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of a continuous bijection between
 \mathbb{R}^n and \mathbb{R}^2 for $n \neq 2$

Hamid Reza Daneshpajouh, Hamed Daneshpajouh and Fereshte Malek



An interesting proof of the nonexistence of a continuous bijection between \mathbb{R}^n and \mathbb{R}^2 for $n \neq 2$

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We show that there is no continuous bijection from \mathbb{R}^n onto \mathbb{R}^2 for $n \neq 2$ by an elementary method. This proof is based on showing that for any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of \mathbb{R}^n ($n \geq 3$) into β arcwise connected dense subsets.

1. Introduction

In 1877 Cantor discovered a bijection of \mathbb{R} onto \mathbb{R}^n for any $n \in \mathbb{N}$. Cantor's map was discontinuous, but the discovery of the Peano curve in 1890 showed that there existed continuous (although not injective) maps of \mathbb{R} onto \mathbb{R}^n . Between then and 1910, several mathematicians showed that there does not exist a bicontinuous bijection (homeomorphism) from \mathbb{R}^m onto \mathbb{R}^n for the cases $m = 2$ and $m = 3$ and $n > m$. Finally in 1911, Brouwer showed that there does not exist a homeomorphism between \mathbb{R}^m and \mathbb{R}^n for $n \neq m$ (for a modern treatment, see [Munkres 1984, p. 109]). The present paper proves the nonexistence of a continuous bijection from \mathbb{R}^n onto \mathbb{R}^2 for $n \neq 2$ by an elementary method.

Rudin [1963] showed that for any countable cardinal $\alpha > 2$, we cannot partition the plane into α arcwise connected dense subsets. In this paper we show that for any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of \mathbb{R}^n ($n \geq 3$) into β arcwise connected dense subsets; then by using this we show that there is no continuous bijection from \mathbb{R}^n onto \mathbb{R}^2 for $n \neq 2$.

Lemma 1. *There is a partition of \mathbb{R}^+ into 2^{\aleph_0} dense subsets.*

Proof. Consider the additive group $(\mathbb{R}, +)$. The quotient group \mathbb{R}/\mathbb{Q} has 2^{\aleph_0} elements which are dense subsets of \mathbb{R} . Intersect them with \mathbb{R}^+ . \square

Theorem 1. *There is a partition of \mathbb{R}^3 into 2^{\aleph_0} arcwise connected dense subsets.*

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Proof. Let $\{S_i \mid i \in I\}$ be a partition of \mathbb{R}^+ into 2^{\aleph_0} dense subsets. The set I is just an index set, so we may suppose that $I = (0, 1)$. Define $L_i = \{(t, it, 0) \mid t > 0\}$ and $M = \bigcup_{i \in I} L_i$ and let A_i be the union of all spheres with center at the origin and radius from S_i , that is, $A_i = \{x \in \mathbb{R}^3 \mid \|x\| \in S_i\}$. Let $B_i = (A_i \setminus M) \cup L_i$. If S is a sphere centered at the origin, then $S \setminus M$ is a sphere with a small arc removed. Therefore $A_i \setminus M$ is the union of some arcwise connected punctured spheres. Open half-line L_i pastes these punctured spheres together, so B_i is arcwise connected. It is obvious that $\{B_i \mid i \in I\}$ is a partition of \mathbb{R}^3 with size 2^{\aleph_0} . Since S_i is dense in \mathbb{R}^+ , A_i and consequently B_i are dense in \mathbb{R}^3 . \square

Corollary 1. *There is a partition of \mathbb{R}^n into 2^{\aleph_0} arcwise connected dense subsets for $n \geq 3$.*

Proof. It is enough to set $B_i^{(n)} = B_i \times \mathbb{R}^{n-3}$, in which B_i is as above. The collection $\{B_i^{(n)} \mid i \in I\}$ is a partition of \mathbb{R}^n satisfying the claim. \square

Note that the union of any number of the sets $B_i^{(n)}$ is an arcwise connected dense subset of \mathbb{R}^n , hence:

Corollary 2. *For any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of \mathbb{R}^n ($n \geq 3$) into β arcwise connected dense subsets.*

Theorem 2. *For any countable cardinal $\alpha > 2$, we cannot partition the plane into α arcwise connected dense subsets.*

Proof. This statement is proved in [Rudin 1963]. \square

Lemma 2. *Let X, Y be metric spaces and $T : X \rightarrow Y$ be a continuous map.*

- (a) *If A is dense in X and T is surjective, then $T(A)$ is dense in Y .*
- (b) *If $B \subset X$ is arcwise connected, then $T(B)$ is also arcwise connected.*

Theorem 3. *There is no continuous bijection from \mathbb{R} onto \mathbb{R}^m for $m \neq 1$.*

Proof. Suppose the contrary: Let $g : \mathbb{R} \rightarrow \mathbb{R}^m$ be a continuous bijective map. We put $B_n = [-n, n]$, and so we have $\mathbb{R}^m = g(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} g(B_n)$. Since \mathbb{R}^m is not in the first category, at least one of the $g(B_n)$, for example $g(B_k)$, has nonempty interior in \mathbb{R}^m . Suppose $B(x, r) \subset g(B_k)$. Since B_k is compact, $f : B_k \rightarrow g(B_k)$ is a homeomorphism. It follows that $B(x, r)$ is homeomorphic with an interval in \mathbb{R} . This is a contradiction, because if we remove 3 points from $B(x, r)$ it remains connected, but this is not the case for the intervals in \mathbb{R} . \square

Theorem 4. *There is no continuous bijection from \mathbb{R}^n onto \mathbb{R}^2 for $n \neq 2$.*

Proof. Suppose the contrary:

- (a) If $n > 2$, then according to Corollary 2 and Lemma 2 we can partition \mathbb{R}^2 into 3 arcwise connected dense subsets, and this contradicts Theorem 2.
- (b) If $n = 1$, then this contradicts Theorem 3. \square

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An interesting proof of the nonexistence of a continuous bijection between \mathbb{R}^n and \mathbb{R}^2 for $n \neq 2$	125
HAMID REZA DANESHPAJOUH, HAMED DANESHPAJOUH AND FERESHTE MALEK	
Analysing territorial models on graphs	129
MARIE BRUNI, MARK BROOM AND JAN RYCHTÁŘ	
Binary frames, graphs and erasures	151
BERNHARD G. BODMANN, BIJAN CAMP AND DAX MAHONEY	
On groups with a class-preserving outer automorphism	171
PETER A. BROOKSBANK AND MATTHEW S. MIZUHARA	
The sharp log-Sobolev inequality on a compact interval	181
WHAN GHANG, ZANE MARTIN AND STEVEN WARUHIU	
Analysis of a Sudoku variation using partially ordered sets and equivalence relations	187
ANA BURGERS, SHELLY SMITH AND KATHERINE VARGA	
Spanning tree congestion of planar graphs	205
HIU FAI LAW, SIU LAM LEUNG AND MIKHAIL I. OSTROVSKII	
Convex and subharmonic functions on graphs	227
MATTHEW J. BURKE AND TONY L. PERKINS	
New results on an anti-Waring problem	239
CHRIS FULLER, DAVID R. PRIER AND KARISSA A. VASCONI	