group $(\mathbb{R}, +)$. The quotient group $\mathbb{R}/\mathbb{Q}$ has $2^{\aleph_0}$ elements which are dense subsets of $\mathbb{R}$. Intersect them with $\mathbb{R}^+$. $\square$

Theorem 1. *There is a partition of $\mathbb{R}^3$ into $2^{\aleph_0}$ arcwise connected dense subsets.*

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Keywords: arcwise connected, dense subset, homeomorphism.

Proof. Let $\{S_i \mid i \in I\}$ be a partition of $\mathbb{R}^+$ into $2^{\aleph_0}$ dense subsets. The set I is just an index set, so we may suppose that $I = (0\ 1)$. Define $L_i = \{(t, it, 0) \mid t > 0\}$ and $M = \bigcup_{i \in I} L_i$ and let A_i be the union of all spheres with center at the origin and radius from S_i, that is, $A_i = \{x \in \mathbb{R}^3 \mid \|x\| \in S_i\}$. Let $B_i = (A_i \setminus M) \cup L_i$. If S is a sphere centered at the origin, then $S \setminus M$ is a sphere with a small arc removed. Therefore $A_i \setminus M$ is the union of some arcwise connected punctured spheres. Open half-line L_i pastes these punctured spheres together, so B_i is arcwise connected. It is obvious that $\{B_i \mid i \in I\}$ is a partition of $\mathbb{R}^3$ with size $2^{\aleph_0}$. Since S_i is dense in $\mathbb{R}^+$, A_i and consequently B_i are dense in $\mathbb{R}^3$. □

Corollary 1. *There is a partition of $\mathbb{R}^n$ into $2^{\aleph_0}$ arcwise connected dense subsets for $n \geq 3$.*

Proof. It is enough to set $B_i^{(n)} = B_i \times \mathbb{R}^{n-3}$, in which B_i is as above. The collection $\{B_i^{(n)} \mid i \in I\}$ is a partition of $\mathbb{R}^n$ satisfying the claim. □

Note that the union of any number of the sets $B_i^{(n)}$ is an arcwise connected dense subset of $\mathbb{R}^n$, hence:

Corollary 2. *For any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of $\mathbb{R}^n$ $(n \geq 3)$ into β arcwise connected dense subsets.*

Theorem 2. *For any countable cardinal $\alpha > 2$, we cannot partition the plane into α arcwise connected dense subsets.*

Proof. This statement is proved in [Rudin 1963]. □

Lemma 2. *Let X, Y be metric spaces and $T : X \to Y$ be a continuous map.*

(a) *If A is dense in X and T is surjective, then $T(A)$ is dense in Y.*

(b) *If $B \subset X$ is arcwise connected, then $T(B)$ is also arcwise connected.*

Theorem 3. *There is no continuous bijection from $\mathbb{R}$ onto $\mathbb{R}^m$ for $m \neq 1$.*

Proof. Suppose the contrary: Let $g : \mathbb{R} \to \mathbb{R}^m$ be a continuous bijective map. We put $B_n = [-n, n]$, and so we have $\mathbb{R}^m = g\big(\bigcup_{n=1}^{\infty} B_n\big) = \bigcup_{n=1}^{\infty} g(B_n)$. Since $\mathbb{R}^m$ is not in the first category, at least one of the $g(B_n)$, for example $g(B_k)$, has nonempty interior in $\mathbb{R}^m$. Suppose $B(x, r) \subset g(B_k)$. Since B_k is compact, $f : B_k \to g(B_k)$ is a homeomorphism. It follows that $B(x, r)$ is homeomorphic with an interval in $\mathbb{R}$. This is a contradiction, because if we remove 3 points from $B(x, r)$ it remains connected, but this is not the case for the intervals in $\mathbb{R}$. □

Theorem 4. *There is no continuous bijection from $\mathbb{R}^n$ onto $\mathbb{R}^2$ for $n \neq 2$.*

Proof. Suppose the contrary:

(a) If $n > 2$, then according to Corollary 2 and Lemma 2 we can partition $\mathbb{R}^2$ into 3 arcwise connected dense subsets, and this contradicts Theorem 2.

(b) If $n = 1$, then this contradicts Theorem 3. □

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