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In this paper, we study numerically the existence and stability of the steady state solutions of the reaction-diffusion equation, $u_t - au_{xx} - u + u^3 = 0$, and the Klein–Gordon equation, $u_{tt} + cu_t - au_{xx} - u + u^3 = 0$, with the boundary conditions: $u(-1) = u(1) = 0$. We show that as $a$ varies, the number of steady state solutions and their stability change.

## 1. Introduction

Reaction-diffusion systems are mathematical models which describe the density/concentration of a substance, a population, etc. The typical form is

$$u_t = a\Delta u + f(u), \tag{1}$$

where $u(x, t)$ is the state variable at position $x$ and at time $t$. $\Delta u$ is the diffusion term with the diffusion constant $a$, and $f(u)$ is the reaction term.

To motivate Equation (1), consider letting $a = 0$, and $f(u) = ku$, and we get perhaps the simplest population growth ordinary differential equation $u_t = ku$. Here $u$ represents the size of a population at time $t$ which is growing with instantaneous growth rate $k$. Now imagine taking a one-dimensional spatially distributed population (such as fish in a long, narrow river) and sectioning it into $N$ subpopulations lined up along the length of the river, each obeying $u_t = ku$. In this case, the fish from one subpopulation cannot move to an adjacent subpopulation. When we add the assumption that the fish can move between adjacent subpopulations and in fact will tend to move from more dense to less dense neighboring subpopulations (as is the case with diffusion), and taking the limit as $N \to \infty$, a one-dimensional linear reaction-diffusion equation $u_t = ku + au_{xx}$ is obtained. The $au_{xx}$ term has the

effect of limiting growth at point $x$ if $u(x, t)$ is concave down as a function of $x$ (for fixed $t$) at that point ($a$ assumed positive). Thus for example, if the neighbors of a subpopulation have smaller population densities, that subpopulation will tend to grow smaller due to emigration.

Thus we take the point of view that we can start with a standard dynamical system represented by a first-order differential equation and convert it into a distributed system represented by a partial differential equation by adding a diffusion term $au_{xx}$. The situation is similar with dynamical systems represented by second-order differential equations. For example, the equation for a damped mass-spring equation (with no driving force) is well known to be $mu_{tt} + cu_t + ku = 0$, where $u$ represents the displacement from rest of a mass attached to a fixed point by a spring and damper as a function of time $t$, and $m$, $c$, and $k$ are parameters representing the mass, the damping constant and the spring constant, respectively. By adding a diffusion term, we get the standard linear Klein–Gordon equation $mu_{tt} + cu_t + ku = au_{xx}$, which one can imagine to be a series of (vertical) harmonic oscillators tied together (horizontally) by more springs.

In this paper we are concerned with nonlinear systems, so in the case of the reaction-diffusion equation, instead of starting with the linear differential equation $u_t = ku$ we start with the nonlinear one: $u_t = u - u^3$ (which has stable fixed points at $u = \pm 1$ and an unstable fixed point at $u = 0$). This could be a model of a population with two stable values (after rescaling). This is similar to the example based on the classic spruce-budworm model studied in [Khain et al. 2010]. Converting this ODE to a PDE with the recipe of adding a diffusion term, we get the reaction-diffusion equation known as the Allen–Cahn equation:

$$u_t = au_{xx} + u - u^3. \tag{2}$$

For the case of the mass-spring system, we replace the linear restoring force by a nonlinear force of the form $f(u) = \alpha u + \beta u^3$ and end up with the Duffing equation:

$$u_{tt} + cu_t + \alpha u + \beta u^3 = 0, \tag{3}$$

where $c$ is damping, and $\alpha$ and $\beta$ are chosen so as to model various physical systems. For example, if $\alpha$ is negative and $\beta$ is positive, the force tends to move the mass away from $u = 0$ and towards $u = \pm 1$ (one system with this property would be when magnets are added above and below the mass, which is assumed to be, say, made of iron, so that the mass is pulled from its equilibrium position either up or down, see Figure 1). Again, adding a diffusion term we get a nonlinear Klein–Gordon equation:

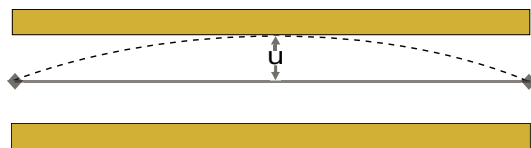$$u_{tt} + cu_t + \alpha u + \beta u^3 = au_{xx}, \, u(-1) = u(1) = 0. \tag{4}$$

**Figure 1.** Damped, thin metal wire, fixed at $x = \pm 1$, placed between two magnets at $u = \pm 1$.

In both equations above when the parameter $a$ is very small, we are back to a bunch of unconnected one-dimensional systems behaving independently. For the fish population in a river model, this would represent the idea that the fish could not migrate up or down the river, but can only reproduce and/or die in one location. For the mass-spring system this would represent totally unconnected, side-by-side oscillators with magnets. Thus we refer to $a$ as the *strength of connection* parameter.

In this paper, by using *spectral methods*, we study the numerical existence and the stability of the steady state solutions of reaction-diffusion equation:

$$u_t = au_{xx} + u - u^3 \tag{5}$$

and Klein–Gordon equation:

$$u_{tt} + cu_t = au_{xx} + u - u^3. \tag{6}$$

For both cases we take the boundary conditions as $u(-1) = u(1) = 0$. We show that as $a$ varies, the number of those special solutions and their stability change.

## 2. Spectral methods

***Introduction.*** Because of the nonlinear terms added to our second order partial differential equations, we chose to use a numerical method of analysis instead of finding exact solutions. We used spectral methods of analysis (see [Trefethen 2000]) instead of more traditional methods, such as finite differences, due to the exponential order of error convergence that the spectral methods demonstrate.

Spectral methods break the second order partial differential equation into a series of first order differential equations. Each first order differential equation lies at a point called a Chebyshev point, which is similar to the equally spaced points used in finite differences. However Chebyshev points are selected by taking the $x$-coordinates of equally spaced points on a half-circle Figure 2, top.

Chebyshev points are closer together towards the endpoints of the equation, which provides a much better polynomial fit and therefore much greater accuracy compared to equally spaced points as seen in Figure 2, bottom (reproduced from [Trefethen 2000]).
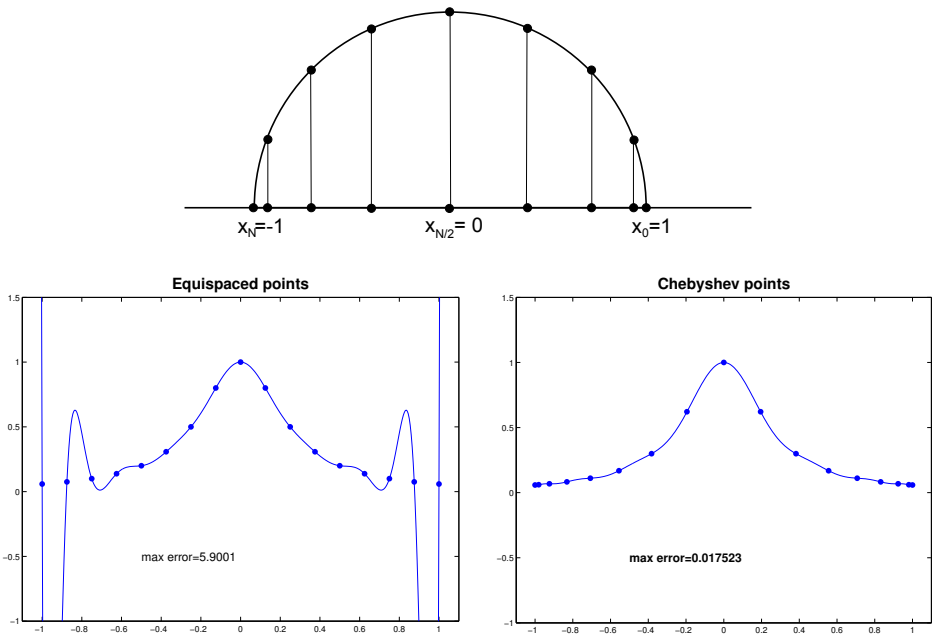
**Figure 2.** Top: Chebyshev points. Bottom: Chebyshev versus equispaced points.

Spectral methods have an exponential order of error convergence, so the error decreases much more rapidly than in other numerical methods. By increasing $N$, defined as the number of Chebyshev points, linearly, the error converges exponentially as in Figure 3 (reproduced from [Trefethen 2000]). However with finite differences method, the number of points are normally increased by a factor of ten to gain just one more decimal point of accuracy.
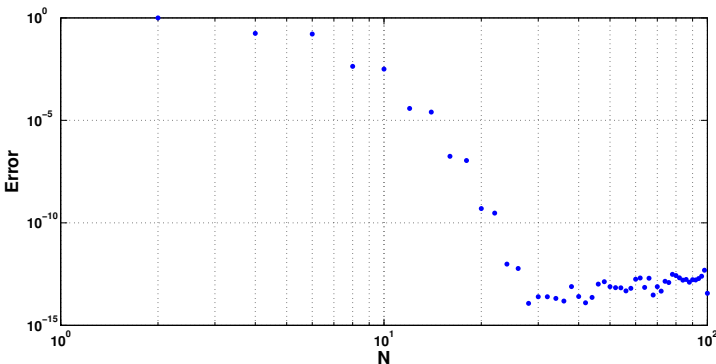


**Figure 3.** Convergence of spectral differentiation.

***Choosing N.*** In order to find the most accurate solution, picking the correct number of Chebyshev points is imperative. As discussed earlier, as $N$ is increased linearly the error decreases exponentially but only to a certain point. Eventually the error reaches a minimum and then begins to increase slowly due to machine error.

**Claim 2.1.** *For the reaction diffusion equation* (5), *the minimum error occurs at approximately* $N = 14$.

*Proof.* In order to select the optimal number of Chebyshev points, we calculate the error by comparing the exact solution of the linear form of (5) to the solution calculated with spectral methods. The linear form of (5) is

$$u_t = au_{xx} + u, \quad u(-1) = u(1) = 0. \tag{7}$$

By using the *separation of variables method*, we get the general solution of (7) as

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{(1-a/4(n\pi)^2)t} \sin(n\pi(x+1)/2). \tag{8}$$

If we pick our initial data as the eigenfunction $u(x, 0) = \sin(\pi(x+1)/2)$, we get $c_1 = 1$ and $c_n = 0$ for $n \geq 2$. So the exact solution of (7) with that initial data is

$$u(x, t) = e^{(1-a/4\pi^2)t} \sin(\pi(x+1)/2). \tag{9}$$

The approximate solution of (7) derived from the spectral methods:

$$u(x, t) = u_0 e^{At},$$

where $A = aD^2 + 1$, $u_0 = \sin \pi(x+1)/2$ and $D^2$ is the Chebyshev matrix derived by spectral methods.

Now we define the error as

$$\text{Error} = \|\text{Exact solution} - \text{Approximate solution}\|$$
$$= \left\| \left( e^{(1-a/4\pi^2)t} - e^{(aD^2+1)t} \right) \sin(\pi(x+1)/2) \right\|.$$

where $\| \cdot \|$ represents the Euclidean norm. In Figure 4, for various $a$ values, we observe that the error reaches its minimum at approximately $N = 14$. □

**Claim 2.2.** *For the Klein–Gordon equation* (6), *the minimum error occurs at approximately* $N = 12$.

*Proof.* Similar to what we did for the reaction-diffusion equation, we calculate the error by comparing the exact solution of the linear form of (6) to the solution calculated with spectral methods. The linear form of (6) is

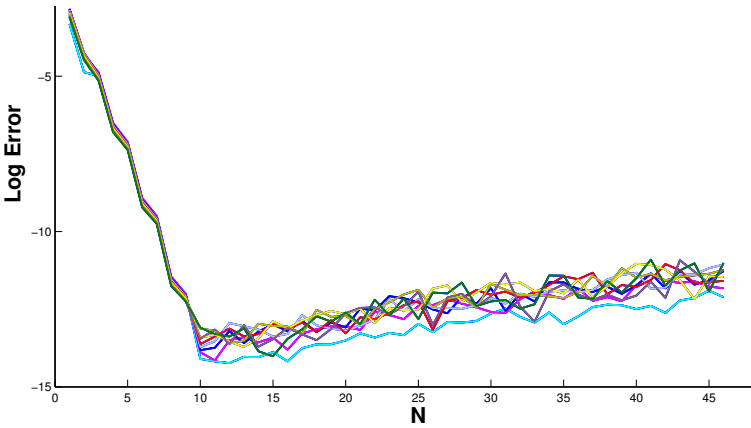$$u_{tt} + cu_t = au_{xx} + u, \quad u(-1) = u(1) = 0. \tag{10}$$

**Figure 4.** Error plots for $a$-values from 0.02 to 0.4.

By using the *separation of variables method* and picking the initial data as

$$u(x, 0) = \sin \frac{\pi(x+1)}{2} \tag{11}$$

and

$$u_t(x, 0) = \left( -\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1 - \frac{a}{4}\pi^2} \right) \sin \frac{\pi(x+1)}{2} \tag{12}$$

and assuming $c^2 + 4 - a\pi^2 > 0$, we get the exact solution of (10) as

$$u(x, t) = \sin \left( \frac{\pi(x+1)}{2} \right) \exp\left( \left( -\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1 - \frac{a}{4}\pi^2} \right) t \right). \tag{13}$$

The approximate solution of (10) derived from the spectral methods is found by using ODE45 by changing the second order differential equation

$$u_{tt} + cu_t = au_{xx} + u = (aD^2 + 1)u$$

into a first order system $y$, defining $z := u_t$,

$$\begin{bmatrix} u \\ z \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ aD^2 + I & -c \end{bmatrix} \begin{bmatrix} u \\ z \end{bmatrix}$$

and using the same initial conditions (11) and (12). We define the error same as we defined for the reaction-diffusion equation and use the Euclidean norm. Figure 5 shows the error for various $a$ values and fixed $c = 1$ (top), an for various $c$ values and fixed $a = 0.1$ (bottom); we observe that the error reaches its minimum at approximately $N = 12$. ☐
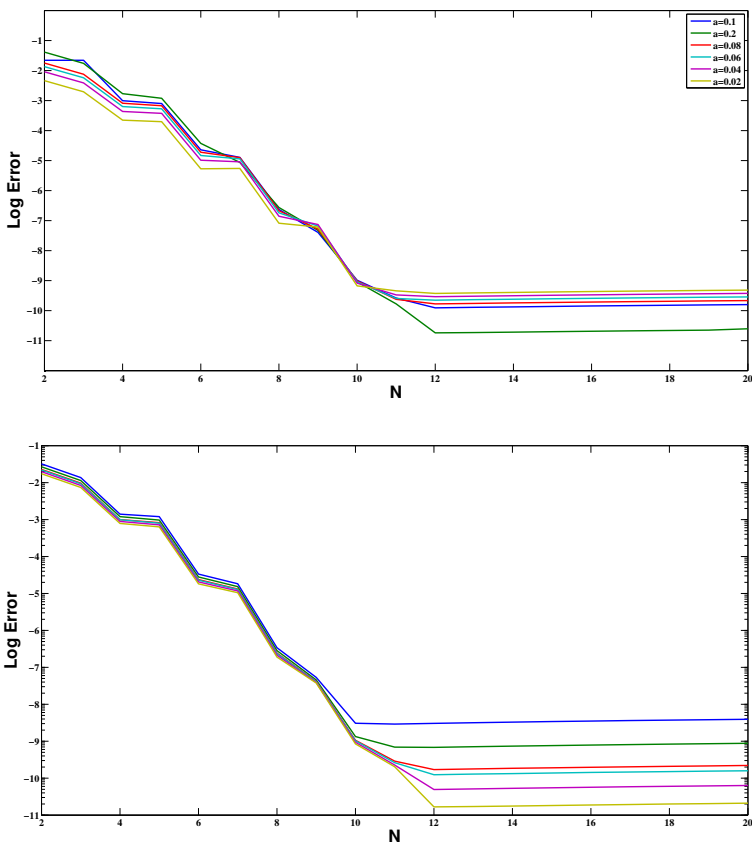
**Figure 5.** Log error. Top: $c = 1$, values of $a$ from 0.02 to 0.2. Bottom: $a = 0.1$, values of $c$ from 0.1 to 2.

## 3. Numerical existence of steady state solutions

***Steady state solutions.*** For both equations, the reaction-diffusion (5) and the Klein–Gordon equation (6), the steady state solutions $u(x, t) = \phi(x)$ satisfy the same equation

$$-a\phi'' - \phi + \phi^3 = 0, \quad \phi(-1) = \phi(1) = 0 \tag{14}$$

since $\phi_t = 0$ and $\phi_{tt} = 0$. Figure 6 shows the steady state solutions for three different $a$-values, $a = 0.2$, $a = 0.1$ and $a = 0.03$. Consecutively (14) has 3, 5 and 7 solutions. The numerical computations show that as $a$ is decreased, two new steady states of opposite sign and increasing number of oscillations occur for each bifurcation. The relation between $a$ and number of steady state solutions will be analytically studied on page 731.
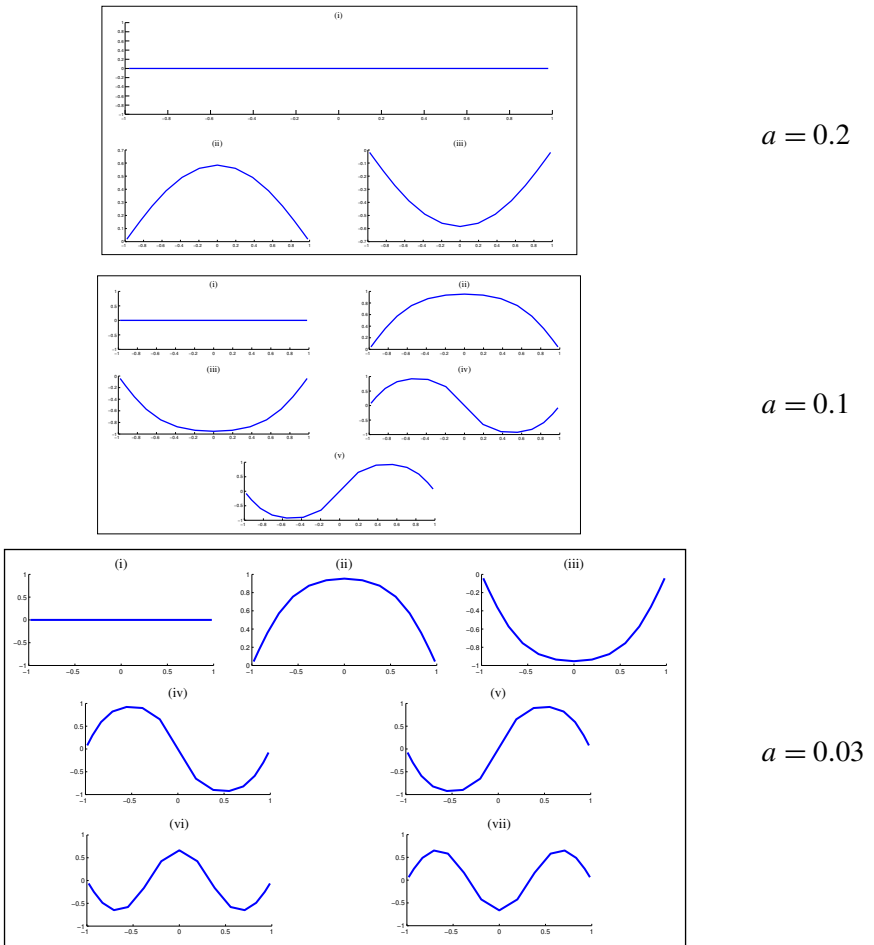
**Figure 6.** Steady state solutions for various values of $a$.

Note that zero is the trivial steady state solution for any $a$. For nonzero solutions, throughout this paper, we will name the steady state solutions. For example, we will name each convex steady state solution in Figure 6 an "n" solution — see graphs labeled (ii); each concave solution a "u" solution — graphs labeled (iii); graphs labeled (iv) for $a = 0.1$ and $a = 0.03$ are the "nu" solutions, and so on.

***Bifurcations.*** The number of steady states for a given $a$ value was verified with the shooting method. In Figure 7 we see shooting method plots for three $a$ values. The number of solutions at an $a$ value is the number of times the plot of $\phi(1)$ versus $\phi'(-1)$ touches the $\phi'(-1)$ axis where $\phi(1) = 0$. The bifurcation values to five decimal places were determined by changing $a$ until a new number of solutions was observed. These bifurcation values are confirmed in the next subsection.
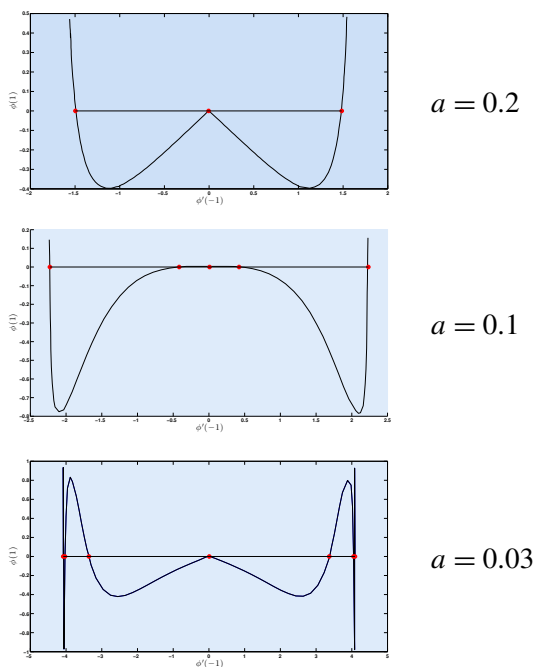
**Figure 7.** Existence of steady states $\phi(1)$ vs $\phi'(-1)$, for various values of $a$.

***Determination of bifurcation values.*** In Section 3 and in the preceding subsection, the steady state solutions to both the nonlinear reaction diffusion equation (5) and the nonlinear Klein–Gordon equation (6) were calculated numerically, and the number of solutions were found for three $a$ values. In particular it was found that for $a = 0.2$ there are three steady state solutions, for $a = 0.1$ there are five steady state solutions and for $a = 0.03$ there are seven steady state solutions.

In this section we find all bifurcation values for the parameter $a$; at each value two new solutions are added. We will show that these bifurcations values are at $a = (2/n\pi)^2$ for $n$ a positive integer. Thus the first few values are 0.4053, 0.1013, 0.0450, 0.0253; for $a > 0.4503$ there is only the zero solution, for $0.1013 < a < 0.4503$ there are three solutions, for $0.0450 < 0.1013$ there are five solutions, for $0.0253 < a < 0.0450$ there are seven solutions, and so on. This is consistent with the numerical results.

Consider the initial value problem

$$x' = y, \quad y' = -\lambda^2(x - x^3), \quad x(-1) = 0, \quad y(-1) = y_0, \qquad (15)$$

which is equivalent to equations (14) with $1/\lambda^2$ substituted for $a$.
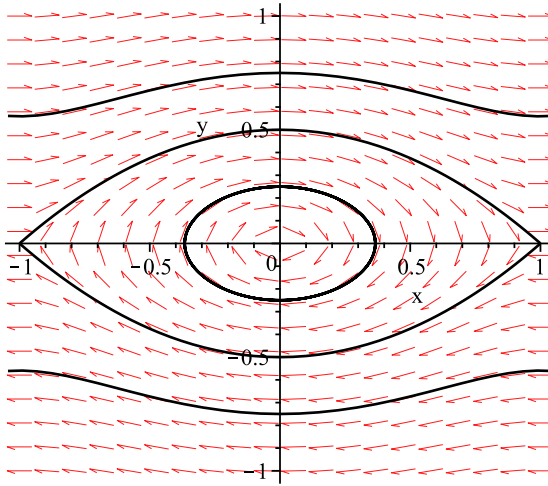
**Figure 8.** Phase portrait for $x' = y$, $y' = -\lambda^2(x - x^3)$ with $\lambda = 0.5$.

Clearly this system has saddle points at $(\pm 1, 0)$. Due to the symmetry of the vector field for this system, the fixed point at $(0, 0)$ is a center. Solution curves closer to the origin than the stable manifolds of $(\pm 1, 0)$ form closed loops. Thus the solution curves to (15) circle the origin in the clockwise direction for $y_0$ sufficiently small. See Figure 8.

Let $(x(y_0, t), y(y_0, t))$ represent the solution to (15) and let $\theta(y_0, t)$ represent the angle that the line segment connecting $(x(y_0, t), y(y_0, t))$ with $(0, 0)$ makes with the positive $x$-axis. Thus $\theta(y_0, t)$ is the angle in the polar coordinate representation of $(x(y_0, t), y(y_0, t))$, and hence $\tan \theta(y_0, t) = y(y_0, t)/x(y_0, t)$. Assume that $\theta$ starts at $\pi/2$, corresponding to the initial condition given in (15), and continues to decrease as the solution curve moves clockwise around the origin. Thus after one loop of the solution curve around the origin $\theta$ is $-3\pi/2$, after two loops $\theta$ is $-7\pi/2$, and so on.

**Theorem 3.1.** $\theta(y_0, t)$ *is an increasing function of* $y_0 > 0$ *for fixed t.*

*Proof.* For convenience we suppress the $y_0$ argument and write $\theta(y_0, t)$ as $\theta(t)$. Differentiating $\tan \theta(t) = y(t)/x(t)$ with respect to $t$ we get

$$\left(1 + \tan^2 \theta(t)\right)\theta'(t) = \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t)}.$$

Solving for $\theta'(t)$ and using $\tan^2 \theta(t) = y(t)^2/x(t)^2$ results in

$$\theta'(t) = \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t) + y^2(t)}.$$

Then using the DE system $x' = y$, $y' = -\lambda^2(x - x^3)$ we get

$$\theta'(t) = \frac{-\lambda^2 x^2(t) + \lambda^2 x^4(t) - y^2(t)}{x^2(t) + y^2(t)}.$$

Switching to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$) on the right side yields

$$\theta'(t) = \frac{-\lambda^2 r^2(t)\cos^2\theta(t) + \lambda^2 r^4(t)\cos^4\theta(t) - r^2(t)\sin^2\theta}{r^2(t)}$$

$$= -\lambda^2\cos^2\theta(t) + \lambda^2 r^2(t)\cos^4\theta(t) - \sin^2\theta(t).$$

Rearranging a bit we get

$$\theta'(t) = \lambda^2\cos^2\theta(t)\big(-1 + r^2(t)\cos^2\theta(t)\big) - \sin^2\theta(t). \tag{16}$$

Using $x = r\cos\theta$ we could also write (16) as

$$\theta'(t) = \lambda^2\cos^2\theta(t)(-1 + x^2(t)) - \sin^2\theta(t), \tag{17}$$

which shows that $\theta' < 0$ for $-1 < x < 1$. This is to be expected as we know that solution curves inside the unstable manifold of the fixed points circle the origin clockwise.

It is clear from (16) that $\theta'$ increases as a function of $r$ for fixed $\theta$. Since $\theta'$ is negative this means that for a given $\theta$, the solution curves farther from the origin are circling the origin at a slower angular rate (smaller absolute value) than those that are closer. See Figure 9.

This implies that for $y_0$ chosen so that the solution curve forms a closed loop, smaller $y_0$ means that the solution curve has wrapped further around the origin in the clockwise direction, and hence $\theta(y_0, t)$ is smaller (for fixed $t$). This means $\theta(y_0, t)$ is an increasing function of $y_0$ as claimed.

Let $\lambda > 0$ be fixed. Let $x_1(y_0, t)$ represent the solution to

$$x' = y, \quad y' = -\lambda^2 x, \quad x(-1) = 0, \quad y(-1) = y_0. \tag{18}$$
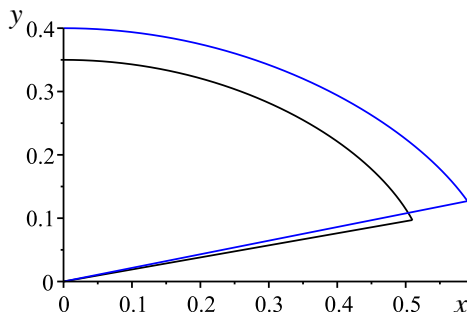


**Figure 9.** Solution curves closer to the origin move faster.

This is just (15) without the $x^3$ term. It is easy to show that

$$x_1(y_0, t) = y_0 \sin(\lambda(t+1))/\lambda \quad \text{and} \quad y_1(y_0, t) = y_0 \cos(\lambda(t+1)).$$

As before, let $(r_1, \theta_1)$ be the polar representation of $(x_1, y_1)$, so that $\theta_1(y_0, t)$ is the polar angle for the point $(x_1(y_0, t), y_1(y_0, t))$. Since

$$\tan \theta_1(y_0, t) = \frac{y_1(y_0, t)}{x_1(y_0, t)} = \frac{y_0 \cos(\lambda(t+1))}{y_0 \sin(\lambda(t+1))/\lambda} = \lambda \cot(\lambda(t+1)),$$

we know that $\theta_1(y_0, t)$ is in fact independent of $y_0$. Thus instead of $\theta_1(y_0, t)$ we will write $\theta_1(t)$.  □

**Theorem 3.2.** *Fix $t$ and $\lambda$. As $y_0 \to 0$, $\theta(y_0, t) \to \theta_1(t)$ monotonically.*

*Proof.* The monotonic part follows from the previous theorem. To prove the limit part, the basic idea is that $x^3$ is negligible compared to $x$ for small $x$, and so the linear and nonlinear vector fields, given in Equations (15) and (18), respectively, are indistinguishable for small $x$. We now flesh out the details of this argument.

For the IVP, $x'' + \lambda^2(x - x^3) = 0$, $x(-1) = 0$, $x'(-1) = y_0$, which is just (15) written in second-order form, we can multiply the DE by $x'$ to get

$$x'x'' + \lambda^2(x - x^3)x' = 0.$$

We can integrate both sides now to get

$$\tfrac{1}{2}(x')^2 + \lambda^2\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right) = C.$$

Then using $x(-1) = 0$ and $x'(-1) = y_0$ we get $C = y_0^2/2$. We now have

$$(x')^2 + \lambda^2\left(x^2 - \tfrac{1}{2}x^4\right) = y_0^2 \tag{19}$$

after substituting and multiplying by 2. Plotting (19) in the phase plane ($x$ on the horizontal axis and $x'$ on the vertical) for various $y_0$ we are back to the closed curves in Figure 8, which shows clearly that $x(t)$ can be "trapped" in an arbitrarily small region $-\varepsilon < x(t) < \varepsilon$ if $y_0$ is taken to be sufficiently small.  □

In order to finish the proof of the theorem we need to invoke a version of the Gronwall inequality:

**Lemma 3.3** (Gronwall's inequality: see for example [Howard 1998, Theorem 2.1]). *Let $X$ be a Banach space and $U \subset X$ an open set in $X$. Let $f, g : [a, b] \times U \to X$ be continuous functions and let $y, z : [a, b] \to U$ satisfy the initial value problems*

$$y'(t) = f(t, y(t)), \ y(a) = y_0$$
$$z'(t) = g(t, z(t)), \ z(a) = z_0. \tag{20}$$

*Also assume there is a constant $C \geq 0$ so that*

$$\|g(t, x_2) - g(t, x_1)\| \leq C \|x_2 - x_1\| \tag{21}$$

*and a continuous function $\phi : [a, b] \to [0, \infty)$ so that*

$$\|f(t, y(t)) - g(t, y(t))\| \leq \phi(t). \tag{22}$$

*Then for $t \in [a, b]$*

$$\|y(t) - z(t)\| \leq e^{C|t-a|} \|y_0 - z_0\| + e^{C|t-a|} \int_a^t e^{-C|s-a|} \phi(s)\, ds.$$

For our purposes the Banach space $X$ is just the real numbers, so that the norm is the absolute value.

For convenience we suppress the $y_0$ and write $\theta(t)$ instead of $\theta(y_0, t)$. We can rewrite (17) as

$$\theta'(t) = f(t, \theta(t)),$$

where

$$f(t, \theta) = -\lambda^2 \cos^2 \theta - \sin^2 \theta + \lambda^2 x^2(t) \cos^2 \theta$$

and where $x(t)$ comes from the solution to the full system in (15). Similarly, $\theta_1$ satisfies

$$\theta_1'(t) = g(t, \theta_1(t)),$$

where

$$g(t, \theta) = -\lambda^2 \cos^2 \theta - \sin^2 \theta.$$

We now apply the Gronwall inequality using the above choices for $f$ and $g$, where the interval $[a, b]$ is $[-1, 1]$ and where we choose the same initial condition $\pi/2$ for the DEs, that is, $\theta(-1) = \pi/2$ and $\theta_1(-1) = \pi/2$ (which corresponds to $x(0) = 0$ and $y(0) = y_0$ in rectangular coordinates). Equation (21) is called a Lipschitz condition and is satisfied by $g(t, \theta)$ for some $C$ because it is continuously differentiable as a function of $\theta$. Finally since

$$\left|f(t, \theta(t)) - g(t, \theta(t))\right| = \left|\lambda^2 x^2(t) \cos^2 \theta(t)\right| \leq \lambda^2 x^2(t),$$

we see that (22) is satisfied with $\phi(t) = \lambda^2 x^2(t)$. The conclusion of the Gronwall inequality follows, which means that for fixed $t \in [-1, 1]$ and $\lambda$

$$|\theta(t) - \theta_1(t)| \leq e^{C|t+1|} \int_{-1}^t e^{-C|s+1|} \lambda^2 x^2(s)\, ds$$

since the initial conditions are the same. If we assume that $|x(t)| < \varepsilon$ then

$$e^{C|t+1|} \int_{-1}^{t} e^{-C|s+1|} \lambda^2 x^2(s) \, ds < e^{C|t+1|} \int_{-1}^{t} e^{-C|s+1|} \lambda^2 \varepsilon^2 \, ds$$
$$= \frac{1}{C} \lambda^2 \varepsilon^2 \left( e^{C|t+1|} - 1 \right).$$

Following the above inequalities we have shown that

$$|\theta(y_0, t) - \theta_1(t)| < \frac{1}{C} \lambda^2 \varepsilon^2 \left( e^{C|t+1|} - 1 \right).$$

The theorem follows by recalling that $|x(t)|$ can be made arbitrarily small for $y_0$ sufficiently small.

**Theorem 3.4.** *The eigenvalues of the linear BVP*

$$x'' + \lambda^2 x = 0, \quad x(-1) = 0, \quad x(1) = 0 \tag{23}$$

*are the bifurcation points for the nonlinear BVP*

$$x'' + \lambda^2 (x - x^3) = 0, \quad x(-1) = 0, \quad x(1) = 0. \tag{24}$$

*Specifically the eigenvalues of the linear problem are $\lambda_n = n\pi/2$, and there is one solution to the nonlinear problem (the zero solution) for $0 \le \lambda \le \lambda_1$, three solutions to the nonlinear problem for $\lambda_1 < \lambda \le \lambda_2$, five solutions to the nonlinear problem for $\lambda_2 < \lambda \le \lambda_3$, and so on.*

*Proof.* If $\lambda_n$ is an eigenvalue for the BVP in (23) it is easy to show that $\lambda_n = n\pi/2$ for $n = 1, 2, \dots$ and that the corresponding eigenfunctions are any multiple of $\sin(\lambda_n(t+1))$. Note that the system of differential equations in (18) is equivalent to the differential equation in (23). Thus if $\lambda = \lambda_n$ for some $n \ge 1$, then a solution to (18) automatically satisfies $x(1) = 0$ and so is a solution to (23) for any $y_0$. This explains the relationship between the IVP in (18) and the BVP in (23).

The relationship between the nonlinear IVP in (15) and the nonlinear BVP in (24) is similar in that the differential equations are equivalent. However, because of the nonlinearity, (24) can be solved for *any* $\lambda$ by solving the IVP in (15) and varying $y_0$ until $x(1) = 0$ is obtained (sometimes called the "shooting method" for solving a BVP). Note that $x(1) = 0$ in the phase plane means that the angle $\theta(y_0, 1)$ is any of $-\pi/2, -3\pi/2, -5\pi/2, \dots$.

Now fix $\lambda$ and start with $y_0$ chosen so that the solution to the IVP in (15) is the stable manifold of $(1, 0)$. As $y_0$ decreases towards zero, $\theta(y_0, 1)$ will wrap clockwise around the origin until it reaches $\theta_1(1)$ in the limit, as shown in the previous theorem. Every time $\theta(y_0, 1)$ passes $-n\pi/2$ for $n$ odd we get a solution to the nonlinear BVP in (24). If $\lambda < \lambda_1$ this never happens (because $\theta_1(1) > -\pi/2$), so the only solution is the zero solution. If $\lambda_1 < \lambda < \lambda_2$ then $-3\pi/2 < \theta_1(y_0, 1) < -\pi/2$

and so $\theta(y_0, 1)$ will pass just $-\pi/2$, in which case we have the zero solution as well as one more solution. By symmetry, a third solution occurs for $y_0$ negative. If $\lambda_2 < \lambda < \lambda_3$, then $\theta(y_0, 1)$ will pass $-\pi/2$ and $-3\pi/2$ yielding two solutions, plus the zero solution, plus the symmetric solutions for $y_0$ negative, for a total of five. Proceeding in this manner, the theorem is proved. □

As stated earlier in this section, we have substituted $1/\lambda^2$ for the parameter $a$ in the nonlinear reaction-diffusion and Klein–Gordon equations, and thus the bifurcation values in terms of $a$ are $a = (2/n\pi)^2$.

## 4. Stability analysis

In order to determine the stability of a steady state solution, we look at the solution in the vicinity of the steady state and observe its behavior over time.

We assume that $u(x, t)$ is the solution such that $u_0 = u(x, 0)$ is in the vicinity of the steady state solution. The difference between the solution and steady state solution is known as the perturbation and is denoted as $v(x, t)$ and, satisfies

$$v(x, t) = u(x, t) - \phi(x). \tag{25}$$

Stability analysis will require us to study the long time behavior of the perturbation. If the solution diverges from the steady state ($\lim_{t\to\infty} \|v\| \to \infty$), then the steady state is unstable. If the solution does not diverge and the perturbation remains small, the steady state is stable.

***Stability of reaction-diffusion equation.*** By substituting (25) into (5), we get

$$(\phi + v)_t - a(\phi + v)_{xx} - (\phi + v) + (\phi + v)^3 = 0.$$

Since $-a\phi'' - \phi + \phi^3 = 0$ and $\phi_t = 0$, we get

$$v_t - av_{xx} - v + 3\phi^2 v + 3\phi v^2 + v^3 = 0.$$

By stability manifold theorem, we can say that the stability of the above equation will be similar to its linearized equation which is as follows:

$$v_t - av_{xx} - v + 3\phi^2 v = 0,$$

which can be rewritten as

$$v_t = (aD^2 + 1 - 3\phi^2)v = 0.$$

So the eigenvalues of the linearized RD operator $aD^2 + 1 - 3\phi^2$ will tell us if $v$ blows up in time, decreases to 0, or stays bounded.

Our numerical results show that the largest eigenvalues of the linearized RD operator about the "u" and "n" solutions are negative. This implies that these
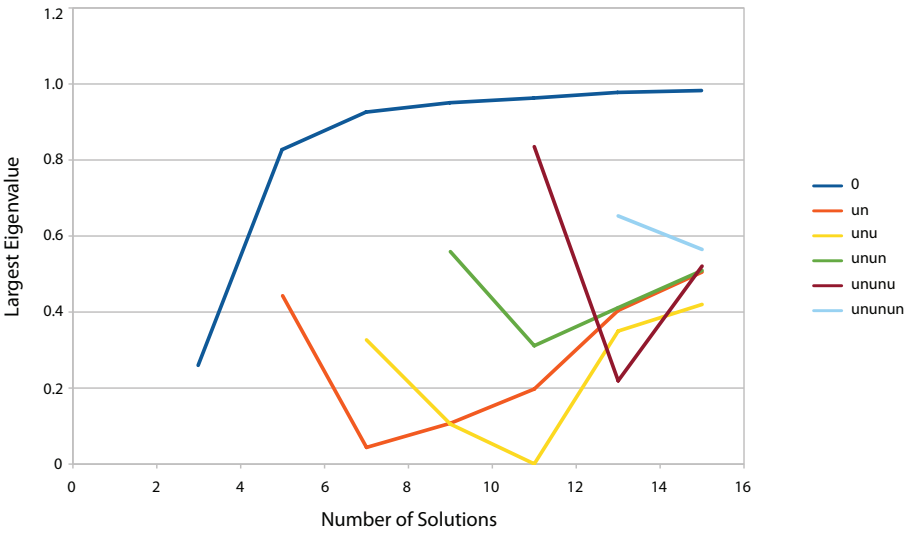
**Figure 10.** Largest eigenvalue of the linearized RD operator for the unstable steady states.

solutions are stable. All other solutions are unstable because the linearized RD operator has at least one positive eigenvalue. Figure 10 shows the largest eigenvalue of the linearized RD operator about each unstable steady state as the number of solutions from a given $a$-value increases (as the $a$-value decreases).

***Stability of Klein–Gordon equation.*** By substituting (25) into (6), we get

$$(\phi + v)_{tt} + c(\phi + v)_t - a(\phi + v)_{xx} - (\phi + v) + (\phi + v)^3 = 0.$$

Since $-a\phi'' - \phi + \phi^3 = 0$, and $\phi_t = 0$ and $\phi_{tt} = 0$, we get

$$v_{tt} - av_{xx} - v + 3\phi^2 v + 3\phi v^2 + v^3 = 0.$$

By the stable manifold theorem, we can say that the stability of the above equation will be similar to its linearized equation, which is as follows:

$$v_{tt} + cv_t - av_{xx} - v + 3\phi^2 v = 0.$$

Let's write it as a first order system by defining $v_t = w$. We get

$$\begin{bmatrix} v \\ w \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ aD^2 + I - 3\phi^2 & -c \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

So we need to find the eigenvalues of the operator matrix in this equation in order to find the long time behavior of $\begin{bmatrix} v \\ w \end{bmatrix}$.

Because all of the new steady states that occur after three solutions are unstable, we then look at how unstable they are. One metric for instability is the magnitude
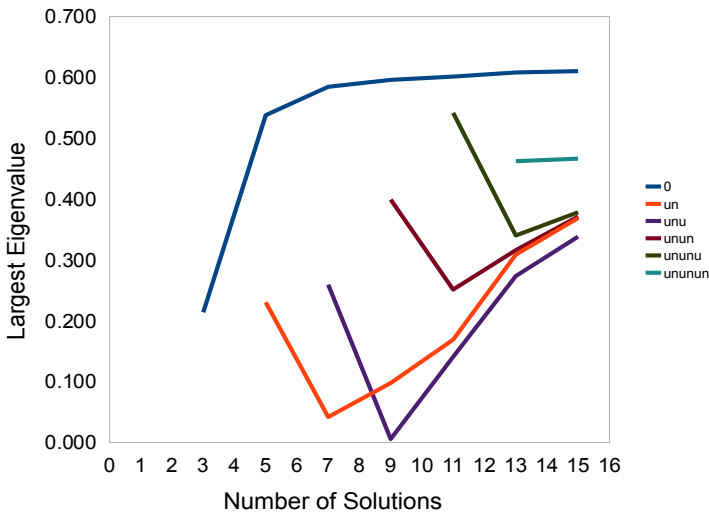
**Figure 11.** Largest eigenvalue of the linearized KG operator for the unstable steady states.

of the largest positive eigenvalue. The larger this eigenvalue is, the more unstable the steady state. Figure 11 shows the magnitude of the largest positive eigenvalue for various solutions as the number of solutions from a given $a$-value increases (as the $a$-value decreases). Note how the solution becomes more stable at first, and then becomes less stable as $a$ decreases. Also note that the zero solution becomes more unstable quickly but then approaches a level of instability asymptotically.

In Figure 12, steady states are indicated stable or unstable and are organized by the bifurcation range that they occur in (number of solutions for a range of values

Number of Solutions

| | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|---|---|
| **0** | stable | unstable | unstable | unstable | unstable | unstable | unstable | unstable |
| ⌒ | | stable | stable | stable | stable | stable | stable | stable |
| ⌵ | | stable | stable | stable | stable | stable | stable | stable |
| ⌒⌵ | | | unstable | unstable | unstable | unstable | unstable | unstable |
| ⌵⌒ | | | unstable | unstable | unstable | unstable | unstable | unstable |
| ⌒⌒⌵ | | | | unstable | unstable | unstable | unstable | unstable |
| ⌵⌵⌒ | | | | unstable | unstable | unstable | unstable | unstable |
| ⌒⌵⌵ | | | | | unstable | unstable | unstable | unstable |
| ⌵⌒⌒ | | | | | unstable | unstable | unstable | unstable |
| ⌒⌒⌵⌵ | | | | | | unstable | unstable | unstable |
| ⌵⌵⌒⌒ | | | | | | unstable | unstable | unstable |
| ⌒⌵⌵⌵ | | | | | | | unstable | unstable |
| ⌵⌒⌒⌒ | | | | | | | unstable | unstable |
| ⌒⌒⌵⌵⌵ | | | | | | | | unstable |
| ⌵⌵⌒⌒⌒ | | | | | | | | unstable |

**Figure 12.** The stability of different types of solutions as the value of $a$ decreases.
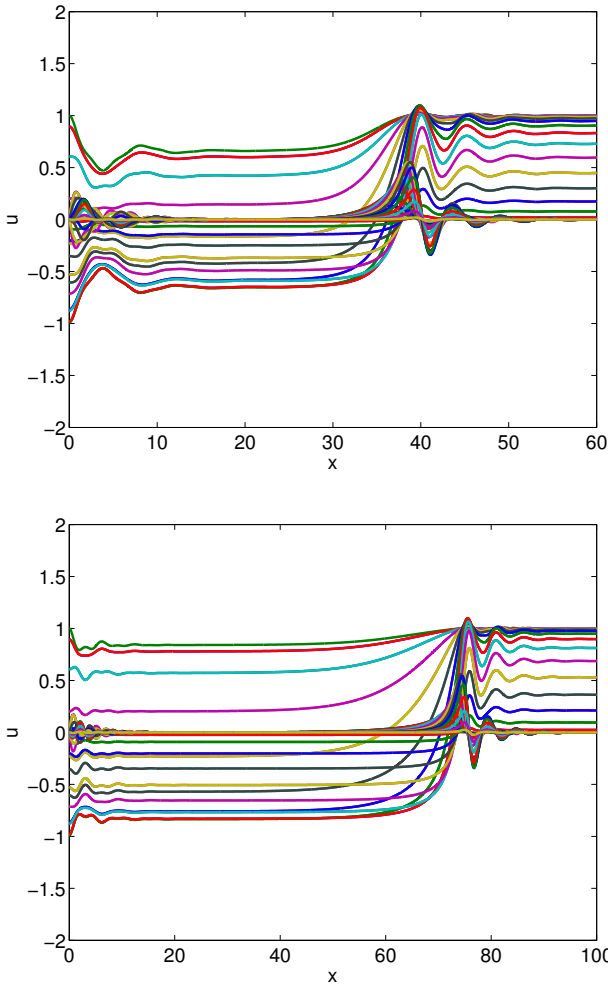
**Figure 13.** Side-view plots. Top: $a = 0.03$, $c = 0.5$. Bottom: $a = 0.02$, $c = 0.5$.

of $a$) and the look of the solution. For $a$ values with only one steady state, the only steady state is the zero solution and it is stable. For all other bifurcation ranges, the zero solution is unstable. The solutions depicted in the second and third row of Figure 12 are stable. All other steady states are unstable.

In Figure 13 we compare two side-view plots to verify that the increase in stability of unstable steady states as observed in Figure 11 actually happens. As expected, the side plot with $a = 0.02$ (9-solution range) took longer to move to a stable solution than the plot with $a = 0.03$ (7-solution range) because it was less unstable (the positive eigenvalues were closer to being negative).

## 5. Simulations

Figure 14 depicts simulations of the Klein–Gordon equation for the comparison made in the previous section between $a = 0.03$ and $a = 0.02$ with $c = 0.5$. The simulation with $a = 0.02$ takes longer than the simulation with $a = 0.03$ to reach a stable steady state solution. These images agree with the side-view plots in Figure 13 and the plot in Figure 11. The simulations can be seen at youtu.be/dlNbTOUUMX8 ($a = 0.02$) and youtu.be/ccdF6tU2Vcw ($a = 0.03$).
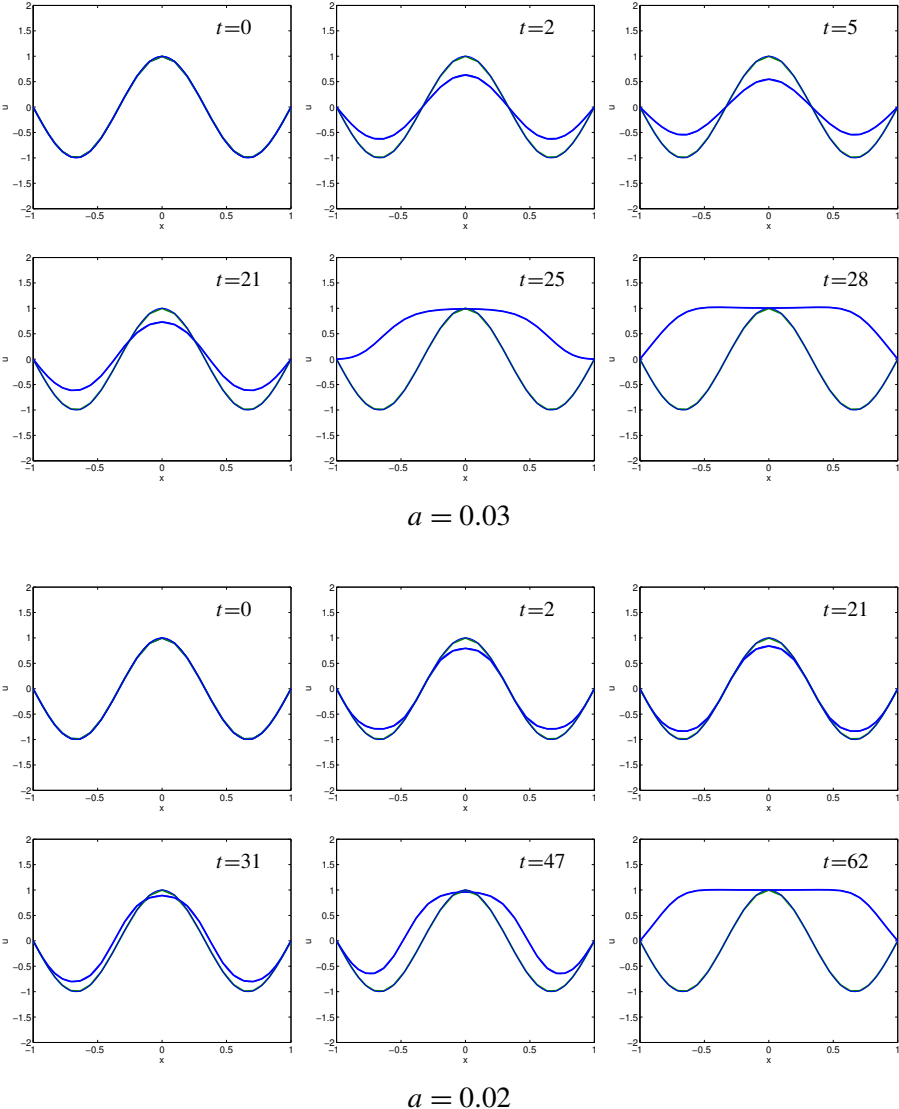


**Figure 14.** Results of the simulations for $a = 0.03$ and $a = 0.02$.

## References

[Howard 1998] R. Howard, "The Gronwall inequality", lecture notes, 1998, available at http://www.math.sc.edu/~howard/Notes/gronwall.pdf.

[Khain et al. 2010] E. Khain, Y. T. Lin, and L. M. Sander, "Fluctuations and stability in front propagation", preprint, 2010. arXiv 1009.5945v1

[Trefethen 2000] L. N. Trefethen, *Spectral methods in MATLAB*, Software, Environments, and Tools **10**, Society for Industrial and Applied Mathematics, Philadelphia, 2000. MR 2001c:65001 Zbl 0953.68643

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