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On commutators of matrices over unital rings

Michael Kaufman and Lillian Pasley



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Let  $R$  be a unital ring and let  $X \in M_n(R)$  be any upper triangular matrix of trace zero. Then there exist matrices  $A$  and  $B$  in  $M_n(R)$  such that  $X = [A, B]$ .

## 1. Introduction

Shoda [1936] proved that every matrix with trace zero over the complex numbers could be expressed as a commutator  $AB - BA$ . Albert and Muckenhoupt [1957] extended this result to matrices over any field. For matrices over commutative rings it is known that matrices of trace zero in general cannot be presented as commutators [Lissner 1961; Rosset and Rosset 2000]. Recently, Khurana and Lam [2012] showed every matrix with trace zero over any field can be expressed as a generalized commutator  $ABC - CBA$ . But the same result does not hold for matrices over commutative rings. Our work is motivated by the following question posed by Khurana and Lam: if  $n \geq 3$ , is every upper triangular matrix a generalized commutator over any ring  $S$  [Khurana and Lam 2012, Question 8.17]. In the case when  $n = 2$  this question has a negative answer as has been shown in [Khurana and Lam 2012, Theorem 8.11]. Using ideas due to Khurana and Lam we will give a simple proof of this case. We will also show that every  $n \times n$  upper triangular matrix of trace zero over any unital ring can be presented as a commutator.

## 2. Results

In this section, the trace of an  $n \times n$  matrix  $M = (x_{i,j})$  is denoted  $\text{tr}(M) = \sum_{k=1}^n x_{k,k}$ . Let  $R$  be any ring and  $S$  any commutative ring. We need some auxiliary results.

**Proposition 1** [Khurana and Lam 2012, Proposition 6.6]. *Let*

$$X = [A, B, C] = ABC - CBA,$$

where  $X, A, B, C \in M_n(S)$ . Then  $\text{tr}(BX) = 0$ .

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**Proposition 2** [Khurana and Lam 2012, Proposition 8.3]. *Let  $D \in R$  such that  $DC = CD \in Z(R)$  (the center of  $R$ ). If  $X = [A, B, C] \in R$ , then*

$$DX = [D, ABC] + [A, BCD] \quad \text{and} \quad XD = [D, CBA] + [A, BCD].$$

If  $X, D \in M_n(S)$ , then  $\text{tr}(XD) = \text{tr}(DX) = 0$ .

Khurana and Lam showed for  $n \geq 2$  there exist  $n \times n$  matrices that can not be expressed as generalized commutators. Now we use the preceding propositions to provide a different proof for the  $n = 2$  case.

**Theorem 3** [Khurana and Lam 2012, Theorem 8.11]. *There exists a  $2 \times 2$  upper triangular matrix that can not be expressed as a generalized commutator (i.e.,  $X \neq ABC - CBA$ ).*

*Proof.* Let  $A = (a_{ij})$ ,  $B = (b_{ij})$ ,  $C = (c_{ij})$ ,  $A, B, C \in M_2(S)$ , where  $S = \mathbb{C}[x, y, z]$  and  $x, y$ , and  $z$  are indeterminates. Now suppose  $X \in M_2(S)$  is the upper triangular matrix  $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$  such that  $X = ABC - CBA$ .

We begin by observing that

$$BX = \begin{pmatrix} b_{11}x & b_{11}y + b_{12}z \\ b_{21}x & b_{21}y + b_{22}z \end{pmatrix}.$$

By Proposition 1,  $\text{tr}(BX) = b_{11}x + b_{21}y + b_{22}z = 0$ . This implies that polynomials  $b_{11}$ ,  $b_{21}$ , and  $b_{22}$  cannot contain constant terms.

We consider the characteristic equation of  $A$ . From  $A^2 + \lambda A + \mu I = 0$  where

$$\lambda = -\text{tr}(A) = -a_{11} - a_{22} \quad \text{and} \quad \mu = \det(A) = a_{11}a_{22} - a_{12}a_{21},$$

we see that  $A(A + \lambda I) = -\mu I$ , and so  $A(A + \lambda I) \in Z(S)$ . Now we examine

$$(A + \lambda I)X = \begin{pmatrix} -a_{22}x & -a_{22}y + a_{12}z \\ a_{21}x & a_{21}y - a_{11}z \end{pmatrix}.$$

By Proposition 2,  $\text{tr}((A + \lambda I)X) = -a_{22}x + a_{21}y - a_{11}z = 0$ . This implies that polynomials  $a_{11}$ ,  $a_{21}$ , and  $a_{22}$  cannot contain constant terms. Similarly, polynomials  $c_{11}$ ,  $c_{21}$ , and  $c_{22}$  cannot contain constant terms. From  $X = ABC - CBA$  we obtain

$$x = a_{12}(b_{21}c_{11} + b_{22}c_{21}) + b_{12}(a_{11}c_{21} - a_{21}c_{11}) + c_{12}(-a_{11}b_{21} - a_{21}b_{22}). \quad (1)$$

Polynomials  $a_{11}$ ,  $a_{21}$ ,  $a_{22}$ ,  $b_{11}$ ,  $b_{21}$ ,  $b_{22}$ ,  $c_{11}$ ,  $c_{21}$ , and  $c_{22}$  contain no constant terms, so the right-hand side of (1) cannot contain a linear term. Since the left-hand side of (1) is a polynomial of degree 1, namely  $x$ , we arrive at a contradiction.  $\square$

Since there exist upper triangular matrices in  $M_n(S)$  that cannot be expressed as generalized commutators, we consider what can be said about upper triangular matrices with respect to commutators.

**Theorem 4.** *Let  $R$  be a unital ring and let  $X \in M_n(R)$  be any upper triangular matrix of trace zero. Then there exist matrices  $A$  and  $B$  in  $M_n(R)$  such that  $X = [A, B]$ .*

This theorem is not true without the assumption that  $R$  is a unital ring. Let  $R$  be the ring of polynomials over  $\mathbb{C}$  with zero constant terms in variable  $x$ . Then

$$X = \begin{pmatrix} x & 0 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & 0 \\ 0 & 0 & \cdots & 0 & -(n-1)x \end{pmatrix}$$

is of trace zero. However, the entries of a nonzero commutator  $[A, B]$  in  $M_n(R)$  do not contain any linear terms.

*Proof of Theorem 4.* Let  $X \in M_n(R)$  be an upper triangular matrix of the form

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & x_{n-1,n-1} & x_{n-1,n} \\ 0 & \cdots & 0 & -\sum_{k=1}^{n-1} x_{k,k} \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We define the matrix  $B$  as follows: for  $1 \leq i-1 \leq j \leq n$ , let

$$b_{ij} = \sum_{k=1}^{i-1} x_{k,j-i+k+1}.$$

All other terms of  $B$  are zero. Our goal is to show that  $X = [A, B]$ . Let  $[A, B] = (t_{i,j})$ . We want to prove  $t_{i,j} = x_{i,j}$  for  $i \geq j$ ,

$$t_{n,n} = -\sum_{k=1}^{n-1} x_{k,k},$$

and  $t_{i,j} = 0$  for  $i < j$ . We will split the proof into four cases.

Case 1. If  $i > j$ , then  $t_{i,j} = b_{i+1,j} - b_{i,j-1} = 0$ .

Case 2. If  $i = j = 1$ , then  $t_{i,j} = b_{21} = x_{11}$ .

Case 3. If  $i = j = n$ , then

$$t_{i,j} = 0 - b_{n,n-1} = 0 - \sum_{k=1}^{n-1} x_{k,k} = - \sum_{k=1}^{n-1} x_{k,k}.$$

Case 4. If  $i < j$  or  $i = j \in \{2, 3, \dots, n-1\}$ , then

$$t_{i,j} = b_{i+1,j} - b_{i,j-1} = \sum_{k=1}^i x_{k,j-i+k} - \sum_{k=1}^{i-1} x_{k,j-i+k} = x_{i,j}.$$

This completes the proof. □

This result may be used to give a proof of the well-known theorem due to Shoda [1936].

**Corollary 5.** *Let  $\mathbb{C}$  be the field of complex numbers and  $M_n(\mathbb{C})$  be the ring of  $n \times n$  matrices. Then every matrix of trace zero is a commutator.*

*Proof.* Let  $P$  be any matrix of trace zero and  $Q$  be Jordan normal form for  $P$ . So we have  $P = C^{-1}QC$  for some invertible  $C$ . Since  $P$  is upper triangular and of trace zero by Theorem 4 there exist  $A, B \in M_n(\mathbb{C})$  such that  $Q = [A, B]$ . Therefore,  $P = C^{-1}QC = C^{-1}[A, B]C = [C^{-1}AC, C^{-1}BC]$ . □

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# involve

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vol. 7

no. 6

<a href="#">A median estimator for three-dimensional rotation data</a>	713
MELISSA BINGHAM AND ZACHARY FISCHER	
<a href="#">Numerical results on existence and stability of steady state solutions for the reaction-diffusion and Klein–Gordon equations</a>	723
MILES ARON, PETER BOWERS, NICOLE BYER, ROBERT DECKER, ASLIHAN DEMIRKAYA AND JUN HWAN RYU	
<a href="#">The <math>h</math>-vectors of PS ear-decomposable graphs</a>	743
NIMA IMANI, LEE JOHNSON, MCKENZIE KEELING-GARCIA, STEVEN KLEE AND CASEY PINCKNEY	
<a href="#">Zero-inflated Poisson (ZIP) distribution: parameter estimation and applications to model data from natural calamities</a>	751
SADIE BECKETT, JOSHUA JEE, THAPELO NCUBE, SOPHIA POMPILUS, QUINTEL WASHINGTON, ANSHUMAN SINGH AND NABENDU PAL	
<a href="#">On commutators of matrices over unital rings</a>	769
MICHAEL KAUFMAN AND LILLIAN PASLEY	
<a href="#">The nonexistence of cubic Legendre multiplier sequences</a>	773
TAMÁS FORGÁCS, JAMES HALEY, REBECCA MENKE AND CARLEE SIMON	
<a href="#">Seating rearrangements on arbitrary graphs</a>	787
DARYL DEFORD	
<a href="#">Fibonacci Nim and a full characterization of winning moves</a>	807
CODY ALLEN AND VADIM PONOMARENKO	