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The sharp log-Sobolev inequality
on a compact interval

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(Communicated by Kenneth S. Berenhaut)

We provide a proof of the sharp log-Sobolev inequality on a compact interval.

1. Introduction

The Gaussian log-Sobolev inequality, due to A. J. Stam [1959, Equation 2.3] or Paul Federbush [1969, Equation (14)], although often attributed to L. Gross [1975, Corollary 4.2], played a crucial role in Perelman's proof [2002] of the Poincaré conjecture. We consider log-Sobolev inequalities for finite Lebesgue measure. F. Maggi [Morgan 2009] observed that the sharp log-Sobolev inequality on the interval follows from an isoperimetric conjecture of Díaz et al. [2012], which remains open, but provided no proof. We found it in [Wang 1999], which cited Deuschel and Stroock [1990], who gave a proof of the sharp log-Sobolev inequality on the circle. We then traced this result back to [Émery and Yukich 1987, page 1; Rothaus 1980, Theorem 4.3; Weissler 1980, Theorem 1]. Our Theorem 2.2 shows that the interval case follows quickly from the circle case.

2. Log-Sobolev inequality on a compact interval

In considering the isoperimetric problem in sectors of the plane with density r^p , Díaz et al. [2012, Corollary 4.24, Conjecture 4.18] conjectured the inequality

$$\left[\int_0^1 r^q d\alpha \right]^{1/q} \leq \int_0^1 \sqrt{r^2 + (q-1) \frac{r'^2}{\pi^2}} d\alpha, \quad (1)$$

where $1 < q \leq 2$. F. Maggi [Morgan 2009] observed that (1) implies the log-Sobolev inequality of Theorem 2.2. Here we observe that Theorem 2.2 follows from a proposition of Weissler.

MSC2010: primary 46; secondary 53.

Keywords: isoperimetric inequality, log-Sobolev inequality.

Proposition 2.1 [Weissler 1980, Theorem 1]. *Let f be a nonnegative C^1 function on the circle S^1 of length 1. Suppose $\int_{S^1} f^2 = 1$. Then we have the sharp inequality*

$$4\pi^2 \int_{S^1} f^2 \log f \leq \int_{S^1} f'^2.$$

Various proofs are discussed in Section 3.

Theorem 2.2. *Let f be a nonnegative C^1 function on the interval $[0, 1]$. Suppose $\int_0^1 f^2 = 1$. Then we have the inequality*

$$\pi^2 \int_0^1 f^2 \log f \leq \int_0^1 f'^2. \quad (2)$$

Proof. Let f be any nonnegative C^1 function on $[0, 1]$ such that $\int_0^1 f^2 = 1$. Define a nonnegative piecewise C^1 function g on S^1 such that

$$g(x) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f(2-2x) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then $\int_{S^1} g^2 = 1$. By smoothing, Proposition 2.1 applies to g . By simple computation, we have that

$$\int_{S^1} g^2 \log g = \int_0^1 f^2 \log f \quad \text{and} \quad \int_{S^1} g'^2 = 4 \int_0^1 f'^2.$$

The conclusion follows. □

Remark 2.3. Feng-Yu Wang [1999, Example 1.2] suggested an alternative proof of (2), but we don't understand his proof. He considered densities $C_\epsilon \exp(\epsilon \cos \pi x)$ and functions $f_\epsilon = \exp(-\epsilon \cos \pi x)$, with C_ϵ chosen to make the integral of f_ϵ^2 equal to 1. Then f_ϵ satisfies the differential equation

$$f_\epsilon'' - \pi \epsilon \sin \pi x f_\epsilon' = -\pi^2 f_\epsilon \log f_\epsilon. \quad (3)$$

He said that it follows that (2) holds for those functions and densities with sharp constant π^2 . This might follow if it were known that functions realizing equality exist, but Wang himself [1999, page 655] admits that “the author is not sure yet whether there always exists [such a function].” Indeed, in the case of the circle with unit density, there apparently is no such function. Of course, the sharp inequality for density 1 would follow as ϵ approaches 0.

A similar result holds on the interval $[a, b]$ for a function with root mean square m .

Corollary 2.4. *Let f be a nonnegative C^1 function on the interval $[a, b]$. Suppose*

$$\frac{1}{b-a} \int_a^b f^2 = m^2, \quad m > 0.$$

Then we have the inequality

$$\frac{\pi^2}{(b-a)^2} \left(\int_a^b f^2 \log f - (b-a)m^2 \log m \right) \leq \int_a^b f'^2. \quad (4)$$

Proof. Let f be a nonnegative C^1 function on the interval $[a, b]$ such that

$$\frac{1}{b-a} \int_a^b f^2 = m^2 > 0, \quad m > 0.$$

Define a function g on the interval $[0, 1]$ as

$$g(x) = \frac{1}{m} f((b-a)x + a).$$

Then g is nonnegative and C^1 . Moreover, we have

$$\int_0^1 g(x)^2 dx = \int_0^1 \frac{1}{m^2} f((b-a)x + a)^2 dx = \frac{1}{(b-a)m^2} \int_a^b f(y)^2 dy = 1.$$

Therefore, we can apply Theorem 2.2 to the function g . We have

$$\frac{\pi^2}{b-a} \int_0^1 g^2 \log g \leq (b-a) \int_0^1 g'^2. \quad (5)$$

Note that

$$g'(x) = \frac{b-a}{m} f'((b-a)x + a).$$

By direct calculation, we have

$$\int_0^1 g'(x)^2 dx = \frac{(b-a)^2}{m^2} \int_0^1 f'((b-a)x + a)^2 dx = \frac{(b-a)^2}{m^2} \int_a^b f'(x)^2 dx.$$

We also have

$$\begin{aligned} \int_0^1 g(x)^2 \log g(x) dx &= \frac{1}{m^2} \int_0^1 f((b-a)x + a)^2 \log \frac{f((b-a)x + a)}{m} dx \\ &= \frac{1}{(b-a)m^2} \int_a^b f(x)^2 \log \frac{f(x)}{m} dx \\ &= \frac{1}{(b-a)m^2} \int_a^b f^2 (\log f - \log m) \\ &= \frac{1}{(b-a)m^2} \left(\int_a^b f^2 \log f - (b-a)m^2 \log m \right). \end{aligned}$$

Therefore, by plugging these identities into (5), we have

$$\frac{\pi^2}{(b-a)m^2} \left(\int_a^b f^2 \log f - (b-a)m^2 \log m \right) \leq \frac{b-a}{m^2} \int_a^b f'^2.$$

This is equivalent to the desired inequality (4). □

Corollary 2.4 can be written in the following form.

Corollary 2.5. *Let f be a nonnegative C^1 function on the interval $[a, b]$. Suppose*

$$\frac{1}{b-a} \int_a^b f = m > 0.$$

Then we have the inequality

$$\frac{2\pi^2}{(b-a)^2} \left(\int_a^b f \log f - m \log m \right) \leq \int_a^b \frac{f'^2}{f}.$$

Proof. Define a nonnegative piecewise C^1 function g on the interval $[a, b]$ as $g = \sqrt{f}$. Plugging g into Corollary 2.4 yields the desired result. □

Proposition 2.6. *In Theorem 2.2, π^2 is the best possible constant.*

Proof. For any $0 < \epsilon < 1$, define

$$f_\epsilon(x) = \sqrt{1 - \epsilon^2} + \sqrt{2}\epsilon \cos \pi x.$$

Then by direct computation, we have

$$\lim_{\epsilon \rightarrow 0^+} \frac{\int_0^1 f_\epsilon'^2}{\int_0^1 f_\epsilon^2 \log f_\epsilon} = \pi^2.$$

Therefore, the constant π^2 cannot be replaced by a larger constant. □

Remark 2.7. The function $\cos \pi x$ comes from the equality case of a Wirtinger inequality which follows from the log-Sobolev inequality [Morgan 2009].

3. Proofs of the sharp log-Sobolev inequality on the circle

We summarize three proofs of Proposition 2.1 given by Rothaus [1980, Theorem 4.3], Weisler [1980, Theorem 1], Émery and Yukich [1987, page 1], and Deuschel and Stroock [1990, Remark 1.14 (i)].

3.1. Weisler’s proof. Weisler proved a stronger result than Proposition 2.1 by Fourier expansion of functions of period 2π .

Proposition 3.1 [Weisler 1980, Theorem 1]. *Let $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be in L^2 and suppose $f(\theta) \geq 0$ almost everywhere. Then*

$$\int f^2 \log f \leq \sum_{n=-\infty}^{\infty} |n| |a_n|^2 + \|f\|_2^2 \log \|f\|_2$$

in the sense that if the right-hand side is finite, then so is the left-hand side and the inequality holds. ($0^2 \log 0$ is taken to be 0.)

Obviously the above inequality is stronger than the inequality

$$\int f^2 \log f \leq \sum_{n=-\infty}^{\infty} |n|^2 |a_n|^2 + \|f\|_2^2 \log \|f\|_2,$$

which is equivalent to Proposition 2.1 by change of variables as in Corollary 2.4.

Weissler [1980] cited [Rothaus 1978] but did not have [Rothaus 1980], where Rothaus gave his proof of Proposition 2.1.

3.2. Rothaus's proof. Rothaus proved Proposition 2.1 by a variational method. (References in this section are relative to [Rothaus 1980].) He considered an equivalent problem with a positive parameter ρ in Section 4. If a related constant b_ρ is zero, then the log-Sobolev inequality on the circle with the constant $2/\rho$ holds. For each b_ρ , he showed in Theorem 4.2 that a minimizing function exists, is positive and satisfies a related differential equation. Moreover, for $\rho > 1/2\pi^2$, the only positive solution to the differential equation is the constant function 1 (Theorem 4.3) and hence b_ρ is zero. Therefore in the limit $b_{1/2\pi^2}$ is zero, and our Proposition 2.1 follows.

Rothaus cited [Weissler 1980], saying that “a result related to Theorem 6.3 appears in” that paper.

3.3. Émery and Yukich's proof. Proposition 2.1 was proved by Émery and Yukich [1987, page 1] by using estimates deploying the Brownian motion semigroup.

Émery and Yukich [1987] cited both Weissler [1980] and Rothaus [1980].

3.4. Deuschel and Stroock's proof. Deuschel and Stroock considered the log-Sobolev inequality in general spaces with densities. As a special case, they proved [Deuschel and Stroock 1990, Remark 1.14 (i)] that the log-Sobolev constant for the circle of length 1 with Lebesgue measure is the first eigenvalue of the Laplacian, namely $4\pi^2$ (corresponding to the first eigenfunction $\sin 2\pi x$).

Deuschel and Stroock [1990] cited [Émery and Yukich 1987].

Acknowledgements

We thank our advisor Frank Morgan for his patience and invaluable input. We thank our friend Andrew Kelly for substantial contributions to this paper. We thank the National Science Foundation for grants to Professor Morgan and the Williams College SMALL Research Experience for Undergraduates, and Williams College for additional funding. We thank the Mathematical Association of America, MIT, the University of Chicago, and Williams College for supporting our trip to talk at MathFest 2012.

The authors were undergraduate students at MIT (Ghang), Williams College (Martin) and University of Chicago (Waruhiu) at the time this work was carried out.

References

- [Deuschel and Stroock 1990] J.-D. Deuschel and D. W. Stroock, “Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models”, *J. Funct. Anal.* **92**:1 (1990), 30–48. MR 91j:58174 Zbl 0705.60066
- [Díaz et al. 2012] A. Díaz, N. Harman, S. Howe, and D. Thompson, “Isoperimetric problems in sectors with density”, *Adv. Geom.* **14**:4 (2012), 589–619.
- [Émery and Yukich 1987] M. Émery and J. E. Yukich, “A simple proof of the logarithmic Sobolev inequality on the circle”, pp. 173–175 in *Séminaire de Probabilités, XXI*, edited by J. Azéma et al., Lecture Notes in Math. **1247**, Springer, Berlin, 1987. MR 89f:26020 Zbl 0616.46023
- [Federbush 1969] P. Federbush, “Partially alternate derivation of a result of Nelson”, *J. Math. Phys.* **10**:1 (1969), 50–52.
- [Gross 1975] L. Gross, “Logarithmic Sobolev inequalities”, *Amer. J. Math.* **97**:4 (1975), 1061–1083. MR 54 #8263 Zbl 0318.46049
- [Morgan 2009] F. Morgan, “Log-Sobolev inequality”, blog entry, 2009, <http://sites.williams.edu/Morgan/2009/06/11/sobolev-type-inequality/>.
- [Perelman 2002] G. Perelman, “The entropy formula for the Ricci flow and its geometric applications”, preprint, 2002. Zbl 1130.53001 arXiv math/0211159
- [Rothaus 1978] O. S. Rothaus, “Lower bounds for eigenvalues of regular Sturm–Liouville operators and the logarithmic Sobolev inequality”, *Duke Math. J.* **45**:2 (1978), 351–362. MR 58 #1368 Zbl 0435.47049
- [Rothaus 1980] O. S. Rothaus, “Logarithmic Sobolev inequalities and the spectrum of Sturm–Liouville operators”, *J. Funct. Anal.* **39**:1 (1980), 42–56. MR 81m:34025 Zbl 0472.47024
- [Stam 1959] A. J. Stam, “Some inequalities satisfied by the quantities of information of Fisher and Shannon”, *Information and Control* **2** (1959), 101–112. MR 21 #7813 Zbl 0085.34701
- [Wang 1999] F.-Y. Wang, “Harnack inequalities for log-Sobolev functions and estimates of log-Sobolev constants”, *Ann. Probab.* **27**:2 (1999), 653–663. MR 2000i:58067 Zbl 0948.58023
- [Weissler 1980] F. B. Weissler, “Logarithmic Sobolev inequalities and hypercontractive estimates on the circle”, *J. Funct. Anal.* **37**:2 (1980), 218–234. MR 81k:42007 Zbl 0463.46024

Received: 2012-09-15 Accepted: 2013-01-06

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
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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

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2014

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