Counting matrices over a finite field
with all eigenvalues in the field

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(Communicated by Kenneth S. Berenhaut)

Given a finite field $\mathbb{F}$ and a positive integer n, we give a procedure to count the $n \times n$ matrices with entries in $\mathbb{F}$ with all eigenvalues in the field. We give an exact value for any field for values of n up to 4, and prove that for fixed n, as the size of the field increases, the proportion of matrices with all eigenvalues in the field approaches $1/n!$. As a corollary, we show that for large fields almost all matrices with all eigenvalues in the field have all eigenvalues distinct. The proofs of these results rely on the fact that any matrix with all eigenvalues in $\mathbb{F}$ is similar to a matrix in Jordan canonical form, and so we proceed by enumerating the number of $n \times n$ Jordan forms, and counting how many matrices are similar to each one. A key step in the calculation is to characterize the matrices that commute with a given Jordan form and count how many of them are invertible.

1. Introduction

Let $\mathbb{F}$ be a field and let $M_n(\mathbb{F})$ denote the set of $n \times n$ matrices with entries in $\mathbb{F}$. As an example, consider $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \in M_2(\mathbb{R})$. The roots of its characteristic polynomial $\det(A - \lambda I)$ are $\lambda = \pm i$, which are the eigenvalues of A. Though the entries in A are real numbers, the eigenvalues are not. This example serves to motivate the following question: If $\mathrm{Eig}_n(\mathbb{F})$ denotes the set of elements of $M_n(\mathbb{F})$ that have all of their eigenvalues in $\mathbb{F}$, what is the cardinality of $\mathrm{Eig}_n(\mathbb{F})$? For a field like $\mathbb{R}$ that is uncountably infinite, this question is trivial, but in this paper we examine the case when $\mathbb{F}$ is a finite field with q elements. This line of research was initiated by Olšavský [2003], who determined that for any prime p,

$$|\mathrm{Eig}_2(\mathbb{Z}_p)| = \tfrac{1}{2}p^4 + p^3 - \tfrac{1}{2}p^2. \tag{1}$$

Here we present a method for determining $|\mathrm{Eig}_n(\mathbb{F})|$ for any n. We use the fact that any matrix $A \in M_n(\mathbb{F})$ with all eigenvalues in $\mathbb{F}$ is similar to a matrix J in Jordan canonical form. Thus we can determine $|\mathrm{Eig}_n(\mathbb{F})|$ using the following procedure:

(1) Enumerate all $n \times n$ Jordan forms.

(2) Enumerate all matrices in $M_n(\mathbb{F})$ that are similar to each Jordan form.

In Section 2, we review the definitions and notation necessary to work with Jordan forms. Then in Section 3 we explain the procedure to determine $|\text{Eig}_n(\mathbb{F})|$, giving a general formula in (3). We illustrate the process for the case $n = 2$, giving a slightly shorter derivation of (1) than was given in [Olšavský 2003].

We group matrices in Jordan form by what we call their double partition type, which is defined in Section 3. In Sections 4 and 5 we find formulas for the quantities required to compute $|\text{Eig}_n(\mathbb{F})|$ for any n. In Section 4 we state a formula for the number of Jordan forms of a given double partition type. Then in Section 5 we prove that the number of matrices similar to any matrix in Jordan form depends only on its double partition type, and give a formula that determines this number for any double partition type. In Section 6, we use these results to give explicit formulas for $|\text{Eig}_3(\mathbb{F})|$ and $|\text{Eig}_4(\mathbb{F})|$ for any finite field $\mathbb{F}$.

Olšavský [2003] also noted that the proportion of matrices in $M_2(\mathbb{Z}_p)$ with all eigenvalues in $\mathbb{Z}_p$ approaches 1/2 as p goes to infinity. In Section 7, we generalize this result to prove that the proportion of matrices in $M_n(\mathbb{F})$ with all eigenvalues in $\mathbb{F}$ approaches $1/n!$ as q approaches infinity. As a corollary, we prove that for large finite fields, if a matrix has all eigenvalues in the field, then almost surely all if its eigenvalues are distinct.

2. Jordan canonical form

Denote the set of invertible matrices in $M_n(\mathbb{F})$ by $\text{GL}_n(\mathbb{F})$. We will repeatedly use the fact that any element of $M_n(\mathbb{F})$ with all eigenvalues in $\mathbb{F}$ is similar to a matrix in Jordan canonical form. Here we review the necessary definitions. For a more thorough introduction, see [Hungerford 1974, Chapter 7.4].

For $1 \leq i \leq n$, let A_i be a square matrix. The *direct sum* of these matrices, denoted $\bigoplus_{i=1}^{n} A_i$, is a block diagonal matrix such that the matrices A_i lie on the diagonal, and all other entries are zero:

$$\bigoplus_{i=1}^{n} A_i = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{pmatrix}.$$

A *Jordan block* of size $k \geq 1$ corresponding to some eigenvalue $\lambda \in \mathbb{F}$ is a $k \times k$ matrix with λs along the diagonal, 1s along the superdiagonal, and 0s everywhere else (see Figure 1). Let $A, J \in M_n(\mathbb{F})$. Then A is *similar* to J if there exists a matrix $P \in \text{GL}_n(\mathbb{F})$ such that $A = PJP^{-1}$. It is well known (see, for example, [Hungerford 1974, Chapter 7.4, Corollary 4.7 (iii)]) that any matrix $A \in M_n(\mathbb{F})$ with all eigenvalues in $\mathbb{F}$ is similar to a matrix J which is the direct sum of Jordan blocks,

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Figure 1. A generic Jordan block and a Jordan block with eigenvalue 3.

$$A = \begin{pmatrix} 1 & 3 & 3 & 4 \\ 3 & 3 & 0 & 2 \\ 0 & 4 & 3 & 1 \\ 3 & 3 & 3 & 2 \end{pmatrix} = PJP^{-1} = \begin{pmatrix} 3 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 3 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 4 & 4 \\ 3 & 3 & 0 & 2 \\ 4 & 0 & 0 & 1 \\ 0 & 2 & 1 & 3 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 3 & 3 & 4 \\ 3 & 3 & 0 & 2 \\ 0 & 4 & 3 & 1 \\ 3 & 3 & 3 & 2 \end{pmatrix} = QJ'Q^{-1} = \begin{pmatrix} 3 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 3 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 4 & 4 \\ 0 & 2 & 1 & 3 \\ 3 & 3 & 0 & 2 \\ 4 & 0 & 0 & 1 \end{pmatrix}.$$

Figure 2. The matrix $A \in M_4(\mathbb{Z}_5)$ is similar to J and J', which are different representations of the same Jordan form.

and this matrix J is unique up to the ordering of the blocks. The *Jordan canonical form* (or simply *Jordan form*) for A is this direct sum of Jordan blocks. We will use $J_n(\mathbb{F})$ to denote the set of Jordan forms in $M_n(\mathbb{F})$. We note that multiple matrices may correspond to a given Jordan form (e.g., $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ are representatives of the same form), so when we wish to work with a particular matrix in $J_n(\mathbb{F})$, we will specify an order for the blocks in the direct sum. For example, in Figure 2, the matrix $A \in M_4(\mathbb{Z}_5)$ has the Jordan form that is the direct sum of the Jordan blocks (1), (2), and $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. In the figure, $J \in J_4(\mathbb{Z}_5)$ and $J' \in J_4(\mathbb{Z}_5)$ are two examples of matrices that are direct sums of these blocks.

3. How to determine $|\text{Eig}_n(\mathbb{F})|$

Since every matrix in $\text{Eig}_n(\mathbb{F})$ is similar to a matrix in Jordan canonical form, we can determine $|\text{Eig}_n(\mathbb{F})|$ by first enumerating all Jordan forms in $J_n(\mathbb{F})$ and then counting how many matrices in $M_n(\mathbb{F})$ are similar to each one. In this section we explain this process and illustrate it with the example of computing $|\text{Eig}_2(\mathbb{F})|$.

For any matrix $A \in M_n(\mathbb{F})$, let $S(A) \subseteq M_n(\mathbb{F})$ be the set of matrices similar to A, and let $C(A)$ denote the subgroup of $\text{GL}_n(\mathbb{F})$ of matrices P such that $PA = AP$.

Lemma 3.1. *For any $J \in M_n(\mathbb{F})$, we have $|S(J)| = |\text{GL}_n(\mathbb{F})|/|C(J)|$.*

Proof. Fix $J \in M_n(\mathbb{F})$; we must find the cardinality of $S(J) = \{AJA^{-1} : A \in \mathrm{GL}_n(\mathbb{F})\}$. For any $A, B \in \mathrm{GL}_n(\mathbb{F})$, $AJA^{-1} = BJB^{-1}$ if and only if $B^{-1}AJ = JB^{-1}A$, that is, if $B^{-1}A \in C(J)$. Now, $B^{-1}A \in C(J)$ if and only if A and B are in the same coset of $C(J)$ in $\mathrm{GL}_n(\mathbb{F})$, and thus $S(J)$ has the same cardinality as the number of cosets of $C(J)$ in $\mathrm{GL}_n(\mathbb{F})$, which is equal to $|\mathrm{GL}_n(\mathbb{F})|/|C(J)|$ by Lagrange's theorem. $\square$

It is well known (see, for example, [Stanley 2012, Proposition 1.10.1]) that if $\mathbb{F}$ has q elements, then $|\mathrm{GL}_n(\mathbb{F})| = \prod_{i=0}^{n-1}(q^n - q^i)$. Thus to find $|S(J)|$ for any J, it suffices to find $|C(J)|$, and we obtain the following formula for $|\mathrm{Eig}_n(\mathbb{F})|$:

$$|\mathrm{Eig}_n(\mathbb{F})| = \sum_{J \in J_n(\mathbb{F})} |S(J)| = \sum_{J \in J_n(\mathbb{F})} \frac{\prod_{i=0}^{n-1}(q^n - q^i)}{|C(J)|}. \tag{2}$$

For $J \in J_n(\mathbb{F})$, it turns out that $|C(J)|$ depends on what we will call the double partition type of J, which we define now.

A *partition* of a positive integer n is a collection of (not necessarily distinct) positive integers $\{n_1, n_2, \ldots n_k\}$ such that $n_1 + n_2 + \cdots + n_k = n$. We define the *partition type* of a matrix $J \in J_n(\mathbb{F})$ as the partition of n given by the size of the Jordan blocks in J. For example, the partition type of the matrix $J \in J_4(\mathbb{Z}_5)$ below is $\{1, 1, 2\}$ since it has two 1×1 Jordan blocks and one 2×2 Jordan block.

$$J = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We define a *double partition* (or partition of a partition) of a positive integer n to be a partition where the numbers in the partition are themselves partitioned into subsets. Denote the set of double partitions of n by $\mathrm{DP}(n)$. For example, $\{\{2, 3\}, \{1, 1, 3\}, \{2, 3\}, \{4\}\} \in \mathrm{DP}(19)$. We define the *double partition type* (or simply *type*) of a matrix $J \in J_n(\mathbb{F})$ by grouping all elements of its partition type into sets where two elements of the partition are placed in the same set if their corresponding eigenvalues are the same. For example, the double partition type of the matrix J above is $\{\{1, 1\}, \{2\}\}$ since the two 1×1 blocks have the same eigenvalue, and the 2×2 block has a different eigenvalue. The study of double partitions dates back at least as far as [Cayley 1855] and [Sylvester 1851], and the values of $|\mathrm{DP}(n)|$ are collected in the *Online Encyclopedia of Integer Sequences* (oeis.org), sequence A001970.

The utility of knowing the double partition type of a matrix in Jordan form is given by the following lemma.

Lemma 3.2. *If $J_1, J_2 \in J_n(\mathbb{F})$ have the same double partition type, then $|C(J_1)| = |C(J_2)|$.*

The proof of Lemma 3.2 is one of the main results of the paper and will be deferred until Section 5. This lemma justifies the following definition: For any double partition type T define $c(T)$ and $s(T)$ so that $c(T) = |C(J)|$ and $s(T) = |S(J)|$, where J is any matrix of type T. Letting $t(T)$ denote the number of Jordan forms of type T, we can now rewrite (2) as

$$|\mathrm{Eig}_n(\mathbb{F})| = \sum_{J \in J_n(\mathbb{F})} |S(J)| = \sum_{T \in \mathrm{DP}(n)} t(T)s(T) = \sum_{T \in \mathrm{DP}(n)} t(T) \frac{\prod_{i=0}^{n-1}(q^n - q^i)}{c(T)}. \quad (3)$$

We now are prepared to illustrate our procedure for determining $|\mathrm{Eig}_n(\mathbb{F})|$ by computing $|\mathrm{Eig}_2(\mathbb{F})|$. The first step is to enumerate all the double partition types in $\mathrm{DP}(n)$. In the case $n = 2$, there are two possible partitions, $2 = 1 + 1$, and $2 = 2$. There are three double partition types: $T_{2,1} = \{\{1\}, \{1\}\}$, $T_{2,2} = \{\{1, 1\}\}$ and $T_{2,3} = \{\{2\}\}$. We give a general formula for $t(T)$ in Lemma 4.1, but for $n = 2$ we can just examine each type. The Jordan forms of type $T_{2,1}$ are represented by matrices of the form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ where $\lambda_1 \neq \lambda_2$ and thus $t(T_{2,1}) = \binom{q}{2}$. Jordan forms of type $T_{2,2}$ are represented by matrices of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, and Jordan forms of type $T_{2,3}$ are represented by matrices of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, so $t(T_{2,2}) = t(T_{2,3}) = q$.

It remains to compute $c(T)$ for each double partition type. A general formula for $c(T)$ is given by (4), but again for $n = 2$ we can argue ad hoc. A 2×2 matrix commutes with a matrix J of type $T_{2,1}$ if and only if it is of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{F}$, and of course this is true regardless of the specific values of λ_1 and λ_2. Since an element of $C(J)$ must be invertible, there are $q - 1$ choices each for a and b, and $c(T_{2,1}) = (q-1)^2$. Similarly, if J is of type $T_{2,2}$, $C(J) = \mathrm{GL}_2(\mathbb{F})$, and $c(T_{2,2}) = (q^2 - 1)(q^2 - q)$. Finally, if J is of type $T_{2,3}$, $C(J) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{F}, a \neq 0 \right\}$, and so $c(T_{2,3}) = q(q-1)$.

To complete the example, we apply (3):

$$|\mathrm{Eig}_2(\mathbb{F})| = \sum_{T \in \mathrm{DP}(2)} t(T) \frac{|\mathrm{GL}_2(\mathbb{F})|}{c(T)} = \sum_{i=1}^{3} t(T_{2,i}) \frac{(q^2 - 1)(q^2 - q)}{c(T_{2,i})}$$

$$= \binom{q}{2} \frac{(q^2 - 1)(q^2 - q)}{(q-1)^2} + q \frac{(q^2 - 1)(q^2 - q)}{(q^2 - 1)(q^2 - q)} + q \frac{(q^2 - 1)(q^2 - q)}{q(q-1)}$$

$$= \tfrac{1}{2}q^4 + q^3 - \tfrac{1}{2}q^2.$$

4. The number of Jordan forms of a given type

Fix n and consider a double partition $T = \{S_1, S_2, \ldots, S_k\}$ of n where for $1 \leq i \leq k$, S_i is a set of positive integers. To determine the number of Jordan forms in $J_n(\mathbb{F})$ of type T, we count the number of ways to assign eigenvalues to the S_i. Of course, if $S_i = S_j$ then assigning eigenvalue λ_1 to S_i and λ_2 to S_j yields the same Jordan

form as assigning λ_2 to S_i and λ_1 to S_j, so we need to account for any repeated subsets in T. For example, if $n = 19$ and $T = \{\{2, 3\}, \{1, 1, 3\}, \{2, 3\}, \{4\}\}$, then the subset $\{2, 3\}$ is repeated twice, and there will be

$$\binom{q}{2}(q-2)(q-3) = \binom{q}{2}\binom{q-2}{1}\binom{q-3}{1} = \frac{q!}{2!\, 1!\, 1!\, (q-4)!}$$

elements of $J_n(\mathbb{F})$ of type T. The value of $t(T)$ in the general case is given by the following lemma.

Lemma 4.1. *Let $T = \{S_1, S_2, \ldots, S_k\}$ be a double partition, where for $1 \le i \le k$, S_i is a set of positive integers. Let $B_1, \ldots, B_l$ be equivalence classes of the sets in T so that S_i and S_j are in the same equivalence class if and only if $S_i = S_j$. Let $b_i = |B_i|$. Then the number of Jordan forms of type T in $J_n(\mathbb{F})$ is 0 if $q < k$ and otherwise is given by the formula*

$$t(T) = \binom{q}{b_1}\binom{q-b_1}{b_2} \cdots \binom{q-b_1-b_2-\cdots-b_{l-1}}{b_l}$$

$$= \frac{q!}{b_1!\, b_2! \cdots b_l!(q - b_1 - b_2 - \cdots - b_l)!}.$$

Proof. There are $\binom{q}{b_1}$ ways to assign eigenvalues to the sets in B_1, $\binom{q-b_1}{b_2}$ ways to assign eigenvalues to the sets in B_2 without repeating any of the b_1 eigenvalues already assigned to sets in B_1, and so forth. □

5. Invertible matrices that commute with a Jordan form

In this section we first characterize the structure of the matrices that commute with Jordan forms, with the ultimate statement of this characterization coming in Corollary 5.3. This characterization depends only on the double partition type, and not the specific eigenvalues, and this observation is sufficient to prove Lemma 3.2. To determine $|C(J)|$, we must determine how many of the matrices of the form specified by Corollary 5.3 are invertible, and this is done starting on page 636.

For any matrix $A \in M_n(\mathbb{F})$, let $(A)_{i,j}$ denote the entry of A in the i-th row and j-th column. We say an $n \times m$ matrix A is *streaky upper triangular* if it has the following three properties (see Figure 3):

  (i) $(A)_{i,1} = 0$ if $i > 1$.

 (ii) $(A)_{n,j} = 0$ if $j < m$.

(iii) $(A)_{i,j} = (A)_{i+1,j+1}$ if $1 \le i \le n-1$, $1 \le j \le m-1$.

We denote the set of $n \times m$ streaky upper triangular matrices over $\mathbb{F}$ by $\mathrm{SUT}_{n,m}(\mathbb{F})$. In Lemmas 5.1 and 5.2 we examine products of the form $J_n A$ and $A J_m$ where A is $n \times m$, and J_n and J_m are $n \times n$ and $m \times m$ Jordan blocks, respectively (see

$$
\begin{pmatrix} 2 & 1 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\qquad
\begin{pmatrix} 0 & 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}
\qquad
\begin{pmatrix} 2 & 5 & 1 & 0 & 5 \\ 0 & 2 & 5 & 1 & 0 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}
\qquad
\begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

Figure 3. Some examples of streaky upper triangular matrices.

$$
J_4 A = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \end{pmatrix}
$$

$$
= \begin{pmatrix} \lambda a_{1,1} + a_{2,1} & \lambda a_{1,2} + a_{2,2} & \lambda a_{1,3} + a_{2,3} & \lambda a_{1,4} + a_{2,4} & \lambda a_{1,5} + a_{2,5} \\ \lambda a_{2,1} + a_{3,1} & \lambda a_{2,2} + a_{3,2} & \lambda a_{2,3} + a_{3,3} & \lambda a_{2,4} + a_{3,4} & \lambda a_{2,5} + a_{3,5} \\ \lambda a_{3,1} + a_{4,1} & \lambda a_{3,2} + a_{4,2} & \lambda a_{3,3} + a_{4,3} & \lambda a_{3,4} + a_{4,4} & \lambda a_{3,5} + a_{4,5} \\ \lambda a_{4,1} & \lambda a_{4,2} & \lambda a_{4,3} & \lambda a_{4,4} & \lambda a_{4,5} \end{pmatrix},
$$

$$
A J_5 = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}
$$

$$
= \begin{pmatrix} \lambda a_{1,1} & \lambda a_{1,2} + a_{1,1} & \lambda a_{1,3} + a_{1,2} & \lambda a_{1,4} + a_{1,3} & \lambda a_{1,5} + a_{1,4} \\ \lambda a_{2,1} & \lambda a_{2,2} + a_{2,1} & \lambda a_{2,3} + a_{2,2} & \lambda a_{2,4} + a_{2,3} & \lambda a_{2,5} + a_{3,4} \\ \lambda a_{3,1} & \lambda a_{3,2} + a_{3,1} & \lambda a_{3,3} + a_{3,2} & \lambda a_{3,4} + a_{3,3} & \lambda a_{3,5} + a_{3,4} \\ \lambda a_{4,1} & \lambda a_{4,2} + a_{4,1} & \lambda a_{4,3} + a_{4,2} & \lambda a_{4,4} + a_{4,3} & \lambda a_{4,5} + a_{3,4} \end{pmatrix}.
$$

Figure 4. Examples of $J_4 A$ and $A J_5$ where A is 4×5 and J_4 and J_5 are Jordan blocks.

Figure 4). We show $J_n A = A J_m$ if and only if J_n and J_m have the same eigenvalue and $A \in \mathrm{SUT}_{n,m}(\mathbb{F})$, or if J_n and J_m have different eigenvalues and A is the all zeros matrix. These lemmas enable us to characterize the matrices that commute with any matrix $J \in J_n(\mathbb{F})$

Lemma 5.1. *Let $J_n \in J_n(\mathbb{F})$ and $J_m \in J_m(\mathbb{F})$ be Jordan blocks with eigenvalue λ, and suppose A is an $n \times m$ matrix. Then $J_n A = A J_m$ if and only if $A \in \mathrm{SUT}_{n,m}(\mathbb{F})$.*

Proof. Suppose $J_n A = A J_m$. We will show $A \in \mathrm{SUT}_{n,m}(\mathbb{F})$.

First, we examine the individual entries in the first column. For $1 \leq i \leq n-1$, we have $(J_n A)_{i,1} = \lambda(A)_{i,1} + (A)_{i+1,1}$ and $(A J_m)_{i,1} = \lambda(A)_{i,1}$. If these two quantities are equal, then $(A)_{i+1,1} = 0$ for $1 \leq i \leq n-1$, which implies A has property (i).

Next, we examine the individual entries in the last (n-th) row. For $2 \leq j \leq m$, we have $(J_n A)_{n,j} = \lambda(A)_{n,j}$ and $(A J_m)_{n,j} = \lambda(A)_{n,j} + (A)_{n,j-1}$. If these two quantities are equal, then $(A)_{n,j-1} = 0$ for $2 \leq j \leq m$, which implies A has property (ii).

Finally, we examine the other entries. If $1 \leq i \leq n-1$ and $2 \leq j \leq m$, we have $(J_n A)_{i,j} = \lambda(A)_{i,j} + (A)_{i+1,j}$ and $(A J_m)_{i,j} = \lambda(A)_{i,j} + (A)_{i,j-1}$. If these two quantities are equal, then $(A)_{i+1,j} = (A)_{i,j-1}$ for $1 \leq i \leq n-1$, $2 \leq j \leq m$, which implies A has property (iii).

Conversely, suppose $A \in \mathrm{SUT}_{n,m}(\mathbb{F})$. Then we verify $J_n A = A J_m$ using four cases.

- For $i = n$, $j = 1$, $(J_n A)_{n,1} = \lambda(A)_{n,1} = (A J_m)_{n,1}$.
- For $i \leq n-1$, $j = 1$, $(J_n A)_{i,1} = \lambda(A)_{i,1} + (A)_{i+1,1} = \lambda(A)_{i,1} = (A J_m)_{i,1}$.
- For $i = n$, $j \geq 2$, $(J_n A)_{n,j} = \lambda(A)_{n,j} = \lambda(A)_{n,j} + (A)_{n,j-1} = (A J_m)_{n,j}$.
- For $i \leq n-1$, $2 \leq j$, $(J_n A)_{i,j} = \lambda(A)_{i,j} + (A)_{i+1,j} = \lambda(A)_{i,j} + (A)_{i,j-1} = (A J_m)_{i,j}$. $\qquad\square$

Lemma 5.2. *Let $J_n \in J_n(\mathbb{F})$ be a Jordan block with eigenvalue λ and $J_m \in J_m(\mathbb{F})$ be a Jordan block with eigenvalue μ, where $\lambda \neq \mu$, and suppose A is an $n \times m$ matrix. Then $J_n A = A J_m$ if and only if $(A)_{i,j} = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$.*

Proof. If $(A)_{i,j} = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$, then $J_n A = A J_m$.

Conversely, suppose $J_n A = A J_m$. First examine the case $i = n$, $j = 1$: $(J_n A)_{n,1} = \lambda(A)_{n,1}$ and $(A J_m)_{n,1} = \mu(A)_{n,1}$ imply $(A)_{n,1} = 0$.

Next we proceed by induction on the first column ($j = 1$). We know $(A)_{n,1} = 0$. For any $k \geq 0$, assume $(A)_{n-k,1} = 0$. Then

$$(J_n A)_{n-(k+1),1} = \lambda(A)_{n-(k+1),1} + (A)_{n-k,1} = \lambda(A)_{n-(k+1),1},$$

while $(A J_m)_{n-(k+1),1} = \mu(A)_{n-(k+1),1}$. If these two quantities are equal, then $(A)_{n-(k+1),1} = 0$. From this we conclude that $(A)_{i,1} = 0$ for all $1 \leq i \leq n$.

Next we apply the same induction argument to the last row ($i = n$): For any $j \geq 1$, assume $(A)_{n,j} = 0$. Then $(J_n A)_{n,j+1} = \lambda(A)_{n,j+1}$, while

$$(A J_m)_{n,j+1} = \mu(A)_{n,j+1} + (A)_{n,j} = \mu(A)_{n,j+1}.$$

If these two quantities are equal, then $(A)_{n,j+1} = 0$. From this we conclude that $(A)_{n,j} = 0$ for all $1 \leq j \leq m$.

To complete the proof, we show that for all $i < n$, $j > 1$, if $(A)_{i,j-1} = 0$ and $(A)_{i+1,j} = 0$ then $(A)_{i,j} = 0$. If $i < n$, $j > 1$, then $(J_n A)_{i,j} = \lambda(A)_{i,j} + (A)_{i+1,j}$, while $(A J_m)_{i,j} = \mu(A)_{i,j} + (A)_{i,j-1}$. If $(A)_{i,j-1} = 0$ and $(A)_{i+1,j} = 0$ then $\lambda(A)_{i,j} =$

$\mu(A)_{i,j}$, which implies $(A)_{i,j} = 0$. We have shown if the entries below and to the left of $(A)_{i,j}$ are both zero, then $(A)_{i,j}$ is zero as well. Since we know the first column and last row of A contain only zeros, by induction all other entries must be zero as well. □

If A is a block matrix, let $A^{i,j}$ denote the block in the i-th row and j-th column.

Corollary 5.3. *Suppose $n = n_1 + n_2 + \cdots + n_k$, and $J \in J_n(\mathbb{F})$ is a Jordan form with partition type $\{n_1, n_2, \ldots, n_k\}$, that is, J is the direct sum of k Jordan blocks $J_1 \in J_{n_1}(\mathbb{F})$, $J_2 \in J_{n_2}(\mathbb{F})$, \ldots, $J_k \in J_{n_k}(\mathbb{F})$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, some of which may be the same. We can represent J as a $k \times k$ block matrix where $J^{i,i} = J_i$ for $1 \le i \le k$, and $J^{i,j}$ is an $n_i \times n_j$ all-zeros matrix if $i \ne j$. Then $JA = AJ$ if and only if A is a $k \times k$ block matrix, where for $1 \le i, j \le k$, the block $A^{i,j}$ is an element of $\mathrm{SUT}_{n_i,n_j}(\mathbb{F})$ if $\lambda_i = \lambda_j$, and otherwise $A^{i,j}$ is an $n_i \times n_j$ matrix containing all zeros.*

Figure 5 contains an illustration of Corollary 5.3. In this example, $J \in J_{11}(\mathbb{F})$, and in the terminology of Corollary 5.3, we have $k = 6$ and

$$(n_1, n_2, n_3, n_4, n_5, n_6) = (3, 2, 2, 1, 1, 2),$$
$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (1, 1, 1, 4, 4, 0).$$

Proof of Corollary 5.3. Decompose $A \in M_n(\mathbb{F})$ as a $k \times k$ block matrix, where block $A^{i,j}$ is $n_i \times n_j$. Then the equation $JA = AJ$ implies $J_i A^{i,j} = A^{i,j} J_j$. If $\lambda_i = \lambda_j$, then $J_i A^{i,j} = A^{i,j} J_j$ if and only if $A^{i,j} \in \mathrm{SUT}_{n_i,n_j}(\mathbb{F})$ by Lemma 5.1. If $\lambda_i \ne \lambda_j$, then $J_i A^{i,j} = A^{i,j} J_j$ if and only if $A^{i,j}$ contains all zeros, by Lemma 5.2. □

We now show that Lemma 3.2 follows from Corollary 5.3.

$$J = \left(\begin{array}{ccc|cc|cc|cc|cc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right), \quad A = \left(\begin{array}{ccc|cc|cc|cc|cc}
a & b & c & l & m & n & p & 0 & 0 & 0 & 0 \\
0 & a & b & 0 & l & 0 & n & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & s & t & d & e & q & r & 0 & 0 & 0 & 0 \\
0 & 0 & s & 0 & d & 0 & q & 0 & 0 & 0 & 0 \\
\hline
0 & u & v & w & x & f & g & 0 & 0 & 0 & 0 \\
0 & 0 & u & 0 & w & 0 & f & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & h & y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z & i & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & j & k \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & j
\end{array}\right).$$

Figure 5. A matrix commutes with the Jordan form J if and only if it has the form of the matrix A.

Proof of Lemma 3.2. Implicit in the statement of Corollary 5.3 is the fact that the characterization of the matrices that commute with $J \in J_n(\mathbb{F})$ does not depend on the specific eigenvalues of the Jordan blocks, only on which blocks share the same eigenvalue. In other words, it depends only on the double partition type of J. The quantity $|C(J)|$ equals the number of invertible matrices that fit this characterization, and since the characterization depends only on the double partition type of J, if two Jordan forms J_1 and J_2 have the same double partition type, then $|C(J_1)| = |C(J_2)|$. $\qquad\square$

How to determine $|C(J)|$. It remains to explain how to determine $|C(J)|$ for $J \in J_n(\mathbb{F})$. Simply characterizing the matrices that commute with J is not enough, since elements of $C(J)$ must be invertible. Thus we need to determine which of the matrices of the form given in Corollary 5.3 are invertible. Examining the example in Figure 5, we note that A can be represented as a block diagonal matrix, where one block is 7×7 and the other two blocks are 2×2. It turns out that for any $J \in J_n(\mathbb{F})$, the matrices in $C(J)$ can be represented as block diagonal matrices, and since the determinant of a block diagonal matrix is the product of the determinants of the matrices on the diagonal, we must characterize when the matrices on the diagonal are invertible. As in the example in Figure 5, the matrices on the diagonal are themselves block matrices where each block is streaky upper triangular, and Lemma 5.4 describes when such a matrix is invertible.

The proof of Lemma 5.4 proceeds by taking any block matrix where each block is streaky upper triangular and permuting the rows and columns to obtain an upper triangular block matrix, which is invertible if and only if the matrices on the diagonal are invertible. We use the fact that switching the position of two rows or columns of a matrix changes the sign of the determinant. Thus if the columns of a matrix are permuted in some way, and the rows are permuted the same way, the sign of the determinant will change an even number of times, and thus will ultimately be unchanged. To describe the diagonal blocks of the rearranged matrix, we need the following notation: If A is a $b \times b$ block matrix, where each block $A^{i,j}$ is an element of $\mathrm{SUT}_{n,n}(\mathbb{F})$, then we denote by A' the $b \times b$ matrix made up of the entries that are on the diagonal of each block $A^{i,j}$, that is, $(A')_{i,j} = (A^{i,j})_{1,1}$. For example, in Figure 6, A is a 2×2 block matrix with each block in $\mathrm{SUT}_{3,3}(\mathbb{F})$ (i.e., $b = 2$, $n = 3$) and D is a 4×4 block matrix with each block in $\mathrm{SUT}_{2,2}(\mathbb{F})$ (i.e., $b = 4$, $n = 2$). In this case, $A' = \left(\begin{smallmatrix} 3 & 0 \\ 1 & 4 \end{smallmatrix}\right)$, and D' is the 4×4 matrix shown in the figure. We write $\{b_1 n_1, \ldots, b_k n_k\}$ for the collection having b_i copies of n_i, for $1 \le i \le k$.

Lemma 5.4. *Let $\{b_1 n_1, \ldots, b_k n_k\}$ be a partition of n, with $n_1 > n_2 > \cdots > n_k$. Suppose $A \in M_n(\mathbb{F})$ is a $k \times k$ block matrix, where $A^{i,j}$ is $b_i n_i \times b_j n_j$, and each $A^{i,j}$ can itself be represented as a $b_i \times b_j$ block matrix, where each block is an element of $\mathrm{SUT}_{n_i,n_j}(\mathbb{F})$. Then A is invertible if and only if $(A^{i,i})'$ is invertible for*

$$A = \begin{pmatrix} 3 & 2 & 4 & 0 & 3 & 0 \\ 0 & 3 & 2 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 4 & 1 & 4 \\ 0 & 1 & 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 & 0 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 0 & 2 & 3 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 2 & 1 & 4 & 3 & 3 & 4 & 3 & 4 \\ 0 & 2 & 0 & 4 & 0 & 3 & 0 & 3 \\ 1 & 1 & 1 & 4 & 2 & 5 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 3 \\ 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}, \quad D' = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

Figure 6. Block matrices A and D where each block is an $n \times n$ streaky upper triangular matrix.

each i, $1 \leq i \leq k$. In fact,

$$\det A = \prod_{i=1}^{k} (\det(A^{i,i})')^{n_i}.$$

Proof. We go by induction on n_1. If $n_1 = 1$, then $A = A^{1,1} = (A^{1,1})'$ and the statement follows. Now suppose $\{b_1 n_1, \ldots, b_k n_k\}$ is a partition of n, with $n_1 > n_2 > \cdots > n_k$. Let $A \in M_n(\mathbb{F})$ be a $k \times k$ block matrix, where $A^{i,j}$ is $b_i n_i \times b_j n_j$, and each $A^{i,j}$ can itself be represented as a $b_i \times b_j$ block matrix, where each block is an element of $\mathrm{SUT}_{n_i, n_j}(\mathbb{F})$ (see Figure 7).

Let $B_0 = BN_0 = 0$, and for $1 \leq j \leq k$, let B_j denote the sum $b_1 + b_2 + \cdots + b_j$, and BN_j denote the sum $b_1 n_1 + b_2 n_2 + \cdots + b_j n_j$. Permute the columns of A so that if $0 \leq i \leq B_k - 1$, and j is the smallest index so that $i < B_j$, column $BN_{j-1} + (i - B_{j-1})n_j + 1$ becomes column $i + 1$, and all other columns become columns $B_k + 1$ to n, in the same order as they were in the original matrix A. Permute the rows the same way to obtain a 2×2 block matrix D with block structure $\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$ (see Figure 7). Since the rows and columns were rearranged symmetrically, $\det D = \det A$. The minor D_1 is a $k \times k$ block upper triangular matrix where for $1 \leq i \leq k$, $(D_1)^{i,i} = (A^{i,i})'$. Since $n_1 > n_2 > \cdots > n_k$ and each block is streaky upper triangular, the $(n - B_k) \times B_k$ matrix D_3 contains only zeros. D_4 can be represented as a $k \times k$ (if $n_k > 1$) or $k - 1 \times k - 1$ (if $n_k = 1$) block matrix where $(D_4)^{i,j}$ is $b_i(n_i - 1) \times b_j(n_j - 1)$, and each $(D_4)^{i,j}$ can itself be represented as a $b_i \times b_j$ block matrix, where each block is an element of $\mathrm{SUT}_{n_i-1, n_j-1}(\mathbb{F})$. Furthermore, $((D_4)^{i,i})' = (A^{i,i})'$ for each i. Thus by the inductive hypothesis, $\det D_4 = \prod_{i=1}^{k} (\det(A^{i,i})')^{n_i - 1}$. Since the determinant of an upper triangular block matrix is the product of the determinants of the blocks on the diagonal,

$$\det A = \det D_1 \det D_4 = \prod_{i=1}^{k} \det(A^{i,i})' \prod_{i=1}^{k} (\det(A^{i,i})')^{n_i - 1} = \prod_{i=1}^{k} (\det(A^{i,i})')^{n_i}. \quad \square$$
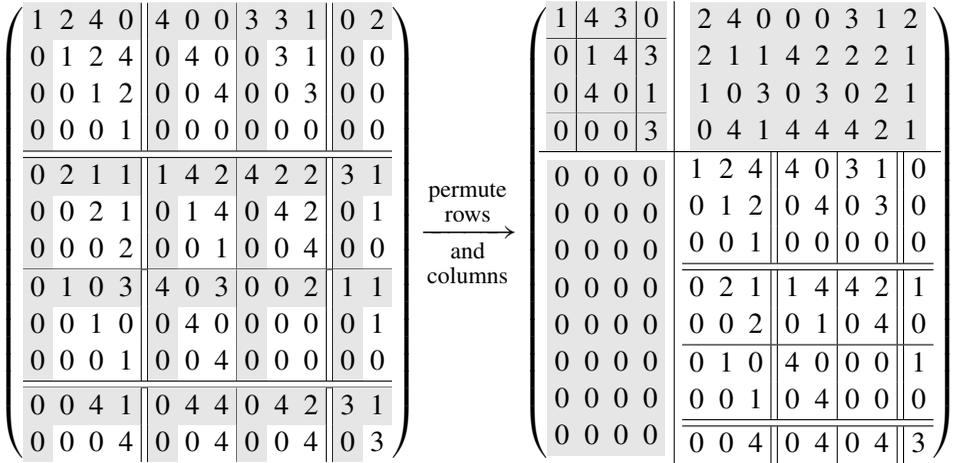
$$
\left(\begin{array}{cccc||cccccc||cc}
1 & 2 & 4 & 0 & 4 & 0 & 0 & 3 & 3 & 1 & 0 & 2 \\
0 & 1 & 2 & 4 & 0 & 4 & 0 & 0 & 3 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 4 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 2 & 1 & 1 & 1 & 4 & 2 & 4 & 2 & 2 & 3 & 1 \\
0 & 0 & 2 & 1 & 0 & 1 & 4 & 0 & 4 & 2 & 0 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 \\
0 & 1 & 0 & 3 & 4 & 0 & 3 & 0 & 0 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 4 & 1 & 0 & 4 & 4 & 0 & 4 & 2 & 3 & 1 \\
0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 3
\end{array}\right)
\xrightarrow[\substack{\text{and}\\ \text{columns}}]{\substack{\text{permute}\\ \text{rows}}}
\left(\begin{array}{cccc|cccccccc}
1 & 4 & 3 & 0 & 2 & 4 & 0 & 0 & 0 & 3 & 1 & 2 \\
0 & 1 & 4 & 3 & 2 & 1 & 1 & 4 & 2 & 2 & 2 & 1 \\
0 & 4 & 0 & 1 & 1 & 0 & 3 & 0 & 3 & 0 & 2 & 1 \\
0 & 0 & 0 & 3 & 0 & 4 & 1 & 4 & 4 & 4 & 2 & 1 \\
\hline
0 & 0 & 0 & 0 & \multicolumn{3}{c}{} & & & & & \\
\end{array}\right)
$$

(The right-hand matrix continues as a block structure:)

$$
\begin{array}{ccc||cccc||c}
1 & 2 & 4 & 4 & 0 & 3 & 1 & 0 \\
0 & 1 & 2 & 0 & 4 & 0 & 3 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 2 & 1 & 1 & 4 & 4 & 2 & 1 \\
0 & 0 & 2 & 0 & 1 & 0 & 4 & 0 \\
0 & 1 & 0 & 4 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 \\
\hline
0 & 0 & 4 & 0 & 4 & 0 & 4 & 3
\end{array}
$$

Figure 7. An illustration of Lemma 5.4 with a 12×12 matrix A (left). Here $n = 12$ is partitioned as $\{1 \cdot 4, 2 \cdot 3, 1 \cdot 2\}$, that is, $k = 3$, $(n_1, n_2, n_3) = (4, 3, 2)$, and $(b_1, b_2, b_3) = (1, 2, 1)$. Then $(B_1, B_2, B_3) = (1, 3, 4)$, $(BN_1, BN_2, BN_3) = (4, 10, 12)$, and in the permutation, columns 1, 5, 8, and 11 become columns 1,2,3, and 4, with the other columns becoming columns 5 through 12, keeping their original order. The rows are permuted the same way. In this example, $(A^{1,1})' = (1)$, $(A^{2,2})' = (1440)$, and $(A^{3,3})' = (3)$, and $\det A = 1^4 \cdot (\det(1440))^3 \cdot 3^2$.

The inductive argument in the proof of Lemma 5.4 implies that the columns of A can be permuted to obtain a block upper triangular matrix, where the diagonal blocks consist of n_i copies each of $(A^{i,i})'$, for $1 \leq i \leq k$. As a corollary, this implies that $A^{i,i}$ is invertible if and only if $(A^{i,i})'$ is invertible. For example, in Figure 6, $\det A = (\det A')^3$ and $\det D = (\det D')^2$.

Corollary 5.5. *Let $g(b_1 n_1, \ldots, b_k n_k)$ denote the number of invertible matrices of the type specified in Lemma 5.4. Then*

$$
g(b_1 n_1, \ldots, b_k n_k) = \left(\prod_{i=1}^{k} q^{b_i^2(n_i - 1)} \prod_{j=0}^{b_i - 1} (q^{b_i} - q^j) \right) \left(\prod_{1 \leq i < j \leq k} q^{2 b_i b_j n_j} \right).
$$

Proof. First we fix i and count the number of different possibilities for $A^{i,i}$. Since $(A^{i,i})'$ must be invertible, and $(A^{i,i})'$ is $b_i \times b_i$, there are $|\mathrm{GL}_{b_i}(\mathbb{F})| = \prod_{j=0}^{b_i - 1} (q^{b_i} - q^j)$ different ways to choose the elements of $(A^{i,i})'$. For each of the b_i^2 blocks in $A^{i,i}$, there are $n_i - 1$ other diagonals whose entries may be nonzero, and they may be chosen arbitrarily, so there are $q^{b_i^2(n_i - 1)}$ ways to choose these other entries. Thus

there are

$$f(b_i, n_i) = \left(q^{b_i^2(n_i-1)} \prod_{j=0}^{b_i-1} (q_i^n - q^j) \right)$$

ways to choose the entries in a diagonal block $A^{i,i}$.

All of the other entries in A which are not required to be zero may be chosen arbitrarily, subject to the constraint that each block is streaky upper triangular. If $i < j$, an element of $\mathrm{SUT}_{n_i, n_j}(\mathbb{F})$ has n_j diagonals that may be nonzero, and for $i \neq j$ there are $b_i b_j$ blocks in each of $\mathrm{SUT}_{n_i, n_j}(\mathbb{F})$ and $\mathrm{SUT}_{n_j, n_i}(\mathbb{F})$. Thus there are a total of

$$\prod_{1 \leq i < j \leq k} q^{2b_i b_j n_j}$$

ways to choose these entries. We conclude that

$$g(b_1 n_1, \ldots, b_k n_k) = \left(\prod_{i=1}^{k} f(b_i, n_i) \right) \left(\prod_{1 \leq i < j \leq k} q^{2b_i b_j n_j} \right)$$

$$= \left(\prod_{i=1}^{k} q^{b_i^2(n_i-1)} \prod_{j=0}^{b_i-1} (q^{b_i} - q^j) \right) \left(\prod_{1 \leq i < j \leq k} q^{2b_i b_j n_j} \right). \quad \square$$

Now we determine $|C(J)|$ for any $J \in J_n(\mathbb{F})$. Any matrix in $C(J)$ can be represented as a block diagonal matrix, where each block on the diagonal is of the type specified in Lemma 5.4. To be precise, let $n = n_1 + n_2 + \cdots + n_l$, and $J \in J_n(\mathbb{F})$ have double partition type T given by

$$\{\{b_{1,1} n_{1,1}, \ldots, b_{1,k_1} n_{1,k_1}\}, \{b_{2,1} n_{2,1}, \ldots, b_{2,k_2} n_{2,k_2}\}, \ldots, \{b_{l,1} n_{l,1}, \ldots, b_{l,k_l} n_{l,k_l}\}\},$$

where, for $1 \leq \alpha \leq l$, we have $n_\alpha = \sum_{i=1}^{k_\alpha} b_{\alpha,i} n_{\alpha,i}$ and $n_{\alpha,i} > n_{\alpha,j}$ if $i < j$. Denote the eigenvalue associated with $\{b_{\alpha,1} n_{\alpha,1}, \ldots, b_{\alpha,k_\alpha} n_{\alpha,k_\alpha}\}$ by λ_α. We can represent J as an $l \times l$ block diagonal matrix where $J^{\alpha,\alpha}$ contains the Jordan blocks with eigenvalue λ_α. If A commutes with J, then A can be represented as an $l \times l$ block matrix where the block $A^{\alpha,\beta}$ is $n_\alpha \times n_\beta$, and Lemma 5.2 guarantees that all off-diagonal blocks $A^{\alpha,\beta}$, $\alpha \neq \beta$ contain all zeros. So A is a block diagonal matrix, which is invertible if and only if each block on the diagonal is itself an invertible matrix. Thus to determine the number of invertible matrices $A \in C(J)$, it suffices to determine for each $1 \leq \alpha \leq l$ the number of invertible matrices $A^{\alpha,\alpha}$.

The matrix $A^{\alpha,\alpha}$ can be represented as a $k_\alpha \times k_\alpha$ block matrix, where $(A^{\alpha,\alpha})^{i,j}$ is $b_{\alpha,i} n_{\alpha,i} \times b_{\alpha,j} n_{\alpha,j}$, and is itself a $b_{\alpha,i} \times b_{\alpha,j}$ block matrix where Lemma 5.1 implies each block is an element of $\mathrm{SUT}_{n_{\alpha,i}, n_{\alpha,j}}(\mathbb{F})$. By Lemma 5.4 it is invertible if and only if the diagonal blocks $(A^{\alpha,\alpha})^{i,i}$ are invertible. Corollary 5.5 implies the number of invertible matrices that have the required block structure is given by

$$|C(J)| = \prod_{\alpha=1}^{l} g(b_{\alpha,1}n_{\alpha,1}, \ldots, b_{\alpha,k_\alpha}n_{\alpha,k_\alpha})$$

$$= \prod_{\alpha=1}^{l} \left(\left(\prod_{i=1}^{k_\alpha} q^{b_{\alpha,i}^2(n_{\alpha,i}-1)} \prod_{j=0}^{b_{\alpha,i}-1} (q^{b_{\alpha,i}} - q^j) \right) \prod_{1 \le i < j \le k_\alpha} q^{2b_{\alpha,i}b_{\alpha,j}n_{\alpha,j}} \right). \quad (4)$$

Using the matrix J from Figure 5 as an example, the variables in (4) are

$$l = 3, \quad (n_1, n_2, n_3) = (7, 2, 2), \quad (k_1, k_2, k_3) = (2, 1, 1),$$

$$n_1 = 7 = 1 \cdot 3 + 2 \cdot 2 = b_{1,1}n_{1,1} + b_{1,2}n_{1,2},$$

$$n_2 = 2 = 2 \cdot 1 = b_{2,1}n_{2,1}, \quad n_3 = 2 = 1 \cdot 2 = b_{3,1}n_{3,1}.$$

Thus for the matrix A in the figure to be invertible, there are $|GL_1(\mathbb{F})| = q - 1$ choices each for a and j, $|GL_2(\mathbb{F})| = (q^2 - 1)(q^2 - q)$ choices each for $\{d, q, w, f\}$ and $\{h, y, z, i\}$, and q choices for each other letter. Thus

$$|C(J)| = (q-1)^2((q^2-1)(q^2-q))^2q^{16}.$$

6. $\mathbf{Eig_3(\mathbb{F})}$ and $\mathbf{Eig_4(\mathbb{F})}$

In this section we compute $|Eig_3(\mathbb{F})|$ and $|Eig_4(\mathbb{F})|$ for any field $\mathbb{F}$ with q elements.

Theorem 6.1. *The number of 3×3 matrices with entries in $\mathbb{F}$ whose eigenvalues are all in $\mathbb{F}$ is*

$$|Eig_3(\mathbb{F})| = \tfrac{1}{6}q^9 + \tfrac{5}{6}q^8 + \tfrac{2}{3}q^7 - \tfrac{1}{6}q^6 - \tfrac{5}{6}q^5 + \tfrac{1}{3}q^4.$$

Proof. There are three partitions of 3: $3 = 1 + 1 + 1 = 1 + 2 = 3$, and 6 double partitions:

$$T_{3,1} = \{\{1\}, \{1\}, \{1\}\}, \quad T_{3,3} = \{\{1, 1, 1\}\}, \quad T_{3,5} = \{\{2, 1\}\},$$

$$T_{3,2} = \{\{1, 1\}, \{1\}\}, \quad T_{3,4} = \{\{2\}, \{1\}\}, \quad T_{3,6} = \{\{3\}\}.$$

For $1 \le i \le 6$, Lemma 4.1 gives the value of $t(T_{3,i})$. Corollary 5.3 gives the form of $C(J)$ for $J \in T_{3,i}$, and the value of $c(T_{3,i})$ is determined by (4). This information is summarized in Table 1.

Using (3), we conclude that $|Eig_3(\mathbb{F})|$ equals

$$\sum_{i=1}^{6} t(T_{3,i}) \frac{(q^3-1)(q^3-q)(q^3-q^2)}{c(T_{3,i})} = \tfrac{1}{6}q^9 + \tfrac{5}{6}q^8 + \tfrac{2}{3}q^7 - \tfrac{1}{6}q^6 - \tfrac{5}{6}q^5 + \tfrac{1}{3}q^4. \quad \square$$

Theorem 6.2. *The number $|Eig_4(\mathbb{F})|$ of 4×4 matrices with entries in $\mathbb{F}$ whose eigenvalues are all in $\mathbb{F}$ equals*

$$= \tfrac{1}{24}q^{16} + \tfrac{3}{8}q^{15} + \tfrac{11}{12}q^{14} + \tfrac{5}{8}q^{13} - \tfrac{1}{4}q^{12} - \tfrac{1}{8}q^{11} - \tfrac{5}{12}q^{10} + \tfrac{3}{8}q^9 + \tfrac{5}{24}q^8 - \tfrac{1}{4}q^7 - \tfrac{1}{2}q^6.$$

$T_{3,i}$	$t(T_{3,i})$	$C(J): J \in T_{3,i}$	$c(T_{3,i})$
$T_{3,1}$ $\left\{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\right\}$	$\binom{q}{3}$	$\left\{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \neq 0\right\}$	$(q-1)^3$
$T_{3,2}$ $\left\{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}\right\}$	$q(q-1)$	$\left\{\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix} : e, ad - bc \neq 0\right\}$	$(q^2-1)(q^2-q)(q-1)$
$T_{3,3}$ $\left\{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right\}$	q	$\mathrm{GL}_3(\mathbb{F})$	$(q^3-1)(q^3-q)(q^3-q^2)$
$T_{3,4}$ $\left\{\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}\right\}$	$q(q-1)$	$\left\{\begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, c \neq 0\right\}$	$q(q-1)^2$
$T_{3,5}$ $\left\{\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right\}$	q	$\left\{\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{bmatrix} : a, e \neq 0\right\}$	$q^3(q-1)^2$
$T_{3,6}$ $\left\{\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\right\}$	q	$\left\{\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} : a \neq 0\right\}$	$q^2(q-1)$

Table 1. Values of $t(T_{3,i})$ and $c(T_{3,i})$.

Proof. There are five partitions of 4:

$$4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2 = 1 + 3 = 4,$$

and 14 double partitions:

$$T_{4,1} = \{\{1\}, \{1\}, \{1\}, \{1\}\}, \qquad T_{4,8} = \{\{2, 1\}, \{1\}\},$$
$$T_{4,2} = \{\{1, 1\}, \{1\}, \{1\}\}, \qquad T_{4,9} = \{\{2, 1, 1\}\},$$
$$T_{4,3} = \{\{1, 1\}, \{1, 1\}\}, \qquad T_{4,10} = \{\{2\}, \{2\}\},$$
$$T_{4,4} = \{\{1, 1, 1\}, \{1\}\}, \qquad T_{4,11} = \{\{2, 2\}\},$$
$$T_{4,5} = \{\{1, 1, 1, 1\}\}, \qquad T_{4,12} = \{\{3\}, \{1\}\},$$
$$T_{4,6} = \{\{2\}, \{1\}, \{1\}\}, \qquad T_{4,13} = \{\{1, 3\}\},$$
$$T_{4,7} = \{\{2\}, \{1, 1\}\}, \qquad T_{4,14} = \{\{4\}\}.$$

The proof proceeds in the same way as the proof of Theorem 6.1, with the relevant information summarized in Table 2. $\qquad \square$

	$T_{4,i}$	$t(T_{4,i})$	$C(J) : J \in T_{4,i}$	$c(T_{4,i})$
$T_{4,1}$	$\left\{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}\right\}$	$\binom{q}{4}$	$\left\{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}\right\}$	$(q-1)^4$
$T_{4,2}$	$\left\{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}\right\}$	$\binom{q}{2}(q-2)$	$\left\{\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix}\right\}$	$(q^2-q)(q^2-1)(q-1)^2$
$T_{4,3}$	$\left\{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}\right\}$	$\binom{q}{2}$	$\left\{\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}\right\}$	$(q^2-q)^2(q^2-1)^2$
$T_{4,4}$	$\left\{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}\right\}$	$q(q-1)$	$\left\{\begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ 0 & 0 & 0 & j \end{bmatrix}\right\}$	$(q^3-1)(q^3-q)(q^3-q^2)(q-1)$
$T_{4,5}$	$\left\{\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right\}$	q	$\mathrm{GL}_4(\mathbb{F})$	$(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)$
$T_{4,6}$	$\left\{\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}\right\}$	$\binom{q}{2}(q-2)$	$\left\{\begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}\right\}$	$q(q-1)^3$
$T_{4,7}$	$\left\{\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}\right\}$	$q(q-1)$	$\left\{\begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & f \end{bmatrix}\right\}$	$q(q-1)(q^2-1)(q^2-q)$
$T_{4,8}$	$\left\{\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}\right\}$	$q(q-1)$	$\left\{\begin{bmatrix} a & b & c & 0 \\ 0 & a & 0 & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix}\right\}$	$q^3(q-1)^3$
$T_{4,9}$	$\left\{\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right\}$	q	$\left\{\begin{bmatrix} a & b & c & d \\ 0 & a & 0 & 0 \\ 0 & e & f & g \\ 0 & h & i & j \end{bmatrix}\right\}$	$(q^2-q)(q^2-1)(q-1)q^5$

Table 2. Values of $t(T_{4,i})$ and $c(T_{4,i})$.

(*Continued on next page*)

$T_{4,i}$		$t(T_{4,i})$	$C(J): J \in T_{4,i}$	$c(T_{4,i})$
$T_{4,10}$	$\left\{ \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right\}$	$\binom{q}{2}$	$\left\{ \begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{bmatrix} \right\}$	$q^2(q-1)^2$
$T_{4,11}$	$\left\{ \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\}$	q	$\left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & 0 & c \\ e & f & g & h \\ 0 & e & 0 & g \end{bmatrix} \right\}$	$(q^2-q)(q^2-1)q^4$
$T_{4,12}$	$\left\{ \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right\}$	$q(q-1)$	$\left\{ \begin{bmatrix} a & b & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \right\}$	$q^2(q-1)^2$
$T_{4,13}$	$\left\{ \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\}$	q	$\left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & e & f \end{bmatrix} \right\}$	$q^4(q-1)^2$
$T_{4,14}$	$\left\{ \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\}$	q	$\left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix} \right\}$	$q^3(q-1)$

Table 2. Values of $t(T_{4,i})$ and $c(T_{4,i})$ (*continued*).

7. The proportion of matrices with all eigenvalues in $\mathbb{F}$

Olšavský [2003] noted that as the size of the field $\mathbb{Z}_p$ increases, the proportion of matrices in $M_2(\mathbb{Z}_p)$ with all eigenvalues in $\mathbb{Z}_p$ approaches 1/2, that is,

$$\lim_{p \to \infty} \frac{|\mathrm{Eig}_2(\mathbb{Z}_p)|}{|M_2(\mathbb{Z}_p)|} = \lim_{p \to \infty} \frac{\frac{1}{2}p^4 + p^3 - \frac{1}{2}p^2.}{p^{2^2}} = \frac{1}{2}.$$

We generalize this result to show the proportion of matrices with all eigenvalues in $\mathbb{F}$ for any fixed n approaches $1/n!$ (Note that the leading coefficients for the polynomials $|\mathrm{Eig}_3(\mathbb{F})|$ and $|\mathrm{Eig}_4(\mathbb{F})|$ are $1/3!$ and $1/4!$, respectively, so the generalization is true for these cases).

Theorem 7.1. *Let $\mathbb{F}$ be a finite field with q elements. Then*

$$\lim_{q \to \infty} \frac{|\mathrm{Eig}_n(\mathbb{F})|}{|M_n(\mathbb{F})|} = \frac{1}{n!}.$$

Proof. We know $|M_n(\mathbb{F})| = q^{n^2}$ and our method for determining $|\mathrm{Eig}_n(\mathbb{F})|$ implies that it is a polynomial in the variable q: Denote the double partitions in $\mathrm{DP}(n)$ by

$T_{n,i}$ for $1 \leq i \leq |\mathrm{DP}(n)|$. For each i, $t(T_{n,i})$ is a polynomial, and since for each $J \in J_n(\mathbb{F})$, $C(J)$ is a subgroup of $\mathrm{GL}_n(\mathbb{F})$, the polynomial $|\mathrm{GL}_n(\mathbb{F})|$ is divisible by $|C(J)|$, and thus $s(T_{n,i})$ is also a polynomial. To evaluate the limit, we must determine the leading coefficient of $|\mathrm{Eig}_n(\mathbb{F})|$. Let $\deg f$ denote the degree (in the variable q) of the polynomial f. Then, since the degree of a sum of a fixed number of polynomials is equal to the maximum degree of the polynomials,

$$
\begin{aligned}
\deg |\mathrm{Eig}_n(\mathbb{F})| = \deg \sum_{i=1}^{|\mathrm{DP}(n)|} t(T_{n,i}) s(T_{n,i}) &= \max_{1 \leq i \leq |\mathrm{DP}(n)|} \deg(t(T_{n,i}) s(T_{n,i})) \\
&= \max_{1 \leq i \leq |\mathrm{DP}(n)|} \deg \left(t(T_{n,i}) \frac{|\mathrm{GL}_n(\mathbb{F})|}{c(T_{n,i})} \right) \\
&= \max_{1 \leq i \leq |\mathrm{DP}(n)|} \deg t(T_{n,i}) + \deg |\mathrm{GL}_n(\mathbb{F})| - \deg c(T_{n,i}) \\
&= \max_{1 \leq i \leq |\mathrm{DP}(n)|} n^2 + \deg t(T_{n,i}) - \deg c(T_{n,i}).
\end{aligned}
$$

Let $T_{n,1}$ be the double partition $\{\{1\}, \{1\}, \ldots, \{1\}\}$, corresponding to the type of Jordan form with all n eigenvalues distinct. For all n, we will show the maximum is attained only for $i = 1$. Indeed, since $\deg t(T_{n,i})$ is equal to the number of distinct eigenvalues in the type, $\deg t(T_{n,1}) = n$, and $\deg t(T_{n,i}) < n$, for all $i > 1$. Furthermore, by (4), $\deg c(T_{n,1}) = n$, and $\deg c(T_{n,i}) \geq n$ for all $i > 1$. Thus $\deg |\mathrm{Eig}_n(\mathbb{F})| = n^2 + \deg t(T_{n,1}) - \deg c(T_{n,1}) = n^2$, and the leading coefficient of $|\mathrm{Eig}_n(\mathbb{F})|$ is equal to the leading coefficient of $t(T_{n,1}) \cdot |\mathrm{GL}_n(\mathbb{F})|/c(T_{n,1})$.

Since $t(T_{n,1}) = \binom{q}{n}$, $|\mathrm{GL}_n(\mathbb{F})| = \prod_{i=0}^{n-1}(q^n - q^i)$, and $c(T_{n,1}) = (q-1)^n$, the leading coefficient of $t(T_{n,i}) \cdot |\mathrm{GL}_n(\mathbb{F})|/c(T_{n,i})$ is $1/n!$. $\square$

Corollary 7.2. *Let* $\mathrm{Ediff}_n(\mathbb{F}) \subseteq \mathrm{Eig}_n(F)$ *denote the set of matrices in* $M_n(\mathbb{F})$ *with all eigenvalues in* $\mathbb{F}$ *and all eigenvalues distinct. Then*

$$
\lim_{q \to \infty} \frac{|\mathrm{Ediff}_n(\mathbb{F})|}{|\mathrm{Eig}_n(\mathbb{F})|} = 1.
$$

Thus, for large enough finite fields, nearly all matrices with all eigenvalues in the field have all different eigenvalues.

Proof. In the notation of the proof of Theorem 7.1,

$$
|\mathrm{Ediff}_n(\mathbb{F})| = t(T_{n,1}) \cdot |\mathrm{GL}_n(\mathbb{F})|/c(T_{n,1}) = q^{n^2}/n! + o(q^{n^2}).
$$

Thus $|\mathrm{Ediff}_n(\mathbb{F})|$ and $|\mathrm{Eig}_n(\mathbb{F})|$ are both polynomials in q with the same leading term. So as q increases the ratio of these cardinalities will approach 1. $\square$

8. Conclusions

Given a finite field $\mathbb{F}$ and a positive integer n, we have given a method for computing $|\text{Eig}_n(\mathbb{F})|$ using (3). In 4.1 and (4) we gave the formulas necessary for computing the pieces of (3). In Section 6 we applied our method in the cases $n = 3$ and $n = 4$, and in Section 7, we showed that as the size of the finite field increases, the proportion of matrices in $M_n(\mathbb{F})$ that have all eigenvalues in $\mathbb{F}$ approaches $1/n!$.

There are a number of interesting directions for future research. For example, Theorem 7.1 describes the asymptotic behavior of $|\text{Eig}_n(\mathbb{F})|$ in the case where n is fixed and q goes to infinity. Is it possible to find an analogous statement in the case that q is fixed and n increases? It could also be interesting to find a geometric or combinatorial interpretation for these numbers. In the case $q = 2$, and $n = 1, 2, 3, 4$, $|\text{Eig}_n(\mathbb{Z}_2)| = 2, 14, 352$, and 33,632, respectively, and this sequence (or even a small piece of it) is not found in the *Online Encyclopedia of Integer Sequences*. Finally the polynomials for $|\text{Eig}_n(\mathbb{F})|$ could be studied further. For example, is the smallest nonzero power in $|\text{Eig}_n(\mathbb{F})|$ always $2(n-1)$ and if so, why? It seems likely that the coefficients will always sum to 1, but will they always be between -1 and 1, and if so, why? Are there patterns in the magnitudes or signs of the coefficients?

Acknowledgement

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