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This paper examines binary codes from a frame-theoretic viewpoint. Binary Parseval frames have convenient encoding and decoding maps. We characterize binary Parseval frames that are robust to one or two erasures. These characterizations are given in terms of the associated Gram matrix and with graph-theoretic conditions. We illustrate these results with frames in lowest dimensions that are robust to one or two erasures. In addition, we present necessary conditions for correcting a larger number of erasures. As in a previous paper, we emphasize in which ways the binary theory differs from the theory of frames for real and complex Hilbert spaces.

1. Introduction

In the last decades, frame theory has matured into a field with relevance in pure and applied mathematics as well as in engineering [Christensen 2003; Kovačević and Chebira 2007a; 2007b]. The simplest examples of frames are finite frames, finite spanning sequences in finite-dimensional real or complex Hilbert spaces. The possibility of having linear dependencies among the frame vectors can be used for error correction when a vector is encoded in terms of its frame coefficients, the inner products with the frame vectors [Goyal et al. 1998]. A common type of error considered in this context is an erasure, when part of the frame coefficients becomes corrupted or inaccessible and one has to recover the encoded vector from partial data [Marshall 1984; 1989]. The performance of frames for decoding erasures was studied, and in certain cases optimal frames could be characterized in a geometric fashion [Casazza and Kovačević 2003; Strohmer and Heath 2003; Holmes and Paulsen 2004; Püschel and Kovačević 2005], which was further extended with graph-theoretic or algebraic methods [Bodmann and Paulsen 2005; Xia et al. 2005; Kalra 2006; Bodmann et al. 2009b; Bodmann and Elwood 2010].

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Apart from the presence of the inner product, one could say that these applications in frame theory are similar to earlier work on error correcting linear codes over finite fields [MacWilliams and Sloane 1977; Betten et al. 2006]. Motivated by the literature in frame theory, a previous paper studied an analogue of Parseval frames in the setting of binary vector spaces [Bodmann et al. 2009a]; see also [Hotovy et al. 2012]. In this paper, we continue this direction of research and ask whether concepts from frame theory yield new insights for binary linear codes. We study how the Gram matrix of a binary frame relates to its robustness, its resilience to erasures. The space spanned by the columns of the Gram matrix is the set of all codewords, so the main question is in which way the robustness of a frame manifests itself. Interpreting the Gram matrix as the adjacency matrix of a graph gives a natural reformulation of conditions for robustness in terms of the connectivity properties of the graph. Note that this graph is different from the so-called Tanner graph of a binary code, which is a bipartite graph associated with the parity check matrix [Betten et al. 2006]. The space of code words is annihilated by the parity check matrix, so one can expect complementary insights from properties of the Gram and Tanner matrices with their associated graphs. While the structure of Tanner graphs has been studied with sophisticated methods in coding theory [Forney 2001; 2003; 2011], the Gram matrix and its role for erasures seems to appear mostly in the literature on frames; see, for example, [Holmes and Paulsen 2004; Bodmann and Paulsen 2005].

The remainder of this paper is structured as follows. In Section 2, we fix notation and define frames and Parseval frames for finite-dimensional binary vector spaces. Section 3 gives a motivation for the use of such frames as binary codes. In Section 4, we study robustness to erasures. Section 5 presents the results on robustness in graph-theoretic terms and gives the smallest frames with robustness to one or two erasures.

2. Preliminaries

We define binary frames and Parseval frames without appealing to the concept of an inner product, as in [Bodmann et al. 2009a]. The vector space that these families of vectors span is of the form $\mathbb{Z}_2^n = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ for some $n \in \mathbb{N}$, with the binary numbers $\mathbb{Z}_2$ as the ground field.

Definition 2.1. A *frame* for $\mathbb{Z}_2^n$ is a family of vectors $\mathcal{F} = \{f_1, \dots, f_k\}$ such that

$$\operatorname{span} \mathcal{F} = \mathbb{Z}_2^n.$$

To define a Parseval frame over $\mathbb{Z}_2^n$, we use a bilinear form that resembles the usual dot product on $\mathbb{R}^n$. For other choices of bilinear forms and a more general theory of binary frames, see [Hotovy et al. 2012].

Definition 2.2. The *dot product* on $\mathbb{Z}_2^n$ is the bilinear map $(\cdot\,,\cdot) : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2$ given by

$$\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}\right) := \sum_{i=1}^{n} a_i b_i .$$

The dot product provides a natural map between vectors and linear functionals on $\mathbb{Z}_2^n$. With the help of this dot product, we define a Parseval frame for $\mathbb{Z}_2^n$.

Definition 2.3. A *Parseval frame* for $\mathbb{Z}_2^n$ is a family of vectors $\mathcal{F} = \{f_1, \ldots, f_k\}$ in $\mathbb{Z}_2^n$ such that

$$x = \sum_{j=1}^{k} (x, f_j) f_j \quad \text{for all } x \in \mathbb{Z}_2^n.$$

According to this definition, a Parseval frame provides a simple, redundant expansion for any vector x in $\mathbb{Z}_2^n$. Unless otherwise noted, when we speak of a frame or of a Parseval frame in this paper, we always mean families of vectors in $\mathbb{Z}_2^n$ with the properties specified in Definitions 2.1 and 2.3, respectively. In the next section, we present a motivating example that explains the design problem of such frames as codes for erasures.

3. Binary frames as codes for erasures

Suppose Alice wants to send Bob a message that consists of a sequence of 0's and 1's. We can represent this message as the column vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{Z}_2^n,$$

where the entries $x_1, x_2, \ldots, x_n$ are the 1st, 2nd, …, n-th digits of the message. Alice is aware that the message is sent through a somewhat unreliable medium, so she decides to *encode* it, that is, convert it into a new message which is generated from a codebook known to both Alice and Bob. The encoded message should have a reasonable chance of withstanding *erasures*, that is, removals of entries in the message that might occur. If the codebook is properly chosen, Bob will be able to recover the original message x from the fragments of the encoded message that remain.

The encoding is a linear map associated with a binary frame. Let the family of vectors $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ be a frame for the vector space $\mathbb{Z}_2^n$, and let

$$\Theta_{\mathcal{F}} = \begin{pmatrix} \leftarrow & f_1 & \rightarrow \\ \leftarrow & f_2 & \rightarrow \\ & \vdots & \\ \leftarrow & f_k & \rightarrow \end{pmatrix} = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,n} \\ \vdots & \vdots & & \vdots \\ f_{k,1} & f_{k,2} & \cdots & f_{k,n} \end{pmatrix},$$

where the entry $f_{i,j}$ is the j-th entry of the i-th vector $f_i \in \mathcal{F}$. Alice encodes her message x by left-multiplying it with the matrix $\Theta_{\mathcal{F}}$. Consequently, Alice's encoded message will be a $k \times 1$ matrix, where the i-th entry is the dot product (x, f_i):

$$\Theta_{\mathcal{F}} x = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,n} \\ \vdots & \vdots & & \vdots \\ f_{k,1} & f_{k,2} & \cdots & f_{k,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (x, f_1) \\ (x, f_2) \\ \vdots \\ (x, f_k) \end{pmatrix}.$$

For convenience, let us abbreviate Alice's encoded message $\Theta_{\mathcal{F}} x$ as

$$\Theta_{\mathcal{F}} x = y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}.$$

A first requirement for the choice of $\mathcal{F}$ is that, if the encoded message arrives unaltered, then Bob can easily extract x from it. If $\mathcal{F}$ is a Parseval frame, then this is indeed the case. In terms of $\Theta_{\mathcal{F}}$, the reconstruction identity in Definition 2.3 is

$$\Theta_{\mathcal{F}}^* \Theta_{\mathcal{F}} = I_n,$$

where $\Theta_{\mathcal{F}}^*$ denotes the transpose of $\Theta_{\mathcal{F}}$.

Imagine at least one entry in the message y gets "erased"; that is, suppose Bob only receives the $r \times 1$ matrix

$$\tilde{y} = \begin{pmatrix} y_{j_1} \\ y_{j_2} \\ \vdots \\ y_{j_r} \end{pmatrix},$$

where $\{j_1, j_2, \dots, j_r\} \subset \{1, 2, \dots, k\}$. For example, if there had been two erasures, then Bob would have received a $(k-2) \times 1$ matrix with two of the original entries in y missing.

The goal is to reconstruct the original message x from the received matrix $\tilde{y}$. This can be achieved by finding an $n \times r$ matrix $\tilde{L}$ such that

$$x = \tilde{L} \begin{pmatrix} y_{j_1} \\ y_{j_2} \\ \vdots \\ y_{j_r} \end{pmatrix}.$$

A notationally more convenient way to formulate this problem is to use the full message without erasures but require reconstruction with a matrix L that has columns of zeros for the erased entries. To see this, let the columns of $\tilde{L}$ be denoted by

$$\tilde{L} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ l_{j_1} & l_{j_2} & \cdots & l_{j_r} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix},$$

and let the entries y_1, y_2, and y_4 in y be erased. Then the matrix L is

$$L = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow \\ 0 & 0 & l_{j_1} & 0 & l_{j_2} & \cdots & l_{j_r} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \end{pmatrix},$$

and there exists L of the above form such that $x = Ly$ if and only if there exists $\tilde{L}$ with $x = \tilde{L}\tilde{y}$. To characterize the requirement on L having columns of zeros, we write $L = LE$, where E is a diagonal 0-1-matrix with a 1 on the diagonal for any digit which gets transmitted and a 0 for every erased digit. With this terminology, we can reformulate the problem of correcting erasures as that of finding *any* L such that $x = LE\Theta_{\mathcal{F}}x$ for each $x \in \mathbb{Z}_2^n$, that is, whether $E\Theta_{\mathcal{F}}$ has a left inverse.

Definition 3.1. Let $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ be a frame for $\mathbb{Z}_2^n$, and let E_J be a diagonal $k \times k$ matrix associated with an erasure of digits indexed by $J \subset \{1, 2, \ldots, k\}$, where $(E_J)_{j,j} = 0$ if $j \in J$ and $(E_J)_{j,j} = 1$ otherwise. We say that the frame $\mathcal{F}$ can *correct* the erasure if $E_J\Theta_{\mathcal{F}}$ has a left inverse. We also say that the erasure of digits indexed by J is *correctable*.

The existence of a left inverse is equivalent to a rank condition and to the spanning property of the family of vectors corresponding to unaffected digits.

Proposition 3.2. *Let $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ be a frame for $\mathbb{Z}_2^n$ and let $J \subset \{1, 2, \ldots, k\}$. The following are equivalent*:

(1) *The erasure of digits indexed by J is correctable.*

(2) *The map $E_J\Theta_{\mathcal{F}}$ is one-to-one.*

(3) *The subfamily $\widetilde{\mathcal{F}} = \{f_j : j \notin J\}$ spans $\mathbb{Z}_2^n$; that is, it is a frame.*

(4) *The matrix $E_J\Theta_{\mathcal{F}}$ has rank n.*

Proof. The equivalence of (1) and (2) is a standard exercise in linear algebra. We next prove the equivalence of (1) and (4). Let $E_J\Theta_F$ have rank n. Since $\mathcal{F}$ is a frame, $k \geq n$. By elementary row operations, $E_J\Theta_F$ can be transformed into reduced row echelon form. However, this sequence of row operations can be obtained by multiplying with a suitable invertible matrix on the left. Thus, there is a $k \times k$ matrix R such that

$$RE_J\Theta_{\mathcal{F}} = \begin{pmatrix} I_n \\ 0_{k-n,n} \end{pmatrix}.$$

Henceforth, we adopt block matrix notation and let I_n denote the $n \times n$ identity matrix and $0_{m,n}$ the $m \times n$ zero matrix with $m, n \in \mathbb{N}$. Next, left multiplying this matrix by $(I_n\ 0_{n,k-n})$ gives

$$(I_n\ 0_{n,k-n})RE_J\Theta_{\mathcal{F}} = I_n.$$

Thus, the required left inverse is $L = (I_n\ 0_{n,k-n})R$. On the other hand, if there is a left inverse for $E_J\Theta_{\mathcal{F}}$ then this matrix must have the maximal possible rank, n.

To see the equivalence of (3) and (4), we observe that $\widetilde{\mathcal{F}}$ is spanning if and only if $\Theta_{\widetilde{\mathcal{F}}}$ has rank n, and the same is true for the matrix $E_J\Theta_{\mathcal{F}}$, where the frame vectors belonging to erased digits have been replaced by zero vectors. $\square$

4. Robustness to erasures

Next, we consider sets of erasures. A natural ordering is to consider erasures of at most one coefficient, then erasures of up to two, etc. A measure for robustness of a frame is how many erasures it can correct.

Definition 4.1. A frame $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ for $\mathbb{Z}_2^n$ is robust to m erasures if $E_J\Theta_{\mathcal{F}}$ has a left inverse for any $J \subset \{1, 2, \ldots, k\}$ of size $|J| \leq m$.

Dimension counting gives a simple necessary condition for the size of a frame robust to m erasures.

Proposition 4.2. *If $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ is a frame for $\mathbb{Z}_2^n$ which is robust to m erasures, then $k \geq n + m$.*

Proof. If $J \subset \{1, 2, \ldots, k\}$ has size $|J| = m$ then by assumption $E_J\Theta_{\mathcal{F}}$ has a left inverse, and the subfamily $\tilde{\mathcal{F}} = \{f_j : j \notin J\}$ spans $\mathbb{Z}_2^n$. Thus, the cardinality of $\tilde{\mathcal{F}}$ is bounded by $|\tilde{\mathcal{F}}| = k - m \geq n$. $\square$

Next, we wish to establish sufficient conditions which ensure robustness. If an erasure indexed by J is not correctable, then $E_J\Theta_{\mathcal{F}}$ is not one-to-one and there exists a nonzero $x \in \mathbb{Z}_2^n$ such that $E_J\Theta_{\mathcal{F}}x = 0$. For Parseval frames, there appears to be a simple condition in terms of an eigenvalue problem for submatrices of the Grammian. We prepare this with a lemma.

Lemma 4.3. *Let A be an $n \times k$ matrix. The matrix AA^* has eigenvalue* 1 *if and only if A^*A has eigenvalue* 1.

Proof. Suppose that A^*A does have an eigenvalue equal to 1. That is, suppose that $A^*Ax = x$. Then $y = Ax$ is nonzero and $AA^*y = AA^*Ax = Ax = y$. Hence, AA^* has an eigenvalue equal to 1. Switching the roles of A and A^* gives the converse. □

Proposition 4.4. *Let $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ be a Parseval frame for $\mathbb{Z}_2^n$ and let $J \subset \{1, 2, \ldots, k\}$. If $E_{J^c}\Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^* E_{J^c}$ does not have eigenvalue one, where J^c is the complement of J in $\{1, 2, \ldots, k\}$, then the erasure is correctable.*

Proof. We use the fact that AA^* has eigenvalue one if and only if A^*A does. Here, $A = E_{J^c}\Theta_{\mathcal{F}} = (I - E_J)\Theta_{\mathcal{F}}$. Assuming there is no eigenvector of eigenvalue one for A^*A means there exists no nonzero x such that

$$\Theta_{\mathcal{F}}^*(I - E_J)(I - E_J)\Theta_{\mathcal{F}}x = \Theta_{\mathcal{F}}^*(I - E_J)\Theta_{\mathcal{F}}x = x.$$

By assumption, $\Theta_{\mathcal{F}}^*\Theta_{\mathcal{F}} = I$, so this implies that there is no $x \neq 0$ with

$$\Theta_{\mathcal{F}}^* E_J \Theta_{\mathcal{F}} x = 0.$$

Consequently, $(\Theta_{\mathcal{F}}^* E_J \Theta_{\mathcal{F}})^{-1}\Theta_{\mathcal{F}}^* E_J \Theta_{\mathcal{F}} = I$ and the required left inverse of $E_J\Theta_{\mathcal{F}}$ is

$$L = (\Theta_{\mathcal{F}}^* E_J \Theta_{\mathcal{F}})^{-1}\Theta_{\mathcal{F}}^*.$$ □

At first glance, robustness against one erasure would motivate the search for frames whose vectors contain only an even number of ones, because then the diagonal of the Gram matrix $\Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^*$ would be zero, avoiding the eigenvalue one condition. However, such frames do not exist because any linear combination of vectors with an even number of ones still has an even number of ones. Thus, a family of such vectors cannot be spanning for all of $\mathbb{Z}_2^n$.

In addition, the above eigenvalue condition is sufficient for recovery, but not necessary. We present an example for this:

Example 4.5. Let $n = 1$, $\mathcal{F} = \{1, 1, 1\}$, and $J = \{2, 3\}$. The encoded "vector" $x \in \{0, 1\}$ is $\Theta_{\mathcal{F}}x = (x\ x\ x)^*$, and thus $E_J\Theta_{\mathcal{F}}$ has the left inverse $(1\,0\,0)$. However, $E_{J^c}\Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^* E_{J^c} = E_{J^c}$ has eigenvalue one.

This motivates the search for a more general condition which ensures robustness. To this end, we introduce a function counting the number of 1's in a vector, the (Hamming) weight.

Definition 4.6. A vector $x \in \mathbb{Z}_2^n$ has *weight* $w(x) = |\{j : x_j = 1\}|$. We also speak of the *parity* of a vector, which is even or odd, depending on whether the weight is an even or an odd number.

Theorem 4.7. *Let $\mathcal{F}$ be a Parseval frame with Gram matrix $G = \Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^*$. The frame $\mathcal{F}$ is robust to m erasures if and only if all the eigenvectors of G corresponding to eigenvalue one have a weight of at least $m+1$.*

Proof. If $\mathcal{F}$ is a Parseval frame, then any eigenvector of eigenvalue one of the Gram matrix is a possible message and vice versa. This is true because if $y = \Theta_{\mathcal{F}}x$ then $\Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^* y = y$ and conversely if y is an eigenvector of eigenvalue one then $y = \Theta_{\mathcal{F}}x$ for $x = \Theta_{\mathcal{F}}^* y$.

Assume that each such eigenvector has weight at least $m+1$. If $|J| \leq m$, then applying E_J to y can only change at most m ones to zero, so $E_J y \neq 0$ and thus $E_J\Theta_{\mathcal{F}}x \neq 0$ unless $x = 0$. This proves that $E_J\Theta_{\mathcal{F}}$ is one-to-one.

Conversely, given a nonzero message y, if for each $J \subset \{1, 2, \ldots, k\}$ with $|J| \leq m$ we have $E_J y \neq 0$, then y must have weight at least $m+1$. □

It is implicit in this characterization that the robustness of a frame against erasures is determined by the Gram matrix. If two frames have the same Gram matrix, then the two frames have identical robustness. Since the weight of a vector is invariant under permutations of its entries, the same holds if the Gram matrices differ only by a permutation of rows and columns. This means that the search for robust frames can be restricted to representatives of equivalence classes introduced in [Bodmann et al. 2009a].

Definition 4.8. Two frames $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ and $\mathcal{G} = \{g_1, g_2, \ldots, g_k\}$ for $\mathbb{Z}_2^n$ are called *switching equivalent* if there is a binary $n \times n$ matrix U such that $U^*U = UU^* = I$ and a permutation σ on the set $\{1, 2, \ldots, k\}$ such that $f_j = Ug_{\sigma(j)}$ for all $j \in \{1, 2, \ldots, k\}$.

Theorem 4.9. *If two frames $\mathcal{F}$ and $\mathcal{G}$ for $\mathbb{Z}_2^n$ are switching equivalent, then $\mathcal{F}$ is robust to m erasures if and only if $\mathcal{G}$ is.*

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are switching equivalent, then the Gram matrices of $\mathcal{F}$ and $\mathcal{G}$ differ by a permutation of rows and columns. The same is true for the eigenvectors corresponding to eigenvalue one. However, the weight of the eigenvectors is invariant under permutation of coordinates. This means, according to the preceding theorem, if $\mathcal{F}$ is robust to m erasures, so is $\mathcal{G}$, and vice versa. □

In the context of real or complex Hilbert spaces, equal-norm frames characterize optimality for one erasure among Parseval frames [Casazza and Kovačević 2003]. In the binary setting, the equal-norm condition would correspond to a frame in which the vectors all have the same parity. Linear combinations of even vectors remain even, so there cannot be a frame consisting only of vectors having even parity, which leaves only the possibility of Parseval frames having only odd vectors. However, we show below that such frames have severe limitations for their robustness. We prepare this with a lemma which is essentially a result in [Haemers et al. 1999, Lemma 2.2].

Lemma 4.10. *Let $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ be a Parseval frame and G the associated Gram matrix; then the vector y with entries $y_j = G_{j,j}$ for $j \in \{1, 2, \ldots, k\}$ is an eigenvector of G corresponding to eigenvalue one.*

Proof. Since G is idempotent, is enough to show that $y_i = G_{i,i}$ defines a vector in the range of G. To see this, let $(\operatorname{ran} G)^\perp = \{x \in \mathbb{Z}_2^k, (x, z) = 0 \text{ for all } z \in \operatorname{ran} G\}$ and recall $((\operatorname{ran} G)^\perp)^\perp = \operatorname{ran} G$ because $\operatorname{ran} G \subset ((\operatorname{ran} G)^\perp)^\perp$ by definition and $\dim(\operatorname{ran} G)^\perp + \dim \operatorname{ran} G = k$. If $x \in (\operatorname{ran} G)^\perp$, then, by setting $z = Gx$ and binary arithmetic, $0 = (z, x) = \sum_{i,j=1}^k G_{i,j} x_i x_j = \sum_{j=1}^k G_{j,j} x_j$. Thus, $(x, y) = 0$ for each such x, and y is necessarily in $\operatorname{ran} G$. □

Next, we examine how many erasures a binary Parseval frame can possibly correct. It turns out that, in some cases, the inequality necessary for correcting all m-erasures, $k \geq n + m$, can be strengthened considerably.

Theorem 4.11. *If $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ is a Parseval frame of which p vectors are odd, then the frame cannot be robust to more than* $\min\{p - 1, k - p/2 - 1\}$ *erasures.*

Proof. We recall that the vector y defined by $y_j = G_{j,j}$, the diagonal of the Gram matrix G, is an eigenvector of G corresponding to eigenvalue one, and that it has weight p. It is clear that the frame cannot correct more than $p - 1$ erasures. On the other hand, assume that the minimal weight q among the vectors in the range of G is assumed by x, so $p \geq q$. The vector $z = x + y$ is then also in the range of G. Define $\Delta = q + p - k$; then the two vectors have at least Δ indices in common for which the entries of both vectors are one. Thus, the weight of z is bounded by $q \leq w(z) \leq q - \Delta + p - \Delta = 2k - q - p$. This inequality gives $q \leq k - p/2$. □

This result shows that binary Parseval frames containing only odd vectors, the binary analogue of real or complex equal-norm Parseval frames, have a severe limitation for robustness.

Corollary 4.12. *If $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ is a Parseval frame which consists only of odd vectors, then it cannot correct more than $k/2 - 1$ erasures.*

Moreover, maximizing the upper bound for robustness yields that a binary Parseval frame achieves the best possible robustness when $p - 1 = k - p/2 - 1$, so $p = 2k/3$.

Corollary 4.13. *If $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ is a Parseval frame for $\mathbb{Z}_2^n$, then it cannot correct more than $2k/3 - 1$ erasures.*

5. Binary Parseval frames, graphs and erasures

With a binary symmetric $k \times k$ matrix A, we associate a graph γ on k vertices. An entry $A_{i,j} = 1$ means there is an edge connecting vertices i and j; otherwise there is no edge between them. If $A_{i,i} = 1$, then vertex i has a loop, and we say that i is adjacent to itself; otherwise, i has no loop. The graph γ determines the matrix A, often called its adjacency matrix. We characterize binary Parseval frames in terms of the adjacency structure of the graph associated with the Gram matrix.

Theorem 5.1. *If $\mathcal{F}$ is a binary frame and $G = \Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^*$ is its Gram matrix, then $\mathcal{F}$ is a Parseval frame if and only if all of the following conditions hold for the graph γ associated with G:*

(1) *Every vertex i has an even number of neighbors in the set $\{1, 2, \dots, k\}\backslash\{i\}$.*

(2) *If two vertices of γ are not adjacent, then the two vertices have an even number of common neighbors.*

(3) *If two vertices of γ are adjacent, then the two vertices have an odd number of common neighbors.*

Proof. First, suppose $\mathcal{F}$ is Parseval. Then $G^2 = \Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^*\Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^* = \Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^* = G$. From this, we conclude that the three properties are true.

(1) Let $G_{i,i} = 1$. Then $\sum_j G_{i,j}G_{j,i} = \sum_j G_{i,j} = 1$. Hence, $\sum_{j,j\neq i} G_{i,j} = 0$. On the other hand, let $G_{i,i} = 0$. Then $\sum_j G_{i,j}G_{j,i} = 0$, and consequently $\sum_{j,j\neq i} G_{i,j} = 0$. Thus, any vertex i has an even number of neighbors in the set of vertices not including i.

(2) If two vertices j and k, $j \neq k$, are nonadjacent then $0 = G_{j,k} = \sum_l G_{j,l}G_{l,k}$. The nodes j and k then necessarily have an even number of common neighbors.

(3) If vertices j and k are adjacent nodes then $1 = G_{j,k} = \sum_l G_{j,l}G_{l,k}$ and they have an odd number of common neighbors.

On the other hand, if these three properties hold then $G^2 = G$ can be verified by a similar discussion of entries on the diagonal and on the off-diagonal: The property (1) implies that $G_{i,i} = (G^2)_{i,i}$, while (2) and (3) imply $G_{j,k} = (G^2)_{j,k}$. If $\mathcal{F}$ is a frame, then the matrix $\Theta_{\mathcal{F}}$ has rank n. Thus by appropriate elementary row operations it can be transformed into the row-reduced echelon form. These row operations amount to left multiplication with an invertible matrix R, $R\Theta_{\mathcal{F}} = \binom{I_n}{0_{n-k,n}}$, and consequently $\Theta_{\mathcal{F}}^* R^* = (I_n \; 0_{n,n-k})$. If $G^2 = G$, then

$$RG^2R^* = \begin{pmatrix} I_n \\ 0_{n-k,n} \end{pmatrix} \Theta_{\mathcal{F}}^*\Theta_{\mathcal{F}}(I_n \; 0_{n,n-k}) = \begin{pmatrix} I_n & 0_{n,n-k} \\ 0_{n-k,n} & 0_{k,k} \end{pmatrix} = RGR^*,$$

and the middle equality shows that $\Theta_{\mathcal{F}}^*\Theta_{\mathcal{F}} = I_n$, so $\mathcal{F}$ is Parseval. $\square$

A graph that satisfies conditions (2) and (3) of Theorem 5.1 is not a strongly regular graph since the exact number of common neighbors may fluctuate between pairs of adjacent vertices and between pairs of nonadjacent vertices. However, since the number of common neighbors remains even or odd between pairs of adjacent or nonadjacent vertices, respectively, we propose the term *strongly parity regular graph* to refer to graphs that satisfy (2) and (3) of Theorem 5.1.

Next, we discuss graph-theoretic criteria for robustness to erasures. With Theorem 4.7, we have a characterization of robustness to m erasures in terms of the weights of the eigenvectors of the Gram matrix G corresponding to eigenvalue one. Because of the relation $G^2 = G$, these are precisely the vectors in the range of G. We can deduce a simple necessary and sufficient condition for the graph associated with a Parseval frame that is robust to one or two erasures.

Theorem 5.2. *Let $\mathcal{F}$ be a Parseval frame for $\mathbb{Z}_2^n$, G its Gram matrix and γ the associated graph. The frame $\mathcal{F}$ is robust to one erasure if and only if every vertex of γ has at least two neighbors other than itself and is part of a cycle of length at most* 4.

Proof. First, we prove that robustness against one erasure implies the graph-theoretic properties. From the Parseval property, we know that each vertex has an even number of neighbors other than itself. If we pick a vertex i then the neighbors of it are encoded in the i-th column of the Gram matrix G. On the other hand, this column vector is in the range of G. If the frame corrects one erasure, then this vector must have at least weight two. Consequently, each vertex has to have at least two neighbors other than itself in order to correct one erasure.

Given a vertex i and two of its neighbors j and l, $i \neq j \neq l \neq i$, then either the vertices j and l are adjacent and i is part of a 3-cycle, or they are not adjacent. In this case, j and l have an even number of common neighbors, so there is another vertex i' adjacent to j and l. Thus i, j, i' and l form a 4-cycle.

Next, we prove that the graph-theoretic properties ensure robustness against one erasure. For this, we only need to make the weaker assumption that each vertex has a neighbor other than itself. We note that a one-erasure not being correctable requires that there is a vector e_l from the standard basis, with some $l \in \{1, 2, \ldots, k\}$, such that $Ge_l = e_l$. This implies that $G_{j,l} = \delta_{j,l}$ for all j, so the l-th vertex is only a neighbor to itself. This is excluded by the assumption. □

Additional conditions characterize robustness against two erasures.

Definition 5.3. We say that a vertex i *discriminates* between two other vertices j and l if it is a neighbor to only one of them. We also say that the pair $\{j, l\}$ has a *discriminating vertex i*.

Theorem 5.4. *Let $\mathcal{F}$ be a Parseval frame for $\mathbb{Z}_2^n$, G its Gram matrix and γ the associated graph. The frame $\mathcal{F}$ is robust to two erasures if and only if the conditions for correcting one erasure hold and if every nonadjacent pair of vertices that are both adjacent to themselves and every adjacent pair of vertices that are both nonadjacent to themselves have a discriminating vertex.*

Proof. We first note that the graph-theoretic conditions in the preceding theorem are implied by robustness against two erasures which is a stronger requirement than correcting all one-erasures.

Next, we recall that Theorem 4.7 characterizes robustness in terms of the existence of certain eigenvectors. Assuming robustness against 1 erasure, an erasure of $m = 2$ digits is not correctable if and only if there is a pair $\{l, l'\}$ and $h = e_l + e_{l'}$ satisfying $Gh = h$. Then, $G_{l,l} = G_{l',l'} = 1$ and $G_{l,l'} = 0$ or $G_{l,l} = G_{l',l'} = 0$ and $G_{l,l'} = 1$. The first case corresponds to two nonadjacent vertices that are neighbors to themselves and the second one is a pair of adjacent vertices that are not neighbors to themselves. In both cases, the eigenvalue equation requires that $G_{j,l} = G_{j,l'}$ for all $j \notin \{l, l'\}$. This means if a vertex j is adjacent to l then it is adjacent to l' and vice versa. We conclude that the eigenvalue equation is satisfied by h if and only if there is no vertex which discriminates between l and l'. Hence, all erasures of $m = 2$ indices are correctable if and only if all one-erasures are and if there is a discriminating vertex for any nonadjacent pair of vertices that are both adjacent to themselves and any adjacent pair of vertices that are both nonadjacent to themselves. □

To illustrate these results, we use them to identify binary Parseval frames in 3 and 4 dimensions that achieve robustness to one or two erasures. We briefly mention that the canonical basis vectors form a Parseval frame that cannot correct any erasure, because they are minimal spanning sets. This means our search starts with 4 vectors in $\mathbb{Z}_2^3$ and 5 vectors in $\mathbb{Z}_2^4$. Removing zero vectors from a frame does not affect the robustness as well as the Parseval property, so we can restrict ourselves to binary Parseval frames which do not contain zero vectors. Apart from zero vectors, identical pairs of vectors do not contribute to the reconstruction identity in Definition 2.3, which can be interpreted as a trivial form of incorporating redundancy in the encoding.

Definition 5.5 [Bodmann et al. 2009a]. A binary Parseval frame $\{f_1, f_2, \ldots, f_k\}$ for $\mathbb{Z}_2^n$ is called *trivially redundant* if there is $j \in \{1, 2, \ldots k\}$ with $f_j = 0$, or if there are two indices $i \neq j$ with $f_i = f_j$.

We restrict our study of robustness to binary Parseval frames that are not trivially redundant. This implies an upper bound on the number of frame vectors:

Theorem 5.6 [Bodmann et al. 2009a]. *Let $n \geq 3$. Let $\mathcal{F} = \{f_i\}_{i=1}^k$ be a family without repeated vectors in $\mathbb{Z}_2^n$ and $\mathcal{G} = \mathbb{Z}_2^n \setminus \mathcal{F}$. If $\mathcal{F}$ is a Parseval Frame for $\mathbb{Z}_2^n$, then $\mathcal{G}$ is also a Parseval frame.*

Corollary 5.7. *If $n \geq 3$ and $\mathcal{F} = \{f_i\}_{i=1}^k$ is not trivially redundant, then $k \leq 2^n - n - 1$.*

Proof. If $\mathcal{F}$ is Parseval, then so is $\mathcal{G}$. Removing the zero vector from $\mathcal{G}$ gives a spanning set $\mathcal{G} \setminus \{0\}$, so it has at least n vectors. The union of $\mathcal{F}$ and $\mathcal{G} \setminus \{0\}$ has a total of $2^n - 1$ vectors, so comparing sizes gives $k + n \leq 2^n - 1$. □

Switching equivalence allows a further simplification of the search. Since the robustness is the same for all representatives of a switching equivalence class, we can extract frames which are robust to one or two erasures from the classification of binary Parseval frames for $\mathbb{Z}_2^3$ and $\mathbb{Z}_2^4$ that are not trivially redundant [Bodmann et al. 2009a].

In $n = 3$ dimensions, the above corollary limits the number of vectors in a binary Parseval frame that is not trivially redundant by $k \leq 2^3 - 3 - 1 = 4$. Up to switching equivalence, there are only two such binary Parseval frames for $\mathbb{Z}_2^3$: the canonical basis with 3 vectors and a binary Parseval frame with 4 vectors [ibid.]. Robustness to one erasure rules out the canonical basis, which leaves the case of 4 vectors. We examine the graph belonging to this Parseval frame, for readability purposes labeling vertices by the corresponding rows in $\Theta_{\mathcal{F}}$.

Example 5.8. The Parseval frame $\mathcal{F}$ for $\mathbb{Z}_2^3$ with encoding matrix

$$\Theta_{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

cannot correct one erasure because the graph associated with $\Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^*$ has an isolated vertex, as shown in Figure 1.

By the limit on the number of vectors, a Parseval frame for $\mathbb{Z}_2^3$ which is robust to one erasure contains at least one repeated vector. We do not pursue this any further because it is a case of trivial redundancy.

We proceed to $n = 4$. Here, the corollary limits the size of the frames we consider to $k \leq 2^4 - 4 - 1 = 11$ vectors. As above, any graph with an isolated vertex prevents robustness to one erasure. This happens for the switching equivalence class of binary Parseval frames of 5 vectors for $\mathbb{Z}_2^4$.

Example 5.9 [Bodmann et al. 2009a]. A Parseval frame $\mathcal{F}$ for $\mathbb{Z}_2^4$ with 5 vectors is, up to switching equivalence, given by

$$\Theta_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

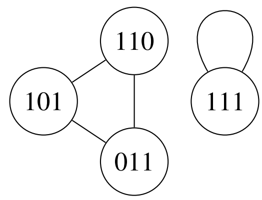

Figure 1. The graph associated with the Parseval frame of 4 vectors in $\mathbb{Z}^3$ given in Example 5.8. Vertices are labeled by the corresponding rows of the encoding matrix. The presence of the isolated vertex (1 1 1) implies that this frame cannot correct one erasure.

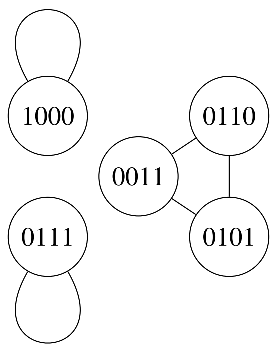

Figure 2. The graph belonging to the Parseval frame of 5 vectors in $\mathbb{Z}^4$ given in Example 5.9 has the isolated vertices (1 0 0 0) and (0 1 1 1), so an erasure of the first frame coefficient or of the last one cannot be corrected.

The graph associated with the Gram matrix has two isolated vertices as shown in Figure 2, so the frame cannot correct one erasure.

Next, we identify a smallest binary Parseval frame for $\mathbb{Z}_2^4$ which is not trivially redundant and can correct one erasure. There is only one switching equivalence class of Parseval frames for $\mathbb{Z}_2^4$ containing 6 vectors [Bodmann et al. 2009a], so it is enough to investigate one representative.

Example 5.10. Let $\mathcal{F}$ be the Parseval frame for $\mathbb{Z}_2^4$ with

$$\Theta_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

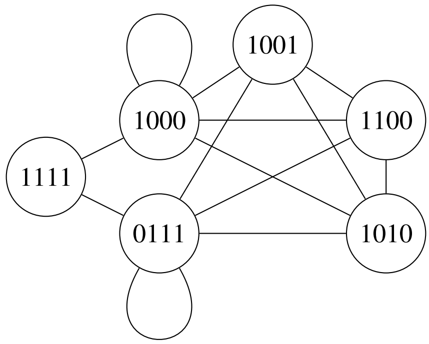

Figure 3. The graph belonging to the Parseval frame of 6 vectors in $\mathbb{Z}^4$ given in Example 5.10. Its adjacency structure satisfies the conditions in Theorem 5.2; thus it can correct one erasure. However, $\mathscr{F}$ is not robust to two erasures since no vertices discriminate between the nonadjacent vertices (0 1 1 1) and (1 0 0 0) which are both adjacent to themselves.

The graph of $\Theta_{\mathscr{F}}\Theta_{\mathscr{F}}^*$ satisfies the conditions for correcting one erasure stated in Theorem 5.2, which can be confirmed by inspecting Figure 3. However, it cannot correct more than one because it fails the requirement of discriminating vertices stated in Theorem 5.4.

The next larger Parseval frames form again a unique switching equivalence class [Bodmann et al. 2009a]. They fail to be robust to two erasures as well.

Example 5.11. Let $\mathscr{F}$ be the binary Parseval frame for $\mathbb{Z}_2^4$ containing seven vectors with

$$\Theta_{\mathscr{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The associated graph shown in Figure 4 satisfies the conditions of Theorem 5.2, but fails the conditions for correcting more than one, as described in Theorem 5.4.

Up to switching equivalence, the next example is the smallest binary Parseval frame for $\mathbb{Z}_2^4$ which is not trivially redundant and can correct 2 erasures.

Example 5.12. Consider the binary Parseval frame $\mathscr{F}$ for $\mathbb{Z}_2^4$ with 8 vectors given

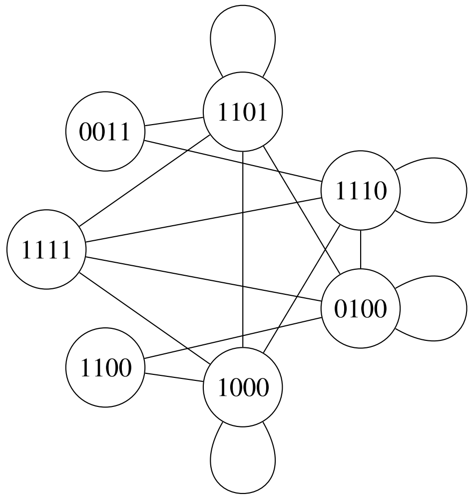

Figure 4. The graph associated with the Parseval frame of 7 vectors in $\mathbb{Z}_2^4$ given in Example 5.11. It satisfies the connectivity conditions for correcting one erasure, but fails to be robust to two erasures because the nonadjacent vertices (1 1 0 1) and (1 1 1 0) are both adjacent to themselves and do not have any discriminating vertex.

by the matrix

$$\Theta_{\mathcal{F}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The associated graph shown in Figure 5 satisfies the conditions of Theorem 5.4, so it can correct up to two erasures.

Finally, we provide necessary conditions for correcting m-erasures, which require increased connectivity.

Theorem 5.13. *Let $\mathcal{F}$ be a Parseval frame for $\mathbb{Z}_2^n$, G its Gram matrix and γ the associated graph. If $\mathcal{F}$ is robust to $m \geq 1$ erasures, then every vertex has at least $m+1$ neighbors, possibly including itself, and it is part of at least $m(m-1)/2$ cycles of length at most* 4.

Proof. This condition follows again from the weights of the columns of G. If a vertex i is adjacent to itself then it needs at least m edges to other vertices. If it is

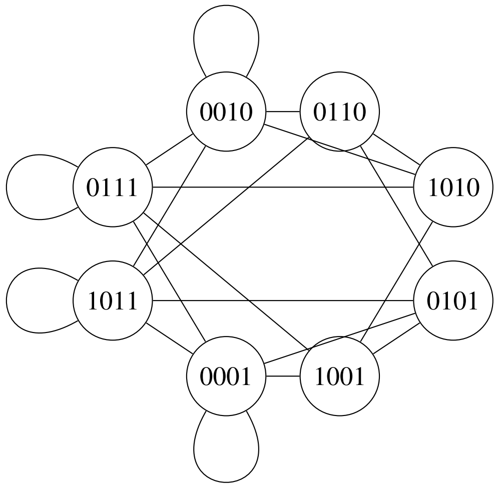

Figure 5. The graph associated with the Parseval frame of 8 vectors in $\mathbb{Z}_2^4$ given in Example 5.12. It can correct up to two erasures because it satisfies the conditions of Theorem 5.4. For example, the vertex (0 1 1 0) discriminates between the nonadjacent vertices (1 0 1 1) and (0 1 1 1), and the vertex (0 0 1 0) discriminates between the adjacent vertices (1 0 1 0) and (1 0 0 1).

not adjacent to itself, it requires $m+1$ edges. Thus, there are at least $m(m-1)/2$ pairs of edges to other vertices. Any pair of such edges leads to either an adjacent pair or to a nonadjacent pair of vertices. If the pair is adjacent, then it forms a 3-cycle with the vertex i. Otherwise, the nonadjacent pair has a common neighbor other than the vertex i, forming a 4-cycle as in Theorem 5.2. □

Such necessary conditions are useful when searching for binary Parseval frames that are maximally robust. This could, in principle, be done by enumerating all Parseval frames and by testing their robustness against erasures exhaustively. The properties of the examples we have examined in $\mathbb{Z}_2^3$ and $\mathbb{Z}_2^4$ would, for example, be accessible by studying linear dependencies among the frame vectors. However, because of the combinatorial nature of robustness, it is advantageous for the search in higher dimensions if testing can be restricted to the subset of Parseval frames satisfying the necessary conditions.

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