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Efficient realization of nonzero spectra by polynomial matrices

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A theorem of Boyle and Handelman gives necessary and sufficient conditions for an n-tuple of nonzero complex numbers to be the nonzero spectrum of some matrix with nonnegative entries, but is not constructive and puts no bound on the necessary dimension of the matrix. Working with polynomial matrices, we constructively reprove this theorem in a special case, with a bound on the size of the polynomial matrix required to realize a given polynomial.

1. Introduction

H. R. Suleĭmanova [1949] posed a question: Given an n-tuple of complex numbers $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$, when is there an $n \times n$ matrix A with nonnegative entries such that $\det(I - At) = \prod_{i=1}^{n}(t - \lambda_i)$? This problem has come to be known as the nonnegative inverse eigenvalue problem, or NIEP. (See [Egleston et al. 2004] for a survey article on the problem.) Although there have been some significant advances, the general NIEP as stated remains open. One major advance was proven by Boyle and Handelman [1991]. They characterized the n-tuples that could be appended with zeros and subsequently realized as the eigenvalues of a nonnegative matrix in the above sense. Their proof relied heavily on results from symbolic dynamics and was not constructive (see [Lind and Marcus 1995] for more on symbolic dynamics and the NIEP). Very recently, Laffey [2012] proved a version of their result by constructive means, although his result is not in quite as general a setting as Boyle and Handelman's. In this paper we provide a construction different from that of Laffey's. The result of our construction is a matrix with polynomial entries, as opposed to real entries, and we describe a simple way to construct a matrix over the reals based on the polynomial matrix. This construction makes use of weighted directed graphs and is described further in [Boyle 1993].

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2. Preliminaries

A nonnegative matrix is *primitive* if it is a square matrix and some power of it is a matrix with strictly positive entries. The nonnegative inverse eigenvalue problem is generally studied in terms of primitive matrices, since given conditions for an n-tuple to be realized by a primitive matrix, one can easily extend to the general nonnegative case; for example, see [Boyle and Handelman 1991; Friedland 2012].

There are several known necessary conditions for an n-tuple of complex numbers σ to be realizable by a primitive matrix:

(1) $\sigma = \bar{\sigma}$. (For every complex number in σ, its complex conjugate is also in σ.)

(2) There exists $\lambda_i \in \sigma$ such that $\lambda_i \in \mathbb{R}_+$ and $\lambda_i > |\lambda_j|$ for $j \neq i$.

(3) For all $k \in \mathbb{N}$, the k-th moment of σ, $s_k = \sum_{i=1}^n \lambda_i^k$, is nonnegative. Moreover, for all $k \in \mathbb{N}$, if $s_k > 0$, then for all $n \in \mathbb{N}$, $s_{nk} > 0$.

The first condition simply reflects the fact that for the polynomial $\prod_{i=1}^n (t-\lambda_i)$ to have real coefficients, any complex roots must come in conjugate pairs. As a result of the first condition, the NIEP can be reformulated as follows: Given a polynomial $p(t) \in \mathbb{R}[t]$, is there a nonnegative matrix A such that $p(t)$ is the characteristic polynomial of A $\big($i.e., $p(t) = \det(It - A) = \prod_{i=1}^n (t-\lambda_i)\big)$? In this case, σ is the list of the roots of the polynomial with multiplicity.

The second comes as a result of the Perron–Frobenius theorem (e.g., see [Berman and Plemmons 1979; Minc 1988]). One of the consequences of this theorem is that a primitive matrix A must have a positive real eigenvalue that exceeds the modulus of all other eigenvalues. This positive real eigenvalue is often referred to as the *Perron eigenvalue* of the primitive matrix A.

The third condition is found by observing that if $\det(It - A) = \prod_{i=1}^n (t-\lambda_i)$ then the trace of A^k is $s_k = \sum_{i=1}^n \lambda_i^k$ for all $k \in \mathbb{N}$. Thus if A is nonnegative, so too must be s_k, and if A^k has a positive trace, then A^{nk} does as well for all $n \in \mathbb{N}$.

Boyle and Handelman [1991] proved that the above necessary conditions are sufficient to find a natural number N such that σ can be augmented by N zeros and then realized by a nonnegative primitive matrix. Restating more precisely, they proved the following, which we'll hereafter refer to as the Boyle–Handelman theorem:

Theorem 2.1 (spectral theorem). *Let $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$. There is an $N \geq 0$ and a nonnegative primitive matrix A such that*

$$\det(It - A) = t^N \prod_{i=1}^n (t - \lambda_i)$$

if and only if:

(1) $\sigma = \bar{\sigma}$.

(2) *There exists* $\lambda_i \in \sigma$ *such that* $\lambda_i \in \mathbb{R}_+$ *and* $\lambda_i > |\lambda_j|$ *for* $j \neq i$.

(3) *For all* $k \in \mathbb{N}$, *the* k*-th moment of* σ, $s_k = \sum_{i=1}^n \lambda_i^k$, *is nonnegative, and if* $s_k > 0$ *then* $s_{nk} > 0$ *for all* $n \in \mathbb{N}$.

Observe that there is an $N \geq 0$ such that $\det(It - A) = t^N \prod_{i=1}^n (t - \lambda_i)$ if and only if $\det(I - At) = \prod_{i=1}^n (1 - \lambda_i t)$, and this will provide us a convenient way to reformulate the theorem. With this, the Boyle–Handelman theorem characterizes which polynomials $q(t) \in \mathbb{R}[t]$ can be the *reverse characteristic polynomial* $\det(I - At)$ for a nonnegative primitive matrix A over the reals.

3. Graphs and polynomial matrices

Let G be a weighted directed graph on N vertices with weights in $\mathbb{R}_+$. Then the *adjacency matrix* A of G is the $N \times N$ matrix in which the (i, j) element is the weight of the edge running from vertex i to vertex j. The *characteristic polynomial* of this matrix (and of the associated graph G) is the polynomial $\chi_A(t) = \det(It - A)$. The *reverse characteristic polynomial* of the graph is the polynomial $\chi_A^{-1}(t) = \det(I - At)$. Of course, the process is reversible. Given a matrix A over $\mathbb{R}_+$, one can easily construct a weighted directed graph G which has A as its adjacency matrix. One simply includes an edge with weight $A(i, j)$ between each pair of vertices i and j.

As we will show, a directed graph G can also be represented by a polynomial matrix $M(t)$ over $t\mathbb{R}_+[t]$, i.e., a matrix $M(t)$ whose entries are polynomials with nonnegative coefficients without constant terms besides 0. This generally allows for presentations of adjacency matrices of smaller size. This process of constructing a polynomial matrix from a graph is also reversible. We begin by describing the reverse process.

Given an $N \times N$ polynomial matrix $M(t)$ over $t\mathbb{R}_+[t]$, the construction of the corresponding weighted digraph G can be carried out as follows. Assign N "primary" vertices with labels $1, 2, \ldots, N$. Then for each term $c_n t^p$ in the polynomial in the (i, j) position of $A(t)$, construct a path of length p from vertex i to vertex j in which the first edge is weighted c_n and each additional edge (if $p > 1$) is weighted 1. If $p > 1$ then the $p - 1$ additional "secondary" vertices in the new path are disjoint from the original N primary vertices and from secondary vertices used in any other path.

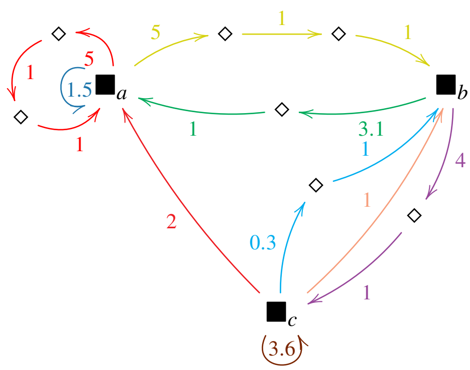
Example 3.1. Take, for example, the matrix over $t\mathbb{R}_+[t]$ given by

$$M(t) = \begin{bmatrix} 5t^3 + 1.5t & 9t^3 & 0 \\ 3.1t^2 & 0 & 4t^2 \\ 2t & 0.3t^2 + t & 3.6t \end{bmatrix}.$$

Note that

$$\det(I - M(t)) = 6t^7 + 48.44t^6 - 29.7t^5 + 22.8t^4 - 9t^3 + 5.4t^2 - 5.1t + 1.$$

From this matrix, by the method described above, we can construct the graph G below (large squares denote the primary vertices, diamonds denote secondary connecting vertices.)



Having constructed this graph, we can now construct the adjacency matrix for the graph A_G, ignoring color and the difference between primary and secondary vertices.

Suppose a directed graph G has m vertices numbered $1, 2, \ldots, m$. Then we can define the $m \times m$ matrix A_G where $A_G(i, j)$ is the weight of the edges from vertex i to j.

Example 3.2. Continuing with the above example, we obtain the adjacency matrix A_G for the graph G:

$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1.5 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3.6 & 0 & 0.3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.1 & 0 & 0 & 4 & 0 \end{bmatrix}.$$

We compute both the characteristic and reverse characteristic polynomials of this adjacency matrix and graph, and obtain

$$\det(It - A_G) = t^{10} - 5.1t^9 + 5.4t^8 - 9t^7 + 22.8t^6 - 29.7t^5 + 48.44t^4 + 6t^3,$$
$$\det(I - A_Gt) = 6t^7 + 48.44t^6 - 29.7t^5 + 22.8t^4 - 9t^3 + 5.4t^2 - 5.1t + 1.$$

We note that for the above example,

$$\det(I - M(t)) = \det(I - A_G t),$$

and this is no coincidence.

Theorem 3.3. *Let $M(t)$ be a matrix over $t\mathbb{R}_+[t]$, and suppose G is the directed graph constructed from $M(t)$ by the aforementioned construction. Suppose A_G is the adjacency matrix for G. Then*

$$\det(I - A_G t) = \det(I - M(t)).$$

Proof. Fix i, j such that $M(t)_{i,j}$ is a polynomial of degree greater than 1. Then for each term $c_{i,j,n}t^n$, where $n > 1$ and $c_{i,j,n} > 0$, there is a path in the graph from vertex i to vertex j of length n, and thus $n+1$ rows (indexed $k_1, k_2, \ldots k_{n+1}$) in the matrix A_G corresponding to each of the $n+1$ vertices along this path. (Note that k_1 corresponds to primary vertex i and k_{n+1} corresponds to primary vertex j) Each of these rows and columns (except k_1 and k_{n+1}) will have only one nonzero term, in the (k_h, k_{h+1}) position, and $c_{i,j,n} = \prod_{h=1}^{n} (A_G)_{k_h, k_{h+1}}$.

Each of these additional $n-1$ rows can be removed from the matrix $I - A_G t$ without changing the determinant by the following row operations, working backwards from $h = n$ to 2:

(1) From row k_{h-1}, subtract row k_h scaled by the entry in position (k_{h-1}, k_h).
(2) From column k_{n+1}, subtract column k_h multiplied by the entry in position (k_h, k_{n+1}).

This sequence results in the product of the terms in positions (k_{h-1}, k_h) and (k_h, k_{n+1}) appearing in position (k_{h-1}, k_{n+1}) and only a 1 remaining in both row and column k_h. Thus, after repeating this process for all the intermediate vertices, there will be a term equivalent to the product of their weights times t raised to the length of the chain added to the (k_1, k_{n+1}) position and a 1 in the primary diagonal for each row/column associated with each intermediate vertex. The determinant can be expanded by minors at each of these 1s, thus reducing the size of the matrix.

Repeating this process for each such $c_{i,j,n}$ term in $I - M(t)$ (and switching rows as necessary at the end) will produce the matrix $I - M(t)$ from $I - A_G t$ without changing the determinant. $\square$

The process of constructing a polynomial matrix $M(t)$ from a weighted directed graph G can be done by simply letting the (i, j) entry of $M(t)$ be $w(i, j)t$, where $w(i, j)$ is the sum of the weights of the edges from i to j. An alternative approach, which could be more efficient in terms of the size of $M(t)$, would be to identify secondary vertices as those which have at most one edge coming in and one out. Then the coefficient of t^k in the (i, j) entry of $M(t)$ is the sum of the weights of the paths of length k from primary vertex i to primary vertex j.

4. Our approach

Our approach is to study the nonnegative inverse eigenvalue problem, and specifically the Boyle–Handelman theorem, in terms of polynomial matrices rather than matrices over $\mathbb{R}_+$. We attempt to reprove the Boyle-Handelman theorem in certain cases by constructing an "efficient" polynomial matrix (in terms of the size of the matrix, without any bound on the degree of polynomials used in that matrix) that realizes a given polynomial. If we were able to bound both the size of the matrix and the degree of the polynomials used then we would be able to bound the size of the corresponding matrix over $\mathbb{R}_+$. In this vein, we will make use of polynomials which are truncations of the power series for $p(t)^{1/N}$.

We proceed forward assuming that $p(t)$ is a polynomial over $\mathbb{R}$ of the form $p(t) = \prod_{i=1}^{n}(1-\lambda_i t)$, where $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies the conditions of the Boyle–Handelman theorem with a strengthened version of the third condition: for all $k \in \mathbb{N}$, $s_k > 0$. We will say that $p(t)$, or perhaps σ, satisfying these conditions satisfies BH+. Below we prove that in such a case, the power series expansion for $p(t)^{1/N}$ has nonpositive coefficients after the constant term. More recently, this result was proven using different means by Laffey, Loewy and Šmigoc [Laffey et al. 2013].

Theorem 4.1. *Assume that* $p(t) = \prod_{i=1}^{d}(1-\lambda_i t)$ *satisfies* BH+. *Then there is an* $N \geq 1$ *such that the power series expansion for* $p(t)^{1/N}$ *is of the form*

$$p(t)^{1/N} = 1 - \sum_{k=1}^{\infty} r_k t^k,$$

where $r_k \geq 0$ *for all* $k \geq 1$.

Proof. Recall that the power series expansion for $(1-t)^{1/N}$ is given by

$$(1-t)^{1/N} = \sum_{k=0}^{\infty} \binom{1/N}{k} t^k,$$

where $\binom{1/N}{k}$ is a generalized binomial coefficient, given by

$$\binom{1/N}{k} = \frac{1/N(1/N-1)(1/N-2)\cdots(1/N-k+1)}{k(k-1)(k-2)\cdots 1}.$$

Then

$$\begin{aligned} p(t)^{1/N} &= \prod_{i=1}^{d}(1-\lambda_i t)^{1/N} \\ &= \prod_{i=1}^{d}\left(\sum_{k=0}^{\infty}\binom{1/N}{k}(-\lambda_i)^k t^k\right) = \prod_{i=1}^{d}\left(1 - \sum_{k=1}^{\infty}\left|\binom{1/N}{k}\right|\lambda_i^k t^k\right). \end{aligned}$$

The k-th coefficient of this series is given by

$$r_k = \left|\binom{1/N}{k}\right|(\lambda_1^k + \lambda_2^k + \cdots + \lambda_d^k) + \sum (-1)^l \left|\binom{1/N}{k_1} \cdots \binom{1/N}{k_d}\right| \lambda_{i_1}^{k_1} \lambda_{i_2}^{k_2} \cdots \lambda_d^{k_d},$$

where the second sum ranges over all combinations of nonnegative k_i such that $k_1 + k_2 + \cdots + k_d = k$, where $l \geq 2$ is the number of nonzero k_i, and where $k_{i_1}, k_{i_2}, \ldots, k_{i_l}$ are these nonzero values.

Factoring $\left|\binom{1/N}{k}\right|\lambda_1^k$ out of this expression (and assuming that λ_1 is the Perron eigenvalue), the first term above becomes

$$1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k + \cdots + \left(\frac{\lambda_d}{\lambda_1}\right)^k,$$

which approaches 1 as $k \to \infty$ and is always positive (by BH+). Therefore, this term has a uniform lower bound $\delta > 0$ (which does not depend on k or N).

The absolute value of the second term is at most

$$\sum \left| \frac{\binom{1/N}{k_1}\binom{1/N}{k_2} \cdots \binom{1/N}{k_d}}{\binom{1/N}{k}} \right| \left|\frac{\lambda_1}{\lambda_1}\right|^{k_1} \left|\frac{\lambda_2}{\lambda_1}\right|^{k_2} \cdots \left|\frac{\lambda_d}{\lambda_1}\right|^{k_d}.$$

Now observe that for $l \geq 2$ and $N \geq 2$,

$$\begin{aligned}
&\left| \frac{\binom{1/N}{k_1}\binom{1/N}{k_2} \cdots \binom{1/N}{k_d}}{\binom{1/N}{k}} \right| \\
&= \left| \frac{\frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k_1+1)}{k_1!} \, \frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k_2+1)}{k_2!} \cdots \frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k_d+1)}{k_d!}}{\frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k+1)}{k!}} \right| \\
&= \left| \left(\frac{1}{N}\right)^{l-1} \frac{k!}{k_1! k_2! \cdots k_d!} \, \frac{\left((\frac{1}{N}-1)\cdots(\frac{1}{N}-k_1+1)\right)\cdots\left((\frac{1}{N}-1)\cdots(\frac{1}{N}-k_d+1)\right)}{(\frac{1}{N}-1)\cdots(\frac{1}{N}-k+1)} \right| \\
&< \left| \left(\frac{1}{N}\right)^{l-1} \frac{k!}{k_1! k_2! \cdots k_d!} \, \frac{(k_{i_1}-1)!(k_{i_2}-1)!\cdots(k_{i_l}-1)!}{(k-1)!} \right| \\
&= \left| \left(\frac{1}{N}\right)^{l-1} \frac{k}{k_{i_1} k_{i_2} \cdots k_{i_l}} \right| \\
&= \left| \frac{1}{N} \, \frac{k}{k_{i_1} k_{i_2} \cdots k_{i_l} N^{l-2}} \right|.
\end{aligned}$$

Since $k_{i_1} k_{i_2} \cdots k_{i_l} N^{l-2}$ is minimized when $l = 2$ and $k_{i_1} = k - 1$, we have

$$\left| \frac{1}{N} \, \frac{k}{k_{i_1} k_{i_2} \cdots k_{i_l} N^{l-2}} \right| < \left| \frac{1}{N} \, \frac{k}{k-1} \right| < \frac{2}{N}.$$

Also note that

$$\sum \left|\frac{\lambda_1}{\lambda_1}\right|^{k_1} \left|\frac{\lambda_2}{\lambda_1}\right|^{k_2} \cdots \left|\frac{\lambda_d}{\lambda_1}\right|^{k_d} < \frac{1}{1-\left|\frac{\lambda_2}{\lambda_1}\right|} \frac{1}{1-\left|\frac{\lambda_3}{\lambda_1}\right|} \cdots \frac{1}{1-\left|\frac{\lambda_d}{\lambda_1}\right|} = M,$$

by expanding the right-hand side into a product of geometric series. Therefore, there is a uniform upper bound of the form $(2/N)M$, where M does not depend on k or N.

Then all we need to do is choose N such that $\delta > (2/N)M$. □

Using this result, we pose the following question:

Question 4.2. Let $p(t)$ be a polynomial which satisfies the condition that there exists $N \geq 1$ such that $p(t)^{1/N} = 1 - \sum_{k=1}^{\infty} r_k t^k$, where $r_k \geq 0$ for all $k \geq 1$. Then does there exist an $N \times N$ polynomial matrix $M(t)$ with nonnegative coefficients such that $\det(I - M(t)) = p(t)$?

As a result of Theorems 3.3 and 4.1, answering in the affirmative would be (nearly) equivalent to proving the Boyle–Handelman theorem (with the exception of the strengthening of the third condition in Theorem 4.1.) Such an answer would further give a constructive proof and would have a bound on the size of the polynomial matrix required to realize a given polynomial. Without putting a bound on the degree of the polynomial matrix, however, this conjecture does not establish any bounds on the size of the regular matrix over $\mathbb{R}_+$. If, however, the size of the polynomial matrix and the degrees of polynomials used in the matrix could both be bounded, then a bound on the size of the realizing regular matrix could be achieved.

At the moment we are able to prove the above conjecture for the cases $N = 1, 2, 3$.

5. Cases $N = 1, 2$

Case $N = 1$. The case where $N = 1$ is trivial. If $p(t)^1 = 1 - r(t)$, where $r(t)$ has no negative coefficients, then the matrix $A(t) = [r(t)]$ suffices, and

$$\det(I - A(t)) = \det([1 - r(t)]) = 1 - r(t) = p(t).$$

Case $N = 2$. Suppose $p(t)^{1/2} = 1 - r(t)$, where $r(t)$ has no negative coefficients. Then let $q(t)$ be the polynomial that results when the power series $r(t)$ is truncated to a degree-n polynomial, where n is greater than or equal to the degree of $p(t)$. Consider the polynomial $(1 - q(t))^2$.

The first n terms of this polynomial will sum to $p(t)$. Let $R(t) = (1 - q(t))^2 - p(t)$. Then $R(t)$ will be a polynomial with lowest-order term of degree $n + 1$ and highest degree of $2n$, and is described by

$$R(t) = \sum_{i=n+1}^{2n} \sum_{j+k=i} q_j q_k t^i,$$

where q_i is the coefficient of the t^i term in $q(t)$. Since all the q_i are nonnegative, $R(t)$ will contain only nonnegative terms.

Then construct the matrix

$$A(t) = \begin{bmatrix} q(t) & R(t)/t \\ t & q(t) \end{bmatrix};$$

we have

$$\det(I - A(t)) = (1 - q(t))^2 - R(t) = p(t).$$

Example 5.1 ($N = 2$). Consider the polynomial $p(t) = 1 - 3t - 2t^2 + 4t^3$. The power series of $p(t)^{1/2}$ is

$$p(t)^{1/2} = 1 - \frac{3t}{2} - \frac{17t^2}{8} - \frac{19t^3}{16} - \frac{517t^4}{128} - \frac{2197t^5}{256} + \cdots .$$

Let $q(t) = \frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16}$. Then

$$(1 - q(t))^2 = 1 - 3t - 2t^2 + 4t^3 + \frac{517t^4}{64} + \frac{323t^5}{64} + \frac{361t^6}{256},$$

and

$$R(t) = (1 - q(t))^2 - p(t) = \frac{517t^4}{64} + \frac{323t^5}{64} + \frac{361t^6}{256}.$$

We can then construct the matrix $A(t)$ as described above:

$$A(t) = \begin{bmatrix} \frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16} & \frac{517t^3}{64} + \frac{323t^4}{64} + \frac{361t^5}{256} \\ t & \frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16} \end{bmatrix},$$

and $A(t)$ realizes the original polynomial $p(t) = 1 - 3t - 2t^2 + 4t^3$.

6. The case $N = 3$

The $N = 3$ case extends the ideas used in the $N = 2$ case, but is much more complicated since the "left over" terms of the $(1 - q(t))^3$ term cannot be assumed to be all positive. In this case, we work with the matrix

$$A(t) = \begin{bmatrix} q(t) & \alpha(t) & \beta(t) \\ 0 & q(t) & t \\ t & 0 & q(t) \end{bmatrix},$$

where $q(t)$ is a truncation of the power series $r(t) = 1 - p(t)^{1/3}$ of some degree n at least as large as the degree of $p(t)$. In this case,

$$\det(I - A(t)) = (1 - q(t))^3 - t^2\alpha(t) - t\beta(t)(1 - q(t)).$$

In what follows, we will denote by b_m, q_m and r_m the coefficients of the term t^m in the polynomials $\beta(t)$, $q(t)$ and power series $r(t)$ respectively, and by $[f(t)]_m$ the coefficient of t^m in a more complicated polynomial expression, $f(t)$.

Were $R(t) = (1-q(t))^{1/3} - p(t)$ strictly positive, then this remainder could be accommodated by the $\alpha(t)$ term, as in the $N = 2$ case, and the $\beta(t)$ term would not be needed. However, this is in fact never the case. Consider the highest-order term of $R(t)$. This term (of degree $3n$) will have coefficient $(-q_n)^3$. Thus $R(t)$ will necessarily contain at least one negative coefficient, and in practice usually has many more.

On the other hand, the lowest-order term of $R(t)$ will always be positive. Since this term has degree $n+1$, greater than the degree of $p(t)$, the coefficient of the term of order $n+1$ in the polynomial $(1-r(t))^3 = p(t)$ must be 0. The only "missing" term of degree $n+1$ when expanding $(1-q(t))^3$ is $3(-r_{n+1})$. Thus the coefficient of the lowest-order term in $R(t)$, $[R(t)]_{n+1} = 3r_{n+1}$, is positive.

Since negative terms exist in $R(t)$, the $\beta(t)$ polynomial term must be used. Any term $b_m t^m$ in $\beta(t)$ is multiplied by $t(1-q(t))$ in the determinant of $I - A(t)$ and thus has the effect of decreasing the $(m+1)$-th coefficient of $(1-q(t))^3 - t\beta(t)(1-q(t))$ and increasing the $(m+2)$-th through $(m+n+1)$-th coefficients. The end goal is to construct the polynomial $\beta(t)$ in such a way that the remainder polynomial $d(t) = (1-q(t))^3 - t\beta(t)(1-q(t)) - p(t)$ has all positive coefficients.

Note that before we include any terms in $\beta(t)$, $\beta(t)$ is zero, so we have $d(t) = R(t)$. We can take the lowest-order term of $d(t)$, which we know to be positive, and include it in $\beta(t)$. This is, in a sense, the largest that this coefficient of $\beta(t)$ can be. If it were any larger then the lowest-order term in the resulting polynomial $d(t)$ would be negative. But it also provides the maximum benefit in terms of increasing the coefficients of terms with higher powers in $d(t)$.

If the next lowest-order term of the resulting $d(t)$ is also positive then we can repeat the process, including this term in $\beta(t)$ as well. This process can be continued either until a negative coefficient is reached or until the entire remaining $d(t)$ is positive. (Success!) In the case that a negative coefficient is reached, one can try again with a larger value of n, meaning that we include more terms in $q(t)$, truncating the power series $r(t)$ at a later point.

Example 6.1 ($N = 3$). Let $p(t) = 1 - 5t + 7t^2 - 3t^3$. Then,

$$p(t)^{1/2} = 1 - \frac{5t}{2} + \frac{3t^2}{8} - \frac{9t^3}{16} - \frac{189t^4}{128} - \frac{891t^5}{256} \cdots .$$

We cannot use a 2×2 matrix since the power series of $p(t)^{1/2}$ is not of the correct form. The power series of $p(t)^{1/3}$ is of the correct form, however, and

$$p(t)^{1/3} = 1 - \frac{5t}{3} - \frac{4t^2}{9} - \frac{76t^3}{81} - \frac{508t^4}{243} - \frac{3548t^5}{729} \cdots .$$

We let $q(t)$ be this power series truncated to 3 terms:

$$q(t) = \frac{5t}{3} + \frac{4t^2}{9} + \frac{76t^3}{81}.$$

Then

$$
\begin{aligned}
(1-q(t))^3 &= 1-5t+7t^2-3t^3 \\
&\quad + \frac{508t^4}{81} - \frac{1532t^5}{243} - \frac{3536t^6}{2187} - \frac{32528t^7}{6561} - \frac{23104t^8}{19683} - \frac{438976t^9}{531441}.
\end{aligned}
$$

Only the first term of $R(t)$ is positive, and

$$
R(t) = \frac{508t^4}{81} - \frac{1532t^5}{243} - \frac{3536t^6}{2187} - \frac{32528t^7}{6561} - \frac{23104t^8}{19683} - \frac{438976t^9}{531441}.
$$

Including this term as the first term in $\beta(t)$, we have

$$
\begin{aligned}
&(1-q(t))^3 - \frac{508t^4}{81}(1-q(t)) \\
&\quad = 1-5t+7t^2-3t^3 + \frac{112t^5}{27} + \frac{2560t^6}{2187} + \frac{6080t^7}{6561} - \frac{23104t^8}{19683} - \frac{438976t^9}{531441}.
\end{aligned}
$$

Thus we now have an additional positive term which can be included in $\beta(t)$. Repeating this process twice more, we eventually get

$$
\begin{aligned}
&(1-q(t))^3 - \left(\frac{508t^4}{81} + \frac{112t^5}{27} + \frac{17680t^6}{2187}\right)(1-q(t)) \\
&\quad = 1-5t+7t^2-3t^3 + \frac{106576t^7}{6561} + \frac{41408t^8}{6561} + \frac{3592064t^9}{531441},
\end{aligned}
$$

which is $p(t)$ plus a polynomial with only positive coefficients, which can then be chosen to be $\alpha(t)$ (after dividing out a factor of t^2) in the matrix. Bringing all of these polynomials together, we can construct the matrix

$$
A(t) = \begin{bmatrix}
\frac{5t}{3}+\frac{4t^2}{9}+\frac{76t^3}{81} & \frac{106576t^5}{6561}+\frac{41408t^6}{6561}+\frac{3592064t^7}{531441} & \frac{508t^3}{81}+\frac{112t^4}{27}+\frac{17680t^5}{2187} \\
0 & \frac{5t}{3}+\frac{4t^2}{9}+\frac{76t^3}{81} & t \\
t & 0 & \frac{5t}{3}+\frac{4t^2}{9}+\frac{76t^3}{81}
\end{bmatrix}
$$

such that $A(t)$ realizes the original polynomial $p(t)$.

At this point in our research a computer program was written which ran through the steps of this "greedy algorithm" to determine whether such a matrix could be constructed for trial polynomials $p(t)$ which satisfied the condition that the power series of $p(t)^{1/3}-1$ had all negative coefficients. All cubic polynomials with integer coefficients less than 100 were tested and no counterexamples were found.

The goal of this algorithm can be reformulated as constructing a polynomial

$$
b(t) = \sum_{i=M+1}^{3n} b_i t^i
$$

such that $p(t)-(1-q(t))^3+b(t)(1-q(t))$ has coefficient 0 for all terms with degree $3n$ or less. Then if $b(t)$ has only positive terms, the realizing matrix can be easily constructed. In the following propositions, we demonstrate that it is always possible to construct such a $b(t)$ with all positive coefficients.

First, we note the following:

Proposition 6.2. *Let $p(t)^{1/3} = 1 - r(t) = 1 - q(t) - s(t)$, where $q(t)$ is polynomial of degree n equal to the power series $r(t)$ truncated to degree n and $s(t)$ is a power series consisting of the remaining terms in $r(t)$. Then*

$$b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m.$$

Proof. By the construction of $b(t)$, for all $m < 3n$,

$$\big[p(t) - (1-q(t))^3 + b(t)(1-q(t))\big]_m = 0,$$

$$\big[b(t)(1-q(t))\big]_m = \big[(1-q(t))^3 - p(t)\big]_m,$$

$$p(t) = ((1-q(t)-s(t))^3 = (1-q(t))^3 - 3s(t)(1-q(t))^2 + 3s(t)^2(1-q(t)) - s(t)^3.$$

Plugging this expression in for $p(t)$ above, we have

$$\big[b(t)(1-q(t))\big]_m = \big[3s(t)(1-q(t))^2 - 3s(t)^2(1-q(t)) + s(t)^3\big]_m.$$

The lowest-order term of $s(t)^3$ will have degree $3n+3$, so it can be dropped, giving

$$\begin{aligned}\big[b(t)(1-q(t))\big]_m &= \big[3s(t)(1-q(t))^2 - 3s(t)^2(1-q(t))\big]_m \\ &= \big[(1-q(t))3s(t)(1-q(t)-s(t))\big]_m \\ &= \big[(1-q(t))3s(t)(1-q(t)-s(t))\big]_m.\end{aligned}$$

Thus,

$$[b(t)]_m = b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m.$$ □

Alternatively, we can write this result in terms of $r(t)$ as

$$b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m = 3\left[r_m + \sum_{i=1}^{m-n} r_i r_{m-i}\right].$$

Proposition 6.3. *Assume $p(t)$ satisfies the conditions of the Boyle–Handelman theorem as well as our strengthened third condition. Let λ_1 be the Perron root of $p(t)$ and suppose $p(t)^{1/3} = 1 - r(t)$. A good estimate of the coefficients r_n of $r(t)$ is*

$$\left|\binom{1/3}{n}\right| \lambda_1^n (a(1/\lambda_1))^{1/3},$$

where $a(t)$ is the polynomial

$$a(t) = \frac{p(t)}{1-\lambda_1 t}$$

and λ_1 is the Perron root of $p(t)$. By a "good estimate" we mean that

$$\lim_{n\to\infty} \frac{r_n}{\left|\binom{1/3}{n}\right| \lambda_1^n (a(1/\lambda_1))^{1/3}} = 1.$$

Proof. We begin with two subclaims.

Subclaim 1. *Let $\epsilon > 0$ be given. Then there exists an $N > 0$ such that for any $n > N$ and for any j with $0 < j < n$,*

$$\left| \frac{\binom{1/3}{n-j}}{\binom{1/3}{n}} \right| < \frac{n}{n-j}(1+\epsilon)^j.$$

Proof. First, note that

$$\begin{aligned}
\left| \frac{\binom{1/3}{n-j}}{\binom{1/3}{n}} \right| &= \left| \frac{\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\cdots\left(\frac{1}{3}-(n-j-1)\right)}{(n-j)!}}{\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\cdots\left(\frac{1}{3}-(n-1)\right)}{n!}} \right| \\
&= \left| \frac{n!}{(n-j)!} \frac{1}{\left(\frac{1}{3}-(n-j)\right)\left(\frac{1}{3}-(n-j+1)\right)\cdots\left(\frac{1}{3}-(n-1)\right)} \right| \\
&= \frac{n!}{(n-j)!} \frac{1}{\left|\frac{1-3(n-j)}{3}\right|\left|\frac{1-3(n-j+1)}{3}\right|\cdots\left|\frac{1-3(n-1)}{3}\right|} \\
&= \frac{n!}{(n-j)!} \frac{1}{(n-j)\left|1-\frac{1}{3(n-j)}\right|(n-j+1)\left|1-\frac{1}{3(n-j+1)}\right|\cdots(n-1)\left|\frac{1}{3(n-1)}\right|} \\
&= \frac{n}{n-j}\prod_{k=1}^{j} \frac{1}{1-\frac{1}{3(n-k)}},
\end{aligned}$$

and

$$\log\left(\prod_{k=1}^{j} \frac{1}{1-\frac{1}{3(n-k)}}\right)^{1/j} = \frac{1}{j}\sum_{k=1}^{j} -\log\left(1-\frac{1}{3(n-k)}\right).$$

Since the denominator $1-1/(3(n-k))$ decreases with k,

$$\begin{aligned}
0 &< \frac{1}{j}\sum_{k=1}^{j} -\log\left(1-\frac{1}{3(n-k)}\right) \\
&\leq \frac{1}{n-1}\sum_{k=1}^{n-1} -\log\left(1-\frac{1}{3(n-k)}\right) = \frac{1}{n-1}\sum_{k=1}^{n-1} -\log\left(1-\frac{1}{3k}\right).
\end{aligned}$$

The last expression above is the average of the first $n-1$ terms of the form $-\log(1-1/3k)$. Since these terms tend to 0 as $k \to \infty$, the average of them does as well. Thus there exists an N such that for all $n \geq N$,

$$\frac{1}{n-1}\sum_{k=1}^{n-1} -\log\left(1-\frac{1}{3k}\right) < \log(1+\epsilon),$$

and for $n \geq N$ and for any j with $1 < j < n$,

$$\log\left(\prod_{k=1}^{j} \frac{1}{1-\frac{1}{3(n-k)}}\right)^{1/j} < \log(1+\epsilon).$$

Therefore,

$$\prod_{k=1}^{j} \frac{1}{1-\frac{1}{3(n-k)}} < (1+\epsilon)^j,$$

and

$$\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| < \frac{n}{n-j}(1+\epsilon)^j. \qquad \square$$

Subclaim 2. *Let $\epsilon > 0$ be given and fix $K > 0$. There exists an $N > K$ such that for any $n > N$ and for any j with $0 < j < K$,*

$$1 < \left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| < 1+\epsilon.$$

Proof. Let $\epsilon_1 = (1+\epsilon)^{1/(k+1)} - 1$. Then by Subclaim 1, there exists an $N_1 > K$ such that for all $n \geq N_1$ and for every j with $0 < j < K < N_1$,

$$\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| < \frac{n}{n-j}(1+\epsilon_1)^j \leq \frac{n}{n-K}(1+\epsilon_1)^K.$$

Since $\lim_{n\to\infty} n/(n-K) = 1$, there exists N_2 such that for all $n \geq N_2$,

$$\frac{n}{n-K} \leq (1-\epsilon_1).$$

Let $N = \max(N_1, N_2)$. Then for all $n \geq N$ and j with $0 < j < K$,

$$\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| < \frac{n}{n-K}(1+\epsilon_1)^K < (1+\epsilon_1)(1+\epsilon_1)^K = (1+\epsilon_1)^{K+1} = (1+\epsilon). \quad \square$$

We use these two subclaims to show that given $\epsilon > 0$, there exists an N such that for all $n > N$,

$$\left|\frac{r_n}{\left|\binom{1/3}{n}\right|\lambda_1^n a(1/\lambda_1)^{1/3}} - 1\right| < \epsilon.$$

Let $\alpha(t) = 1 + \sum_{i=1}^{\infty} \alpha_i t^i$ denote the power series expansion for $a(t)^{1/3} = (p(t)/(1-\lambda_1 t))^{1/3}$ at $t = 0$. Then

$$p(t)^{1/3} = 1 - r(t) = (1-\lambda_1 t)^{1/3}\alpha(t) = \left(1 - \sum_{i=1}^{\infty}\left|\binom{1/3}{i}\right|\lambda_1^i t^i\right)\left(1 + \sum_{i=1}^{\infty}\alpha_i t^i\right).$$

We can then write r_n as

$$\begin{aligned} r_n &= \left|\binom{1/3}{n}\right|\lambda_1^n - \alpha_n + \sum_{k=1}^{n-1}\left|\binom{1/3}{n-k}\right|\alpha_k\lambda_1^{n-k} \\ &= \left|\binom{1/3}{n}\right|\lambda_1^n\left(1+\sum_{k=1}^{n-1}\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right|\alpha_k\lambda_1^{-k} - \frac{\alpha_n}{\left|\binom{1/N}{n}\right|}\lambda_1^{-n}\right). \end{aligned}$$

Let $\delta = \frac{1}{5}\epsilon a(1/\lambda_1)^{1/3}$. If λ_2 is the root of $a(t) = p(t)/(1-\lambda_1 t)$ with the greatest modulus (i.e., for all λ_i roots of $a(t)$, $|\lambda_2| \geq |\lambda_i|$) then the power series $\alpha(t) = a(t)^{1/3}$ has radius of convergence $1/|\lambda_2|$, which is greater than $1/\lambda_1$. Now, for some $K > 0$ and $n > K_1$, we can write

$$1+\sum_{k=1}^{n-1}\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right|\alpha_k\lambda_1^{-k} - \frac{s_n}{\left|\binom{1/3}{n}\right|}\lambda_1^{-n} = a(1/\lambda_1)^{1/3} \tag{6-1}$$

$$+\left(1+\sum_{k=1}^{K}\alpha_k\lambda_1^{-k} - a(1/\lambda_1)^{1/3}\right) \tag{6-2}$$

$$+\sum_{k=1}^{K}\left(\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right| - 1\right)\alpha_k\lambda_1^{-k} \tag{6-3}$$

$$+\sum_{k=K+1}^{n-1}\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right|\alpha_k\lambda_1^{-k} \tag{6-4}$$

$$-\frac{\alpha_n}{\left|\binom{1/3}{n}\right|}\lambda_1^{-n}. \tag{6-5}$$

We can now make each of the terms (6-2) through (6-5) small as follows:
(6-2): Since $1/\lambda_1$ lies in the radius of convergence of $\alpha(t)$, $1+\sum_{k=1}^{K_1}\alpha_k\lambda_1^{-k}$ converges to $a(1/\lambda_1)^{1/3}$. So for some $K_1 > 0$,

$$\left|1+\sum_{k=1}^{K_1}\alpha_k\lambda_1^{-k} - a(1/\lambda_1)^{1/3}\right| < \delta.$$

(6-4): This term is less than

$$\sum_{k=K+1}^{n-1}\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right||\alpha_k|\lambda_1^{-k}.$$

Fix ϵ_2 such that $(1+\epsilon_2)/\lambda_1 < 1/|\lambda_2|$. Then by Subclaim 1, there exists a K_2 such that for all $n > K_2$ and $j < n$,

$$\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| < \frac{n}{n-j}(1+\epsilon)^j.$$

Then for all $n > K_2$,

$$\begin{aligned}
\sum_{k=K_2+1}^{n-1} \left| \frac{\binom{1/3}{n-k}}{\binom{1/3}{n}} \right| |\alpha_k| \lambda_1^{-k} &< \sum_{k=K_2+1}^{n-1} \frac{n}{n-k} (1+\epsilon_2)^k |\alpha_k| \lambda_1^{-k} \\
&= \sum_{k=K_2+1}^{n-1} \left(1 - \frac{k}{n-k}\right) |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k \\
&< \sum_{k=K_2+1}^{n-1} |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k + \sum_{k=K_2+1}^{n-1} k |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k.
\end{aligned}$$

Since $\alpha(t)$ converges absolutely, $(1+\epsilon_2)/\lambda_1$ lies within the radius of convergence of both of these series. Thus there exists a $K_3 \geq K_2$ such that for all $n > k_3$, both

$$\sum_{k=K_3+1}^{n-1} |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k < \delta$$

and

$$\sum_{k=K_3+1}^{n-1} k |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k < \delta.$$

Thus,

$$\sum_{k=K_3+1}^{n-1} \left| \frac{\binom{1/3}{n-k}}{\binom{1/3}{n}} \right| |\alpha_k| \lambda_1^{-k} < 2\delta.$$

(6-5): For sufficiently large n,

$$\begin{aligned}
\frac{\alpha_n}{\left|\binom{1/3}{n}\right|} \lambda_1^{-n} &\leq \left| \frac{n!}{\frac{1}{3}\left(\frac{1}{3}-1\right) \cdots \left(\frac{1}{3}-(n-1)\right)} \right| |\alpha_n| \lambda_1^{-n} \\
&= \left| \frac{1}{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)} \right| \left| \frac{2}{\frac{1}{3}-3} \right| \cdots \left| \frac{n-2}{\frac{1}{3}-(n-1)} \right| (n-1)(n) |\alpha_n| \lambda_1^{-n} \\
&< \tfrac{27}{10}(n-1)(n)|\alpha_n|\lambda_1^{-n}.
\end{aligned}$$

The series $\sum (n-1)(n)\alpha_n t^n$ has radius of convergence greater than $1/\lambda_1$ and converges absolutely, so the sequence $(n-1)(n)\alpha_n t^n$ is Cauchy. Thus there exists K_5 such that for all $n > K_5$,

$$\frac{\alpha_n}{\left|\binom{1/3}{n}\right|} \lambda_1^{-n} < \tfrac{27}{10}(n-1)(n)|\alpha_n|\lambda_1^{-n} < \delta.$$

At this point we fix K in the equation above so that $K = \max(K_1, K_2, K_3, K_4, K_5)$ and look at the remaining term.

(6-3): This term is less than

$$\sum_{k=1}^{K_1}\left(\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right|-1\right)|s_k|\lambda_1^{-k}.$$

Let

$$\epsilon_2=\frac{\delta}{\sum_{k=1}^{K_1}|\alpha_k|\lambda_1^{-k}}.$$

Then by Subclaim 2, since K is fixed, there exists an $N>K$ such that for all $n>N$ and j with $0<j\leq K$,

$$1<\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right|<1+\epsilon_2.$$

Thus, for all $n>N$,

$$\sum_{k=1}^{K_1}\left(\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right|-1\right)|\alpha_k|\lambda_1^{-k}<\sum_{k=1}^{K_1}((1+\epsilon_2)-1)|\alpha_k|\lambda_1^{-k}=\epsilon_2\sum_{k=1}^{K_1}|\alpha_k|\lambda_1^{-k}=\delta.$$

Combining the above, for $K=\max(K_1, K_2, K_3, K_4, K_5)$ and $n>N$,

$$1+\sum_{k=1}^{n-1}\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right|\alpha_k\lambda_1^{-k}-\frac{\alpha_n}{\left|\binom{1/3}{n}\right|}\lambda_1^{-n}<a(1/\lambda_1)^{1/3}+5\delta=a(1/\lambda_1)^{1/3}(1+\epsilon).$$

So,

$$\begin{aligned}&\left|\frac{r_n}{\left|\binom{1/3}{n}\right|\lambda_1^n a(1/\lambda_1)^{1/3}}-1\right|\\ &\qquad=\left|\frac{\left|\binom{1/3}{n}\right|\lambda_1^n\left(1+\sum_{k=1}^{n-1}\left|\binom{1/N}{n-k}/\binom{1/N}{n}\right|\alpha_k\lambda_1^{-k}-\alpha_n/\left|\binom{1/N}{n}\right|\lambda_1^{-n}\right)}{\left|\binom{1/3}{n}\right|\lambda_1^n a(1/\lambda_1)^{1/3}}-1\right|\\ &\qquad<\left|\frac{\left|\binom{1/3}{n}\right|\lambda_1^n\left(a(t)^{1/3}(1+\epsilon)\right)}{\left|\binom{1/3}{n}\right|\lambda_1^n a(1/\lambda_1)^{1/3}}-1\right|=\epsilon.\end{aligned}$$

□

Proposition 6.4. *Let $1-c(t)=(p(t))^{2/3}$. Then there exists an N such that for $k>N$, we have $c_k\geq 0$.*

Proof. By the same method as above, a good approximation for c_n is

$$\left|\binom{2/3}{n}\right|\lambda_1^n(q(1/\lambda_1))^{2/3}.$$

Note that $q(1/\lambda_1)$ must be positive since $q(0)=1$ and $q(t)$ has no root between 0 and $1/\lambda_1$. □

We can now return to our polynomial $b(t)$, which was constructed such that $p(t)-(1-q(t))^3+b(t)(1-q(t))$ has coefficient 0 for all terms with degree $3n$ or less (n is the degree of $q(t)$).

From Proposition 6.2,

$$[b(t)]_m = b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m,$$

where $p(t)^{1/3} = 1-r(t) = 1-q(t)-s(t)$. We can write

$$\begin{aligned} p(t)^{2/3} = 1-c(t) &= (1-q(t)-s(t))^2 \\ &= 1-2q(t)-2s(t)+2q(t)s(t)+q(t)^2+s(t)^2. \end{aligned}$$

Thus for $n < m \le 2n$,

$$b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m = \tfrac{3}{2}(c_n + [q(t)^2]_n),$$

and for $2n < m \le 3n$,

$$b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m = \tfrac{3}{2}(c_n - [s(t)^2]_n).$$

So if n is large enough that $c_m \ge 0$ for $m \ge n$, we have

$$b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m = \tfrac{3}{2}(c_m + [q(t)^2]_m) \ge 0.$$

Then it remains to show that

$$b_m = 3\big[s(t)(1-q(t)-s(t))\big]_m = \tfrac{3}{2}(c_m - [s(t)^2]_m) \ge 0$$

for $2n < m \le 3n$.

From Propositions 6.3 and 6.4 above, we can use the approximations

$$s_n \approx \left|\binom{1/3}{n}\right|\lambda_1^n (q(1/\lambda_1))^{1/3} \quad \text{and} \quad c_n \approx \left|\binom{2/3}{n}\right|\lambda_1^n (q(1/\lambda_1))^{2/3}.$$

Note that for $2n < m \le 3n$,

$$\begin{aligned} \sum_{i,m-i>n}\left|\binom{1/3}{i}\binom{1/3}{m-i}\right| &\le \sum_{i,m-i>n}\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right| \\ &= (m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|. \end{aligned}$$

Proposition 6.5. *For $2n < m \le 3n$, there exists d with $0 < d < 1$ such that*

$$\frac{(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|}{\left|\binom{2/3}{m}\right|} \le 1-d.$$

Proof. We first prove a couple of subclaims.

Subclaim 3. *For a fixed value of n, the expression*

$$\frac{(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|}{\left|\binom{2/3}{m}\right|}$$

is strictly increasing in the range $2n < m < 3n$.

Proof. First note that the denominator of this term, $\left|\binom{2/3}{m}\right|$, is strictly decreasing for increasing m. We can now show that the numerator of this term is strictly increasing by looking at the ratio of consecutive terms. We have

$$\begin{aligned}\frac{(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|}{(m-2n)\left|\binom{1/3}{n+1}\binom{1/3}{m-n}\right|} &= \left|\frac{(m-2n-1)(m-n)}{(m-2n)(1/3-(m-n-1))}\right| \\ &= \left|\left(1-\frac{1}{m-2n}\right)\frac{m-n}{4/3-(m-n)}\right| \\ &= \frac{1-\frac{1}{m-2n}}{1-\frac{4/3}{m-n}}. \end{aligned} \tag{6-6}$$

Then we can compare $1/(m-2n)$ to $(4/3)/(m-n)$ by looking at their ratio,

$$\frac{\frac{4/3}{m-n}}{\frac{1}{m-2n}} = \frac{4(m-2n)}{3(m-n)}.$$

This term is strictly increasing in the range $2n < m < 3n$. It is equal to 0 when $m = 2n$ and equal to 2/3 when $m = 3n$. Thus for $2n < m < 3n$, we have $1/(m-2n) > (4/3)/(m-n)$ and $1-1/(m-2n) < 1-(4/3)/(m-n)$. So the ratio in (6-6) is less than 1, demonstrating that for $2n < m < 3n$,

$$(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right| < (m-2n)\left|\binom{1/3}{n+1}\binom{1/3}{m-n}\right|. \qquad \square$$

Thus it suffices to consider the largest possible value of m, $3n$, which gives us

$$\frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1}\right|}{\left|\binom{2/3}{3n}\right|}.$$

Subclaim 4. *For all $n \geq 1$,*

$$\frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1}\right|}{\left|\binom{2/3}{3n}\right|} < \frac{n\left|\binom{1/3}{n}\binom{1/3}{2n}\right|}{\left|\binom{2/3}{3n}\right|}.$$

Proof. Again, by looking at their ratio, the denominators cancel, leaving

$$\begin{aligned}\frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1}\right|}{n\left|\binom{1/3}{n}\binom{1/3}{2n}\right|} &= \frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1}\right|}{n\left|\binom{1/3}{n}\binom{1/3}{2n}\right|} \\ &= \frac{n-1}{n}\,\frac{1/3-n}{n+1}\,\frac{2n}{1/3-2n-1} \\ &= \frac{n-1}{n}\,\frac{n(1-\frac{1}{3n})}{n+1}\,\frac{2n}{(2n-1)\left(1-\frac{1}{3(2n-1)}\right)} \\ &= \frac{(n-1)(2n)}{(n+1)(2n-1)}\,\frac{\left(1-\frac{1}{3n}\right)}{\left(1-\frac{1}{3(2n-1)}\right)}.\end{aligned}$$

Now, we can observe that

$$\frac{1-\frac{1}{3n}}{1-\frac{1}{3(2n-1)}} < 1$$

and

$$\frac{(n-1)(2n)}{(n+1)(2n-1)} = \frac{2n^2-2n}{2n^2+n-1} < 1, \quad (n \geq 1),$$

and thus their product is less than 1. $\square$

Subclaim 5. *The terms*

$$\frac{n\left|\binom{1/3}{n}\binom{1/3}{2n}\right|}{\left|\binom{2/3}{3n}\right|}$$

are strictly decreasing for increasing values of n.

Proof. We again compute the ratio of consecutive terms, and find

$$\begin{aligned}\left(\frac{(n+1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1}\right|}{\left|\binom{2/3}{3n}\right|}\right) \Big/ \left(\frac{n\left|\binom{1/3}{n}\binom{1/3}{2n-1}\right|}{\left|\binom{2/3}{3n}\right|}\right) \\ &= \frac{n+1}{n}\,\frac{1/3-n}{n+1}\,\frac{(1/3-2n)(1/3-(2n+1))}{(2n+1)(2n+2)} \\ &\quad \times \frac{(3n+1)(3n+2)(3n+3)}{(2/3-3n)(2/3-(3n+1))(2/3-(3n+2))} \\ &= \frac{\left(1-\frac{1}{3n}\right)\frac{2n}{2n+2}\left(1-\frac{1}{6n}\right)\left(1-\frac{1}{6n+3}\right)}{\frac{3n}{3n+3}\left(1-\frac{2}{9n}\right)\left(1-\frac{2}{9n+3}\right)\left(1-\frac{2}{9n+6}\right)}.\end{aligned}$$

We now define

$$\begin{aligned} f(x) &= \frac{\left(1-\frac{1}{3x}\right)\left(1-\frac{1}{6x}\right)\left(1-\frac{1}{6x+3}\right)}{\left(1-\frac{2}{9x}\right)\left(1-\frac{2}{9x+3}\right)\left(1-\frac{2}{9x+6}\right)} \\ &= \frac{3(3x-1)(6x-1)(3x+1)(3x+1)(3+2)}{x(2x+1)(9x-1)(9x-2)(9x+4)}. \end{aligned}$$

We can compute the derivative of this function,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{3(3x-1)(6x-1)(3x+1)(3x+1)(3+2)}{x(2x+1)(9x-1)(9x-2)(9x+4)}\right) \\ &= \frac{6(104976x^7+130491x^6+49167x^5-1485x^4-4239x^3-258x^2+140x+8)}{x^2(1458x^4+1215x^3+135x^2-70x-8)^2}. \end{aligned}$$

The numerator of this function factors as

$$6(3x+1)(8+116x-606x^2-2421x^3+5778x^4+31833x^5+34992x^6).$$

Thus, the only roots of $f'(x)$ can be at $x=-1/3$ or where

$$8+116x-606x^2-2421x^3+5778x^4+31833x^5+34992x^6=0.$$

This has no solutions in $[1,\infty)$, since

$$\begin{aligned} &8+116x-606x^2-2421x^3+5778x^4+31833x^5+34992x^6 \\ &\qquad = (34992x^6-2421x^3)+(31833x^5-606x^2)+5778x^4+116x+8 \end{aligned}$$

and each term above is strictly positive for $n \geq 1$. By a similar argument, the denominator of $f'(x)$ has no roots in $[1,\infty)$. We can calculate $f'(1) = \frac{1672800}{7452900} \approx 0.2244 > 0$, and thus $f'(x) > 0$ for all $x \in [1,\infty)$. Thus $f(x)$ is strictly increasing on $[1,\infty)$ and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\left(1-\frac{1}{3x}\right)\left(1-\frac{1}{6x}\right)\left(1-\frac{1}{6x+3}\right)}{\left(1-\frac{2}{9x}\right)\left(1-\frac{2}{9x+3}\right)\left(1-\frac{2}{9x+6}\right)} = 1.$$

So $f(x) < 1$ for all $x \in [1,\infty)$. The terms

$$\frac{(n)\left|\binom{1/3}{n}\binom{1/3}{2n}\right|}{\left|\binom{2/3}{3n}\right|}$$

are strictly decreasing for increasing values of n. □

We can then evaluate this expression at $n=1$ and find

$$\frac{\left|\binom{1/3}{1}\binom{1/3}{2}\right|}{\left|\binom{2/3}{3}\right|} = \frac{3}{4}.$$

Thus for all n, m with $2n < m \leq 3n$,

$$\frac{(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|}{\left|\binom{2/3}{m}\right|} \leq \frac{3}{4} = 1 - \frac{1}{4}.$$

So, we can choose $d = 1/4$ and the proposition is valid. □

Now, write

$$c_m - [s(t)^2]_m = c_m - \sum_{i, m-i>n} s_i s_{m-i}.$$

For convenience we define $A = \left|\binom{2/3}{m}\right|$ and $B = (m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|$. Choose $\delta > 0$ such that $Ad > \delta(A - 2B - \delta B)$.

By the propositions above, we can choose n such that for all $m > n$,

$$c_m > (1-\delta)\left|\binom{2/3}{m}\right|\lambda_1^m (q(1/\lambda_1))^{2/3}$$

and

$$s_m < (1+\delta)\left|\binom{1/3}{m}\right|\lambda_1^m (q(1/\lambda_1))^{1/3}.$$

So,

$$\begin{aligned}
&c_m - [s(t)^2]_m \\
&> (1-\delta)\left|\binom{2/3}{m}\right|\lambda_1^m (q(1/\lambda_1))^{2/3} \\
&\quad - \sum_{i, m-i>n} \left((1+\delta)\left|\binom{1/3}{i}\right|\lambda_1^i (q(1/\lambda_1))^{1/3}\right)\left((1+\delta)\left|\binom{1/3}{m-i}\right|\lambda_1^{m-i} (q(1/\lambda_1))^{1/3}\right) \\
&= (1-\delta)\left|\binom{2/3}{m}\right|\lambda_1^m (q(1/\lambda_1))^{2/3} - \sum_{i, m-i>n} (1+\delta)^2 \left|\binom{1/3}{i}\right|\left|\binom{1/3}{m-i}\right|\lambda_1^m (q(1/\lambda_1))^{2/3} \\
&= \left(\lambda_1^m (q(1/\lambda_1))^{2/3}\right)\left((1-\delta)\left|\binom{2/3}{m}\right| - (1+\delta)^2 \sum_{i, m-i>n} \left|\binom{1/3}{i}\right|\left|\binom{1/3}{m-i}\right|\right) \\
&\geq \left(\lambda_1^m (q(1/\lambda_1))^{2/3}\right)\left((1-\delta)\left|\binom{2/3}{m}\right| - (1+\delta)^2 (m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|\right).
\end{aligned}$$

In terms of A and B defined above, the term in parentheses in the last line can be expanded to

$$A - \delta A - B - 2\delta B - \delta^2 B.$$

Then since $B/A \leq 1 - d$, we have $A - B \geq Ad$, and the expression above is greater than or equal to

$$Ad - \delta A - 2\delta B - \delta^2 B = Bd - \delta(A - 2B - \delta B).$$

By our choice of δ above, this is strictly greater than or equal to 0, so we are done.

Namely, this demonstrates that we can construct a polynomial $b(t)$ of degree at most $3n$ with positive coefficients such that $p(t)-(1-q(t))^3+b(t)(1-q(t))$ has coefficient 0 for all terms with degree $3n$ or less.

Let $d(t)=(1-q(t))^3-b(t)(1-q(t))-p(t)$. Since n is the degree of $q(t)$, $q(t)^3$ will have degree $3n$ as well, so any remaining terms in $d(t)$ will be the result of trailing terms in the product of $b(t)$ and $q(t)$. Since both of these polynomials contain only positive coefficients, $d(t)$ will as well. As a result, $p(t)=(1-q(t))^3-b(t)(1-q(t))-d(t)$ and we can construct the matrix

$$A(t)=\begin{bmatrix} q(t) & d(t)/t^2 & b(t)/t \\ 0 & q(t) & t \\ t & 0 & q(t) \end{bmatrix}$$

such that $I-A(t)$ has determinant $p(t)$.

7. Further work

The obvious next step in this research would be to continue to study this problem for larger values of N and to develop constructions for correspondingly larger polynomial matrices. Already for the case $N=4$ at least a slightly new method will be required. The logical progression to a 4×4 matrix would be to construct

$$M(t)=\begin{bmatrix} q(t) & \alpha(t) & \beta(t) & \gamma(t) \\ 0 & q(t) & t & 0 \\ 0 & 0 & q(t) & t \\ t & 0 & 0 & q(t) \end{bmatrix}.$$

In this case $I-M(t)$ has determinant

$$(1-q(t))^4-\alpha(t)t^3-\beta(t)(1-q(t))t^2-\gamma(t)(1-q(t))^2t.$$

Ignoring the $\gamma(t)$ term (i.e., letting $\gamma(t)=0$) results in a problem identical to the $N=3$ case; however, it does not appear that this method will suffice for all polynomials which satisfy the condition that $p(t)^{1/4}$ has all negative coefficients. Thus it is likely that a solution will require use of the $\gamma(t)$ polynomial; however, the same "greedy" algorithm cannot be used. Whereas $1-q(t)$ had all negative coefficients except for the leading 1, meaning that each coefficient of $\beta(t)$ "helped" all of the coefficients of higher order by making them more positive, $(1-q(t))^2$ will not in general have that property. So coefficients of $\lambda(t)$ would correct some terms while "hindering" others by making them more negative. It is also possible that a different matrix configuration, utilizing more of the positions occupied by t or 0 is required.

Clearly, the ideal result would be a general proof that demonstrated this result for all N.

Another interesting possibility for research would be to look at the degrees of polynomials required in this construction and to attempt to constrain them. As mentioned before, if a given construction could control the size of both the polynomial matrix and the degrees of the polynomials used in the matrix, then it would put a constraint on the required size of the matrix over $\mathbb{R}_+$ described in the original problem.

Interestingly, the results given here for $N = 1$ and 2 already constrain the degree of the polynomials used. (For a polynomial of degree d, the $N = 1$ requires only a polynomial of degree d and the $N = 2$ case requires a matrix with polynomials of degree at most $2d$.) However, the polynomials required in the $N = 3$ case may currently have arbitrarily high degree.

References

[Berman and Plemmons 1979] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Academic Press, New York, 1979. MR 82b:15013 Zbl 0484.15016

[Boyle 1993] M. Boyle, "Symbolic dynamics and matrices", pp. 1–38 in *Combinatorial and graph-theoretical problems in linear algebra* (Minneapolis, MN, 1991), edited by R. A. Brualdi et al., IMA Vol. Math. Appl. **50**, Springer, New York, 1993. MR 94g:58062 Zbl 0844.58023

[Boyle and Handelman 1991] M. Boyle and D. Handelman, "The spectra of nonnegative matrices via symbolic dynamics", *Ann. of Math.* (2) **133**:2 (1991), 249–316. MR 92d:58057 Zbl 0735.15005

[Egleston et al. 2004] P. D. Egleston, T. D. Lenker, and S. K. Narayan, "The nonnegative inverse eigenvalue problem", *Linear Algebra Appl.* **379** (2004), 475–490. MR 2005b:15040 Zbl 1040.15009

[Friedland 2012] S. Friedland, "A note on the nonzero spectra of irreducible matrices", *Linear Multilinear Algebra* **60**:11-12 (2012), 1235–1238. MR 2989758 Zbl 1257.15006

[Laffey 2012] T. J. Laffey, "A constructive version of the Boyle–Handelman theorem on the spectra of nonnegative matrices", *Linear Algebra Appl.* **436**:6 (2012), 1701–1709. MR 2890950 Zbl 1241.15007

[Laffey et al. 2013] T. J. Laffey, R. Loewy, and H. Šmigoc, "Power series with positive coefficients arising from the characteristic polynomials of positive matrices", preprint, 2013. arXiv 1205.1933

[Lind and Marcus 1995] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, 1995. MR 97a:58050 Zbl 1106.37301

[Minc 1988] H. Minc, *Nonnegative matrices*, Wiley, New York, 1988. MR 89i:15001 Zbl 0638.15008

[Suleĭmanova 1949] H. R. Suleĭmanova, "Stochastic matrices with real characteristic numbers", *Doklady Akad. Nauk SSSR* (*N.S.*) **66** (1949), 343–345. In Russian. MR 11,4d Zbl 0035.20903

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