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We exhibit a combinatorial model based on seating rearrangements, motivated by some problems proposed in the 1990s by Kennedy, Cooper, and Honsberger. We provide a simpler interpretation of their results on rectangular grids, and then generalize the model to arbitrary graphs. This generalization allows us to pose a variety of well-motivated counting problems on other frequently studied families of graphs.

1. Introduction

1.1. ***Background.*** In this section we describe the original motivation for our problems and the original interpretations that are present in the literature.

1.1.1. *Original problem.* Our interest in this combinatorial model begins with a problem presented by Honsberger [1997]:

> A classroom has 5 rows of 5 desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one on his right (of course not all these options are possible for all students). Determine whether or not this directive can be carried out.

It can easily be shown that this directive is impossible [Honsberger 1997; Kennedy and Cooper 1993]. Consider coloring the classroom like a checkerboard. Then every student initially placed on a "white desk" must move to a "black desk" and vice versa. However, our chessboard coloring has 13 white squares and 12 black squares. Thus, were such a rearrangement to exist, by the pigeonhole principle there must be at least one black desk that receives two students from white squares and this violates the terms of the directive. More generally, this proof obviously generalizes to any rectangular classroom that has both an odd number of rows and columns [Otake et al. 1996].

MSC2010: 05C30.
Keywords: matrix permanents, cycle covers, tilings, recurrence relations.

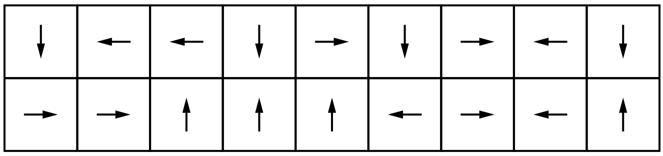

Figure 1. A 2×9 seating rearrangement.

1.1.2. *Early work.* Curtis Cooper and Robert Kennedy [1993] explored some basic extensions to this rearrangement problem by applying some traditional combinatorial and linear algebraic techniques (see also [Otake et al. 1996]). Their goal was to solve the following more general problem:

> A classroom has m rows of n desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one on his right (of course not all these options are possible for all students). ***In how many ways*** can this directive be carried out?

They began by solving the $2 \times n$ and $3 \times n$ cases by classifying all possible endings and constructing matrix systems that represented the interactions among these endings. For example, Figure 1 shows a 2×9 seating rearrangement. Then, the principle of mathematical induction can be used to show that the constructed matrix systems faithfully represent the counting problem. Of particular interest is the fact that the number of rearrangements on a $2 \times n$ grid is equal to the square of the $(n+1)$-st Fibonacci number. In Section 2.1 we will give a combinatorial proof of this fact. However, this method quickly becomes unwieldy, and they were forced to seek more powerful tools to solve the general case.

In order to count the $2m \times n$ seating rearrangements, Cooper and Kennedy turned to the theory of matrix permanents [Marcus and Minc 1965; Otake et al. 1996]. By modifying the adjacency matrix of the underlying grid graph and taking a symbolic determinant of the resulting block matrix they obtained the following representation of the number of seating rearrangements of a $2m \times n$ classroom:

$$2^{2mn} \prod_{t=1}^{2m} \prod_{s=1}^{n} \left(\cos^2\left(\frac{s\pi}{n+1} \right) + \cos^2\left(\frac{t\pi}{2m+1} \right) \right). \tag{1-1}$$

This formula is very similar to the expression derived in 1961 by Kasteleyn [Harary 1967; Kasteleyn 1961], and Temperley and Fisher [1961], that counts the number of domino tilings of a $m \times n$ grid. In Section 2.1 we will justify this correspondence while in Section 4 we will prove a general theorem that gives this relationship as an immediate corollary.

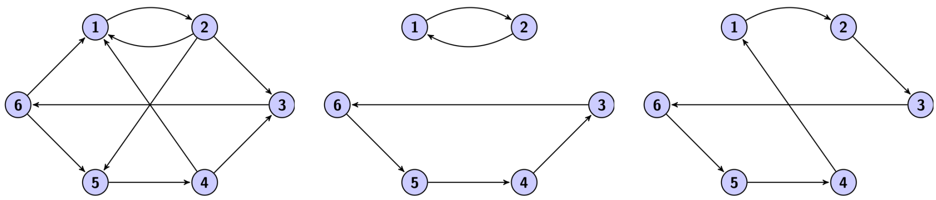

Figure 2. Two cycle covers of a digraph. Left: digraph; middle: even cover; right: odd cover.

1.2. ***Mathematical preliminaries.*** The proofs and results in this paper rely on techniques from combinatorics, linear algebra, and graph theory. Basic definitions and notation not presented here can be found in [Chartrand et al. 2011; van Lint and Wilson 2001; Shilov 1977].

1.2.1. *Cycle covers.* Given a digraph $D = \{V, E\}$, a cycle cover is defined as a subset of the edges, $C \subseteq E$, such that the induced digraph on C contains each vertex of V and each of those vertices lies on exactly one cycle [Harary 1969]. It is easy to see that each cycle cover of a digraph can be considered a permutation of the set of vertex labels, and more specifically a derangement, if no self-loops occur in the digraph. Thus, it is reasonable to consider the parity of a given cycle cover, defined as the parity of the permutation it represents.

Hence, a cycle cover that contains an even number of even cycles is considered even, while a cycle cover with an odd number of even cycles is considered odd. Figure 2 shows a digraph and two of its cycle covers, one of each parity.

1.2.2. *Matrix permanents.* The permanent of a matrix, M, with elements, $M_{u,v}$, is defined as the unsigned sum over all of the permutations of the matrix [Harary 1969; Marcus and Minc 1965]. Thus,

$$\operatorname{per} M = \sum_{\pi \in S_n} \prod_{i=1}^{n} M_{i,\pi(i)}, \tag{1-2}$$

is a symbolic representation of the matrix permanent. It is computationally difficult to calculate the permanent of a general 0–1 matrix (technically the problem of computing the permanent is $\#P$ complete) [Aaronson 2011; Lundow 1996; Valiant 1979]. Although the definition of the permanent looks very similar to that of the determinant, the permanent shares very few of the determinant's useful algebraic properties or relations to eigenvalues. Also, the determinant of a matrix can be calculated in polynomial time by Gaussian elimination, while the permanent cannot. However, interchanging rows or columns of the matrix does not affect the value of the permanent of that matrix [Marcus and Minc 1965].

We are interested in the concept of matrix permanents because the permanent of the adjacency matrix of a digraph is equal to the number of cycle covers of that digraph [Harary 1969]. A survey of results in combinatorics based on this method can be found in [Kuperberg 1998]. However, since the permanent is often infeasible to compute, a natural question is to ask whether we can change the signs of some elements of a given adjacency matrix, A, to form a new matrix, A', with the property that:

$$\operatorname{per} A = \det A'. \tag{1-3}$$

This question of "convertible" matrices was originally posed by Pólya [1913]. Beineke and Harary [1966] showed that digraphs whose adjacency matrix admits an orientation satisfying (1-3) are exactly those that contain no odd cycle covers. Later, Vazarani and Yannakakis [1988] proved that this problem is equivalent to finding pfaffian orientations of bipartite graphs. The pfaffian of a skew-symmetric matrix is a sum over signed products of entries in the matrix that can be used to count the number of perfect matchings in some graphs. For a complete discussion of pfaffians and their relation to perfect matchings see [Loehr 2011, Chapter 12.12].

This problem of pfaffians was characterized by Little [1975], who showed that a given bipartite graph, B, admits a pfaffian orientation if and only if B contains no subgraph homeomorphic to $K_{3,3}$ (the complete bipartite graph with three vertices in each partite set). An obvious extension of this question is to ask how difficult it is to construct such a matrix A' given A. Finally Roberston, Seymour, and Thomas [Robertson et al. 1999] settled the issue by giving a polynomial time algorithm that takes a given graph and either constructs an orientation of its adjacency matrix that satisfies (1-3), or demonstrates a subgraph of G proving that (1-3) cannot be satisfied.

2. Seating rearrangements

In this section we motivate and present our basic model through some simple counting problems.

2.1. ***Domino tilings.*** The original problem studied by Cooper and Kennedy can easily be expressed in terms of perfect matchings or domino tilings, both of which are very familiar combinatorial objects. We showed previously that if m and n are both odd there can be no legitimate rearrangements in an $m \times n$ classroom, so we will only consider the cases where at least one of m and n are even. However, note that the case where there are no legitimate rearrangements trivially satisfies the following lemma as there are no perfect matchings on $P_m \times P_n$ when m and n are both odd, where P_k is the path graph on k vertices.

Lemma 1. *The number of legitimate seating rearrangements in a $2m \times n$ classroom is equal to the square of the number of domino tilings of a $2m \times n$ grid.*

Proof. Begin by coloring the classroom like a chessboard. Note that we may consider the rearrangements of the students initially sitting in white desks separately from the rearrangements of those sitting in black desks since the two groups cannot interfere with each other. Since there are exactly as many black desks as white desks, arranged in the same fashion, the total number of rearrangements is equal to the square of the number of either the black or white rearrangements computed separately.

To complete the proof, consider tiling a $2m \times n$ board with mn dominoes. We can construct a bijection between the rearrangements of students initially placed in black (white) desks with domino tilings by placing a domino in the tiling for each student that covers that student's initial desk and their destination desk. Thus, any seating rearrangement can be deconstructed into two independent domino tilings, one for each initial color. Figure 3 gives an example of this process.

In order to construct a seating rearrangement from an independently selected pair of domino tilings we may perform the operation in reverse. Without loss of generality, associate one of the tilings with movements from white desks to black desks, and associate the other tiling with movements from black desks to white desks. Hence, we can combine any two domino tilings to create a unique seating rearrangement and the proof is complete. □

It is well known (and can be easily seen by comparison to $1 \times n$ tilings with squares and dominoes), that the number of domino tilings of a $2 \times n$ rectangle is equal to the $(n+1)$-st Fibonacci number. This observation, combined with the preceding lemma, provides a combinatorial explanation for the inductive-matrix result of Cooper and Kennedy mentioned in the introduction:

Corollary 2. *The number of seating rearrangements on a $2 \times n$ classroom is equal to the square of the $(n+1)$-st Fibonacci number.*

Another natural corollary to this lemma is a special case of Theorem 8, which will be proved in Section 4.

Corollary 3. *The number of legitimate seating rearrangements in an $m \times 2n$ classroom is equal to the square of the number of perfect matchings on $P_{2m} \times P_n$.*

2.2. ***Arbitrary graphs.*** In order to extend this notion of seating rearrangements to arbitrary graphs we constructed the following modified problem statement:

Problem. Given a graph, place a marker on each vertex. We want to count the number of legitimate "rearrangements" of these markers subject to the following rules:

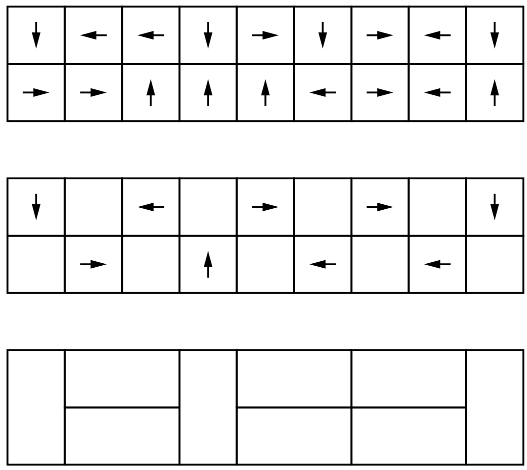

Figure 3. A rearrangement/tiling correspondence.

- Each marker must move to an adjacent vertex.
- After all of the markers have moved, each vertex must contain exactly one marker.

Thus, we define the number of rearrangements on an arbitrary graph to be the number of ways to satisfy the requirements given above. A related, interesting problem is to consider rearrangements where the markers are allowed to *either* remain in place or move along an edge to an adjacent vertex. To formulate this problem extension in graph-theoretic terms, we can add a self-loop to each vertex in the graph and proceed with the problem statement given above, where a vertex with a self-loop is considered adjacent to itself.

2.3. ***Digraphs.*** Given any graph G, we can construct a digraph $\overleftrightarrow{G}$, by replacing each simple edge of G by a pair of directed edges, one in each orientation. Then, the following lemma shows that there is a one-to-one correspondence between rearrangements on G and cycle covers on $\overleftrightarrow{G}$.

Lemma 4. *The number of rearrangements on any simple graph G is equal to the number of cycle covers on* $\overleftrightarrow{G}$.

Proof. Consider a legitimate rearrangement on a graph G, under the rules presented above. To construct a unique cycle cover on $\overleftrightarrow{G}$, place a directed edge in the cycle cover beginning at each markers initial vertex and ending at that markers terminal vertex. By the first rule, each vertex must have out-degree equal to 1. Similarly, by the second rule, each vertex must have in-degree equal to 1. Hence, the constructed cycle cover spans all vertices of G and has $d^+(v) = d^-(v) = 1$ for all $v \in V(G)$, and so is a legitimate cycle cover.

A unique rearrangement on G can be constructed from a given cycle cover on $\overleftrightarrow{G}$ in a similar fashion. Thus, there exists a bijection between these rearrangements and cycle covers, which implies that their magnitudes are equal. □

This gives us the following method for counting rearrangements on arbitrary graphs as well as a combinatorial interpretation of a matrix permanent of the adjacency matrix of a simple graph.

Lemma 5. *Given a graph G, with adjacency matrix $A(G)$, the number of rearrangements on G is equal to* $\operatorname{per}(A(G))$.

Proof. By construction, the adjacency matrices of G and $\overleftrightarrow{G}$ are equal, and the permanent of the adjacency matrix of $\overleftrightarrow{G}$ is equal to the number of cycle covers on $\overleftrightarrow{G}$. Since, by Lemma 4, there is a one-to-one correspondence between cycle covers on $\overleftrightarrow{G}$ and legitimate rearrangements on G, this proof is complete. □

Hence, we have a numerical method to compute the number of rearrangements on any graph. This method is computationally inefficient in general, but can provide numerical values of initial conditions for recurrence relations and generating functions, as well as providing empirical evidence of growth rates and divisibility properties.

2.4. ***Notation.*** For the rest of this paper we will use the notation $R(G)$ to represent the number of legitimate rearrangements on a given graph G. Similarly, $R_s(G)$ will represent the number of rearrangements where each marker is allowed to remain in place. Thus, the statement in the previous lemma could be rewritten as $R(G) = \operatorname{per}(A(G))$.

Several times throughout this paper, we will use the Fibonacci numbers in our counting. In these instances we will use the combinatorial Fibonacci numbers $f_n = F_{n+1}$, indexed as $f_0 = 1$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. This indexing is motivated by the traditional counting interpretation of the Fibonacci numbers as the number of ways to tile a $1 \times n$ board with squares and dominoes. Similarly, we will also employ the Lucas numbers, l_n, with $l_n = f_n + f_{n-2}$ defined as the number of ways to tile a $1 \times n$ bracelet with "rounded" squares and dominoes [Benjamin and Quinn 2003].

From graph theory, K_n will represent the complete graph on n vertices, while $K_{m,n}$ will be the complete bipartite graph with bipartite sets of order m and n. In addition, P_n and C_n will respectively represent the traditional path and cycle graphs on n vertices.

3. Basic graphs

We begin by demonstrating our model on some of the simplest possible graphs. Many more complex and interesting structures in graph theory can be constructed from these basic graphs. For many of these problems the number of rearrangements "with stays", $R_s(G)$ is the more interesting problem.

The simplest graph we consider is the path graph on n vertices. By comparison with the Fibonacci tilings of a $1 \times n$ board it is easy to see that $R_s(P_n) = f_n$. Similarly, we can construct a natural correspondence between Lucas tilings and rearrangements on C_n that accounts for all rearrangements except the two oriented cycles where each marker moves in the same direction. Thus, $R_s(C_n) = l_n + 2$.

Counting the rearrangements on the complete graph of order n is also a simple counting problem. Considering each rearrangement as a permutation, we see that if each marker must move to a new vertex we have that $R(K_n)$ is equal to the n-th derangement number, while if the markers are permitted to stay we have $R_s(K_n) = n!$.

Rearrangements on complete bipartite graphs are slightly more complex, yet still yield nice closed form representations.

Proposition 6. *The number of rearrangements on $K_{n,n}$ is equal to $(n!)^2$.*

Proof. We begin by coloring the vertices of $K_{n,n}$ black or white according to the bipartition. To construct a rearrangement on $K_{n,n}$ we note that much like the rectangular classroom problem, we can consider the movements of all of the vertices in each bipartition independently. Without loss of generality, we may order the white vertices. Then, the first white marker may move to any of n black vertices, while the k-th white marker can select any of the $n-k+1$ remaining black vertices. A similar method can be independently applied to the markers initially placed on black vertices.

Thus, the number of rearrangements of the markers that begin on a particular color is equal to $\prod_{i=1}^{n}(n-i+1) = n!$. Hence, $R(K_{n,n}) = (n!)^2$. □

In order to simplify the statement of the following result, we define some additional notation. Specifically, let $(n)_i = n(n-1)(n-2)\cdots(n-i+1)$ represent the standard falling factorial.

Proposition 7. *The number of rearrangements with stays on $K_{m,n}$ is equal to $\sum_{i=0}^{m}(m)_i(n)_i$.*

Proof. Without loss of generality we can assume that $m \le n$ and color the vertices in the m partition white and the vertices in the n partition black. We can count the rearrangements by conditioning on the number of markers that move from a white vertex to a black vertex. Let i represent the number of markers that move from white to black. Then there are $\binom{m}{i}$ ways to choose which white markers to move.

For any $1 \le k \le i$ the k-th moving white marker may select to move to any of $n-k+1$ black vertices. This gives us $(n)_i$ ways to move the $\binom{m}{i}$ selected white markers.

Graph	Rearrangements	With stays
P_n	$0, 1, 0, 1, 0, \ldots$	f_n
C_n	$0, 1, 2, 4, 2, 4, \ldots$	$l_n + 2 = f_n + f_{n-2} + 2$
K_n	$D(n)$	$n!$
$K_{n,n}$	$(n!)^2$	$\sum_{i=0}^{n}((n)_i)^2$
$K_{m,n}$ with $m < n$	0	$\sum_{i=0}^{m}(m)_i(n)_i$

Table 1. Rearrangements on basic graphs.

At this point there are i empty white vertices and i black vertices that contain a marker that must be moved. There are $i!$ ways to construct a legitimate rearrangement from this scenario. Summing over all possible $i \leq m$ gives

$$\sum_{i=0}^{m}\binom{m}{i}(n)_i i! = \sum_{i=0}^{m}\frac{m!}{i!(m-i)!}(n)_i i!, \\ = \sum_{i=0}^{m}(m)_i(n)_i. \qquad \square$$

Table 1 summarizes the results of this section, some of which will be referenced later in this paper.

4. Theorems

In this section we present some theoretical results related to our seating rearrangement model. The first theorem generalizes our earlier results on the original rectangular seating rearrangement problem and $R(K_{n,n})$.

Theorem 8. *Let $G = (\{U, V\}, E)$ be a bipartite graph. The number of rearrangements on G is equal to the square of the number of perfect matchings on G.*

Proof. We may construct a bijection between pairs of perfect matchings on G and cycle covers on $\overleftrightarrow{G}$. Without loss of generality, select two perfect matchings of G, m_1 and m_2. For each edge (u_1, v_1) in m_1, place a directed edge in the cycle cover from u_1 to v_1. Similarly, for each edge (u_2, v_2) in m_2, place a directed edge in the cycle cover from v_2 to u_2. Since m_1 and m_2 are perfect matchings, by construction, each vertex in the cycle cover has in-degree and out-degree equal to 1.

Given a cycle cover C on $\overleftrightarrow{G}$ construct two perfect matchings on G by taking the directed edges from vertices in U to vertices in V separately from the directed edges from V to U. Each of these sets of (undirected) edges corresponds to a perfect matching by the definition of cycle cover and the bijection is complete.

Since there is a one-to-one correspondence between cycle covers on $\overleftrightarrow{G}$ and rearrangements on G, the theorem is proved. □

Our next result considers the case where we are counting the number of rearrangements with stays on a bipartite graph.

Theorem 9. *The number of rearrangements on a bipartite graph G, when the markers on G are permitted to remain on their vertices, is equal to the number of perfect matchings on $P_2 \times G$.*

Proof. Observe that $P_2 \times G$ can be considered as two identical copies of G where each vertex is connected to its copy by a single edge. To construct a bijection between cycle covers on G and perfect matchings on $P_2 \times G$, associate each self-loop in a cycle cover with an edge between a vertex and its copy in the perfect matching.

Since the graph is bipartite, the remaining cycles in the cycle cover can be decomposed into matching edges from U to V and from V to U as in the previous theorem. □

Applying Theorem 9 to the original problem of seating rearrangements gives that the number of rearrangements in a $m \times n$ classroom, where the students are allowed to remain in place or move, is equal to the number of perfect matchings in $P_2 \times P_m \times P_n$. The $2 \times n$ case is included in the OEIS as A006253 [OEIS 2012]. These matchings are equivalent to tiling a $2 \times m \times n$ rectangular prism with $1 \times 1 \times 2$ tiles. This is a well-known problem that is contained in books on combinatorics, for example [Graham et al. 1994].

A more direct proof of this equivalence between rectangular seating rearrangements with stays and three-dimensional tilings can be given by associating each possible student move type — up/down, left/right, or remain in place — with a particular tile orientation in space. Then, a tiling can be directly constructed from a given seating rearrangement in a one-to-one fashion.

5. Counting examples

We conclude by presenting some examples of the types of counting problems that may be generated with this model. Especially noteworthy are the number of different techniques that may be used to solve these problems.

5.1. ***Prism graphs.*** The prism graph of order n, denoted prism n, is isomorphic to $C_n \times P_2$. Rearrangements on prism n can be considered as $2 \times n$ classroom seating rearrangements on a cylinder.

Example 10. *The number of rearrangements on* prism n *is equal to* $(l_n + 2)^2$ *when n is even.*

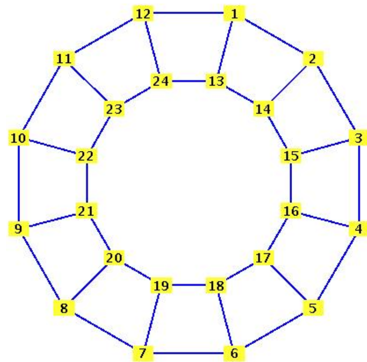

Figure 4. The prism graph of order 12.

Proof. Let n be an even natural number. Then it is easy to see that prism n is a bipartite graph, since each C_n is bipartite, and any cycle that includes edges in both C_n must also be of even length. Since the graph is bipartite, by Theorem 8, the number of rearrangements is equal to the square of the number of perfect matchings. Furthermore, by Theorem 9, the number of perfect matchings on $C_n \times P_2$ is equal to the number of rearrangements with stays on C_n, which we showed in Section 3 was equal to $l_n + 2$. Squaring this quantity gives the result. □

Example 11. *The number of rearrangements on* prism n *is equal to* $l_{2n} + 2$ *when* n *is odd.*

Proof. Let n be an odd natural number. In this case prism n is not bipartite, so we must make a different argument. First note that we can divide the rearrangements into two classes by whether a marker moves between the two C_n in the rearrangement. There are exactly four rearrangements for each n where no markers move between the two C_n, as these correspond to simple cycles where each marker on a C_n moves exactly one square in one direction.

The remaining rearrangements can be placed into a bijection with two independently selected Lucas tilings of order n where a square in a Lucas tiling represents a move between the C_n. Note that since n is odd, any Lucas tiling of order n must contain at least one square so we are not counting the rearrangements in the first class twice.

Combining these two cases, we have $R(\text{prism}\, n) = l_n^2 + 4$. Using a well-known Lucas identity we can simplify this expression as:

$$l_n^2 + 4 = (l_n^2 + 2) + 2 = l_{2n} + 2. \qquad \square$$

Computing the number of rearrangements with stays on a prism graph is a much more difficult problem. Considering all of the possible ways to rearrange

n	3	4	5	6	7	8
No stays	20	81	125	400	845	2401
With stays	82	272	890	3108	11042	39952
n	9	10	11	12	13	14
No stays	5780	15625	39605	104976	271445	714025
With stays	146026	537636	1988722	7379216	27436250	102144036

Table 2. Rearrangements on prism graphs.

an arbitrary pair of adjacent markers each in a separate C_n gives a system of 11 homogeneous, linear recurrence relations. This system is fully derived and demonstrated in Appendix A. This system can then be solved, using the successor operator method due to DeTemple and Webb [2014], to give the following solution:

$$\begin{aligned} a_n =& 10a_{n-1} - 36a_{n-2} + 50a_{n-3} + 11a_{n-4} - 108a_{n-5} + 96a_{n-6} \\ &+ 20a_{n-7} - 75a_{n-8} + 34a_{n-9} + 4a_{n-10} - 6a_{n-11} + a_{n-12}, \end{aligned}$$

with initial conditions given in Table 2.

Using these initial conditions we were further able to construct a generalized power sum by solving a linear equation in the eigenvalues of the recurrence to determine the coefficients:

$$R_s(\text{prism}\, n) = 6 + 4(-1)^n + (2+\sqrt{3})^n + (2-\sqrt{3})^n + 2(1+\sqrt{2})^n + 2(1-\sqrt{2})^n.$$

Since the repeated eigenvalues have coefficients of 0 in the generalized power sum, our sequence must also satisfy a recurrence of order 6. By computing the implied characteristic polynomial, we get the following minimal recurrence for this sequence:

$$a_n = 6a_{n-1} - 7a_{n-2} - 8a_{n-3} + 9a_{n-4} + 2a_{n-5} - a_{n-6}.$$

5.2. *Dutch windmills.* A Dutch windmill, Dw_n^m, consists of m copies of an n cycle all joined at a single vertex. For example, the friendship graphs are $F_k = Dw_3^k$. Counting the rearrangements on Dutch windmills highlights some of the Fibonacci relations of these counting problems.

Example 12. *The number of rearrangements on Dw_n^m is* 0 *when n is even and* $2m$ *when n is odd.*

Proof. We may condition on the movement of the marker initially positioned on the center vertex. The center vertex is adjacent to $2m$ other vertices, and every rearrangement on Dw_n^m must consist of a single n-cycle containing the center vertex

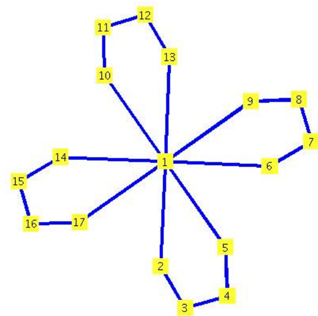

Figure 5. A Dutch windmill Dw_5^4.

and $(m-1)(n-1)/2$ two-cycles pairing up the remaining vertices as there are no other cycles remaining in the graph.

When n is even, removing the center vertex from all but one of the n-cycles leaves an odd number of vertices, which cannot be satisfactorily paired together. Thus, there can be no legitimate rearrangements when n is even.

In the case where n is odd, the movement of the center marker onto one of its $2m$ neighbors completely determines the rearrangement. □

Example 13. *The number of rearrangements with stays permitted on* Dw_n^m *is* $(f_{n-1})^m + 2m(f_{n-2}+1)(f_{n-1})^{m-1}$.

Proof. We may again condition on the behavior of the center marker. There are two cases: either the center marker moves to an adjacent vertex or it remains in place.

When the center marker does not move, the remaining markers form m copies of P_{n-1}, which may each be rearranged independently in f_{n-1} ways.

When the center marker moves onto one of the $2m$ adjacent vertices, it either lies on a two-cycle, in which case there are f_{n-2} ways for the other vertices on that cycle to rearrange themselves, or it lies on the entire n-cycle. The $m-1$ remaining n-cycles that were not selected are again each reduced to P_{n-1}, contributing $(f_{n-1})^{m-1}$ to the rearrangement total.

Combining these two cases gives the desired result:

$$R_s(Dw_n^m) = (f_{n-1})^m + 2m(f_{n-2}+1)(f_{n-1})^{m-1}. \qquad \square$$

5.3. ***Hypercubes.*** Hypercubes are a commonly studied mathematical object, and enumerating the perfect matchings on an arbitrarily large hypercube is an open problem in combinatorics [Lundow 1996]. Rearrangements, both with and without stays, have interesting connections to this problem.

The hypercube of order n can be constructed as a graph whose vertices are labeled with the 2^n binary strings of length n, with an edge between two vertices

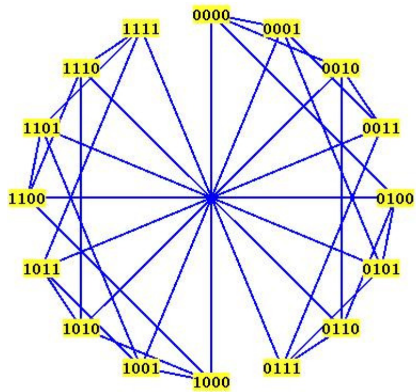

Figure 6. The hypercube of order 4.

	1	2	3	4	5
$R(H_n)$	1	4	81	73984	347138964225
$R_s(H_n)$	2	9	272	589185	16332454526976

Table 3. Hypercube rearrangements.

when the respective labels differ in only one location. More importantly for our purposes, if H_n represents the hypercube of order n, then $H_n \cong H_{n-1} \times P_2$.

Thus, the relations below follow directly from Theorem 8 and Theorem 9.

Corollary 14. *The number of rearrangements on H_n is equal to the square of the number of perfect matchings on H_n.*

Corollary 15. *The number of rearrangements with stays on H_n is equal to the number of perfect matchings on H_{n+1}.*

Appendix A. Computing $R_s(\text{prism}\, n)$

In this appendix, we give the full derivation of the generalized power sum for $R_s(\text{prism}\, n)$. Recall that prism n is isomorphic to $C_n \times P_2$ and may be considered a discrete $2 \times n$ cylinder. Thus, this problem is equivalent to the original seating rearrangement problem in a cylindrical classroom. Our goal is to construct a system of linear recurrences representing the ways that the (arbitrarily chosen) first column of desks can be filled.

We begin by letting a_n represent the number of rearrangements on prism n. Figure 7 shows all of the possible endings that we need to account for in our system. The dots in the figure represent students that have not moved, while the crosses represent students that have already moved. Note that the endings are representations of classes of endings, up to symmetry. Thus, for example, an ending counts as a c_n regardless of whether the completed desk is in the top or bottom row, since the number of rearrangements is the same. To see how the system is constructed consider the possible movements of the students in the first column of a b_n:

- The two students may elect to either remain in their seats or swaps seats with each other; either of these choices leaves a b_{n-1}.
- Both students may swap seats with the next student in their row, leaving a b_{n-2}.
- One of the students may remain in his seat, while the other swaps with his horizontal neighbor. This can happen in two ways, so we have $2c_{n-1}$.
- One of the students may move vertically, while the other moves horizontally. Again this can happen in two ways, and our sum gains a term of $2d_{n-1}$.

Similarly, consider the possibilities for a classroom ending set-up as g_n. As shown in Figure 7, we will assume that the desk with two students is in the upper left while the empty desk is in the lower right. However, this analysis extends to any rotation or reflection of g_n.

- The student yet to move in the upper left may move vertically forcing the student in the bottom left to move horizontally. This leaves a f_{n-1}.
- The student yet to move in the upper left may move horizontally while the student below remains in place. The remaining situation is a g_{n-1}.
- The student yet to move in the upper left may move horizontally while the student below swaps places horizontally, which forces a g_{n-2}.

Extending this reasoning to all of the endings under consideration leads to the following system of recurrences:

$$\begin{aligned}
a_n &= 2b_{n-1} + 2b_{n-2} + 4c_{n-1} + 2e_{n-1} \\
&\qquad + 4f_{n-1} + 4g_{n-1} + 4h_{n-1} + 2i_{n-1} + 2j_{n-1} + 2k_{n-1},\\
b_n &= 2b_{n-1} + b_{n-2} + 2c_{n-1} + 2d_{n-1}, & c_n &= b_{n-1} + c_{n-1},\\
d_n &= b_{n-1} + d_{n-1}, & e_n &= c_{n-1} + e_{n-1},\\
f_n &= f_{n-1} + f_{n-2} + g_{n-1}, & g_n &= f_{n-1} + g_{n-1} + g_{n-2},\\
h_n &= b_{n-1} + h_{n-1}, & i_n &= i_{n-1},\\
j_n &= f_{n-1}, & k_n &= k_{n-1} + h_{n-1}.
\end{aligned}$$

Applying the successor operator, E, to this system gives us the following symbolic matrix whose determinant is the characteristic polynomial of the recurrence relation we are seeking.

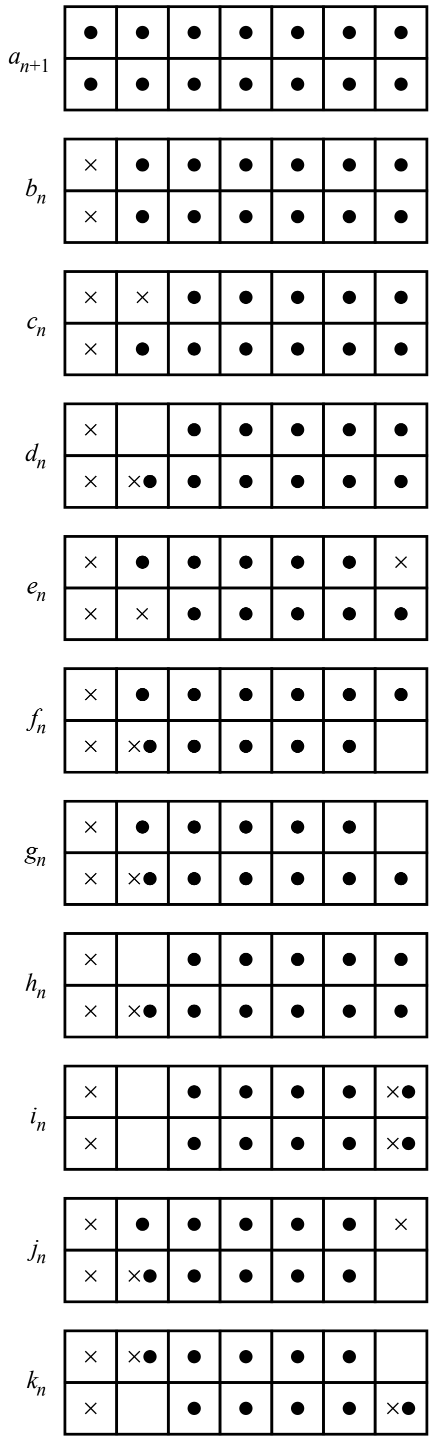

Figure 7. Prism endings.

$M =$

$$\begin{bmatrix}
E^2 & -2E-2 & -4E & 0 & -2E & -4E & -4E & -4E & -2E & -2E & -2E \\
0 & E^2-2E-1 & -2E & -2E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & E-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & E-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & E-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E^2-E-1 & -E & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -E & E^2-E-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & E-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & E & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & E-1
\end{bmatrix}$$

Note that M is defined to satisfy the following equation as the successor operator acts on each sequence in turn:

$$M \begin{bmatrix} a_n & b_n & c_n & d_n & e_n & f_n & g_n & h_n & i_n & j_n & k_n \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$$

We can now calculate the determinant of M and construct our recurrence relation,

$$\det M = E^{15} - 10E^{14} + 36E^{13} - 50E^{12} - 11E^{11} + 108E^{10} \\ -96E^9 - 20E^8 + 75E^7 - 34E^6 - 4E^5 + 6E^4 - E^3.$$

The coefficients of this characteristic polynomial give us our first recurrence relation (Section 5), while the roots of the polynomial are the eigenvalues of our recurrence. After removing the zeros, these eigenvalues and their multiplicities are $\{1^6, -1^2, 1+\sqrt{2}, 1-\sqrt{2}, 2+\sqrt{3}, 2-\sqrt{3}\}$. Thus, our characteristic polynomial factors to:

$$E^3(E-1)^6(E+1)^2(E^2-4E+1)(E^2-2E-1).$$

To find the generalized power sum, we solve the linear system $Ax = b$, where A represents the eigenvalues matrix (with elements multiplied by powers of n where necessary to preserve linear independence), x represents the coefficients vector, and b the initial conditions as shown in Table 2. The coefficients obtained as a solution to this system give the generalized power sum described previously in Section 5.

Taking a product of only the factors corresponding to the eigenvalues in the generalized power sum gives us the following characteristic polynomial of degree 6:

$$(E^2-4E+1)(E^2-2E-1)(E-1)(E+1) = E^6-6E^5+7E^4+8E^3-9E^2-2E+1.$$

Since this polynomial also annihilates our sequence, its corresponding recurrence relation must also be satisfied by our sequence. This gives the second recurrence relation in Section 5. By exhaustively examining the factors of this polynomial we find that it is the polynomial of minimal degree that represents our sequence.

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