The number of convex topologies
on a finite totally ordered set

Tyler Clark and Tom Richmond

The number of convex topologies
on a finite totally ordered set

Tyler Clark and Tom Richmond

(Communicated by Kenneth S. Berenhaut)

We give an algorithm to find the number $T_{\mathrm{cvx}}(n)$ of convex topologies on a totally ordered set X with n elements, and present these numbers for $n \leq 10$.

1. Introduction

A subset B of poset $(X, \leq)$ is *increasing* if $x \in B$ and $y \geq x$ imply $y \in B$, and is *convex* if $x, z \in B$ and $x \leq y \leq z$ imply $y \in B$. An n-point totally ordered set X may be labeled $X = \{1, 2, \ldots, n\}$, where $1 < 2 < \cdots < n$. This set will be denoted $[1, n]$, and in general, $[a, b]$ will denote $\{a, a+1, \ldots, b\} \subset \mathbb{N}$ with the natural order from $\mathbb{N}$. A topology on $(X, \leq)$ is *convex* if it has a base of convex sets, or equivalently, if each point has a neighborhood base of convex sets. Because of these equivalent characterizations, convex topologies are often called *locally convex* topologies. (See [Nachbin 1965]). For finite sets, every point j has a minimal neighborhood $\mathrm{MN}(j)$, which is the intersection of all neighborhoods of j. It is convenient to identify a topology on $[1, n]$ with its base $\{\mathrm{MN}(j) : j \in [1, n]\}$ of minimal neighborhoods of each point. Finite topological spaces are used in computer graphics, where the Euclidean plane is modeled by a topology on a finite set of pixels. If $a < b < c$ in a finite poset with a topology, if c is "near" a and there is any compatibility between the topology and order, we would expect b to also be near a. This is the convexity condition, which is a natural, weak compatibility condition between a topology and order assumed in most applications. We will consider the number of convex topologies on a finite totally ordered set $[1, n]$.

An excellent reference on finding the number $T(n)$ of topologies on an n-element set is [Erné and Stege 1991]. Currently, $T(n)$ is known for $n \leq 18$. A standard approach to counting topologies on a finite set X is to employ the one-to-one correspondence between a topology τ on X and the associated specialization quasiorder defined by $x \leq y$ if and only if x is in the closure of y. This correspondence

dates back to [Alexandroff 1937]. (See [Richmond 1998] for a survey of this connection.) One approach to counting the convex topologies would be to find a (biordered) characterization of convex topologies using some compatibility between the specialization order and the given total order. Fruitful results in this direction have not been found.

For $j \in [1, n]$, a convex subset $N(j)$ of $[1, n]$ containing j has the form $[a, b]$, where $1 \le a \le j \le b \le n$. There are j choices for a and $n - j + 1$ choices for b, and thus $j(n + j - 1)$ choices for $N(j)$. Since a base of minimal neighborhoods for a locally convex topology on $[1, n]$ consists of one convex subset $N(j)$ for each $j \in [1, n]$, we see that

$$\prod_{j=1}^{n} (j)(n + j - 1) = (n!)^2$$

gives an upper bound on $T_{\text{cvx}}(n)$. Of course, arbitrarily selecting a convex set $N(j)$ containing j for each $j \in [1, n]$ is unlikely to give a base for a topology, so this upper bound is not sharp.

2. Nested convex topologies

Stephen [1968] gave a recursive formula for the number of nested topologies (or equivalently, ordered partitions) on an n-point set X, generating the sequence $1, 3, 13, 75, 541, 4683, 47293, \ldots$, which is A000670 in *The On-Line Encyclopedia of Integer Sequences* (OEIS); see [Sloane 2014]. If $X = [1, n]$ is a totally ordered set with n elements, let $T_{\text{Nest}}(n)$ be the number of nested convex topologies on X, and let $T_{\text{Nest}}(n, k)$ be the number of those convex topologies consisting of k nested nonempty open sets $U_1, U_2, \ldots, U_k$, where $X = U_1 \supset U_2 \supset \cdots \supset U_k \neq \varnothing$. Since the indiscrete topology is the only nested topology with one nonempty open set, $T_{\text{Nest}}(n, 1) = 1$. Suppose we have found $T_{\text{Nest}}(m, j)$ for all $m \le n$ and $j \le k$. To find $T_{\text{Nest}}(n, k + 1)$, note that $X = U_1 \supset U_2 \supset U_3 \supset \cdots \supset U_{k+1} \neq \varnothing$ implies that U_2 must contain at least k elements and at most $n - 1$ elements. If $|U_2| = j$, there are $n - j + 1$ ways to choose U_2 as a convex subset of X, and $T_{\text{Nest}}(j, k)$ ways to complete the nested convex topology $\{U_2, \ldots, U_{k+1}\}$ on the j-point totally ordered set U_2. Thus, we have

$$T_{\text{Nest}}(n, k + 1) = \sum_{j=k}^{n-1} (n - j + 1) T_{\text{Nest}}(j, k) = \sum_{m=2}^{n-k+1} m \cdot T_{\text{Nest}}(n - m + 1, k),$$

where the second equality follows from the substitution $m = n - j + 1$. In Table 1, we tabulate the values of $T_{\text{Nest}}(n, k)$ for $n, k \le 10$.

n \ k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1	5	4							
4	1	9	16	8						
5	1	14	41	44	16					
6	1	20	85	146	112	32				
7	1	27	155	377	456	272	64			
8	1	35	259	833	1,408	1,312	640	128		
9	1	44	406	1,652	3,649	4,712	3,568	1,472	256	
10	1	54	606	3,024	8,361	14,002	14,608	9,312	3,328	512

Table 1. $T_{\mathrm{Nest}}(n, k)$, the number of topologies on a totally ordered n-point set consisting of k nested convex sets.

This table (sequence A056242 in the OEIS [Mallows 2014]) is also used by Hwang and Mallow [1995] to count the number of *order-consecutive partitions* of $X = \{1, 2, \ldots, n\}$, which they define as follows: An ordered list $S_1, S_2, \ldots, S_m$ of subsets of X is an order-consecutive partition of X if $\{S_1, \ldots, S_m\}$ is a partition of X and each of the sets $\bigcup_{j=1}^{k} S_j$ $(1 \leq k \leq m)$ is a consecutive set of integers. If $\{S_1, \ldots, S_m\}$ is an order-consecutive partition, clearly $\{S_1, S_1 \cup S_2, S_1 \cup S_2 \cup S_3, \ldots, X\}$ is a nested convex topology on X. Conversely, any nested convex topology $\tau = \{U_1, U_2, \ldots, U_k\}$ on $X = \{1, 2, \ldots n\}$ generates the order-consecutive partition $U_1, U_2 \setminus U_1, U_3 \setminus U_2, \ldots, U_k \setminus U_{k-1}$.

It is easy to confirm from our formula for $T_{\mathrm{Nest}}(n, k)$ that $T_{\mathrm{Nest}}(n, n) = 2^{n-1}$ and $T_{\mathrm{Nest}}(n, 2) = \Delta_n - 1$, where Δ_n is the n-th triangular number.

Now, we note that

$$T_{\mathrm{Nest}}(n) = \sum_{k=1}^{n} T_{\mathrm{Nest}}(n, k).$$

This sequence, whose first few elements are

$$(T_{\mathrm{Nest}}(n))_{n=1}^{10} = (1, 3, 10, 34, 116, 396, 1352, 4616, 15760, 53808),$$

appears as A007052 in the OEIS [Mallows et al. 2014], where it is noted that

$$T_{\mathrm{Nest}}(n) = 4\, T_{\mathrm{Nest}}(n - 1) - 2\, T_{\mathrm{Nest}}(n - 2) \quad \text{for } n > 2.$$

Solving this recurrence relation by standard techniques gives

$$T_{\mathrm{Nest}}(n) = \frac{(2 + \sqrt{2})^n + (2 - \sqrt{2})^n}{4}.$$

Nested convex topologies have as much inclusion as possible. Not only are they totally ordered by inclusion, but they maximize "overlap". The other extreme would be to have as little inclusion and overlap as possible. This suggests considering mutually disjoint collections. A collection $\mathcal{D}$ of mutually disjoint convex subsets of X is not a basis for a topology if $\bigcup \mathcal{D} \neq X$, but $\mathcal{D} \cup \{X\}$ is always a basis for a convex topology on X. The authors have shown that the number of topologies on an n-element totally ordered set having a base consisting of a mutually disjoint collection $\mathcal{D}$ of convex sets, or such a collection $\mathcal{D}$ together with X, is $F_{2n+1} - 1$, where F_k is the k-th Fibonacci number [Clark and Richmond 2010].

3. An algorithm for $T_{\mathrm{cvx}}(n)$

We now present a recursive algorithm to find the number $T_{\mathrm{cvx}}(n)$ of convex topologies on a totally ordered set $[1, n]$. It is easy to check that $T_{\mathrm{cvx}}(1) = 1 = T(1)$ and $T_{\mathrm{cvx}}(2) = 4 = T(2)$. That is, the only topology on a 1-point set is convex, as are all four topologies on a 2-point set.

Suppose $T_{\mathrm{cvx}}(n)$ is known. To find $T_{\mathrm{cvx}}(n+1)$, note that each convex topology on $[1, n+1]$, when restricted to $[1, n]$, gives a unique convex topology on $[1, n]$. Thus, we may count $T_{\mathrm{cvx}}(n)$ by looping through each topology τ counted in $T_{\mathrm{cvx}}(n)$, adding $n+1$ as the greatest point, adjusting the minimal neighborhoods of $j \in [1, n]$, and defining the minimal neighborhood of $n+1$ so that the subspace topology on $[1, n]$ is still τ. That is, considering how each topology on $[1, n]$ may be appropriately expanded to $[1, n+1]$ gives a complete, unduplicated count of the convex topologies on $[1, n+1]$.

Step 1: Redefining minimal neighborhoods of $j \in [1, n]$. We loop through all convex topologies τ on $[1, n]$. The simplest way to extend τ to $[1, n+1]$ so that the restriction of the extension is still τ would be to keep the minimum neighborhoods of each $j \in [1, n]$ unchanged. However, we may also expand some of the minimal neighborhoods of points $j \in [1, n]$ to include $n+1$. To maintain convexity and to guarantee a topology on $[1, n+1]$ whose restriction to $[1, n]$ agrees with τ, the minimal neighborhood $\mathrm{MN}(j)$ of j can be expanded to include $n+1$ if and only if $\mathrm{MN}(j)$ already includes n. If $n \in \mathrm{MN}(j) \subseteq \mathrm{MN}(k)$ and $\mathrm{MN}(j)$ is expanded to include $n+1$, then $\mathrm{MN}(k)$ must also be expanded to include $n+1$, for otherwise $\mathrm{MN}(k)$ would be a neighborhood of j not including $n+1$, contrary to the hypothesis that the minimal neighborhood of j was to include $n+1$.

As an immediate consequence, if $n \in \mathrm{MN}(j) = \mathrm{MN}(k)$, then $\mathrm{MN}(j)$ is expanded to include $n+1$ if and only if $\mathrm{MN}(k)$ is. That is, a single basis element which happens to be the minimal neighborhood of distinct points j and k is still treated as a single entity in the expansion process.
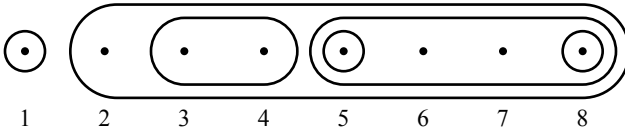
Figure 1. A sample topology on [1, 8].
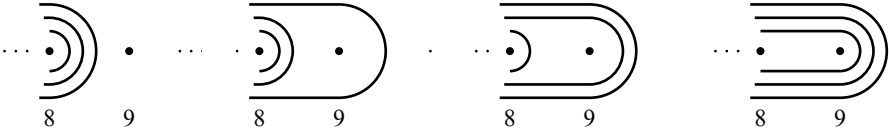
Figure 2. Possible expansions of minimal neighborhoods containing previous right endpoint: none, outermost one, outermost two, outermost three.

Thus, if $\mathcal{B} = \{\text{MN}(1), \text{MN}(2), \ldots, \text{MN}(n)\}$ has m distinct sets containing n, we expand the outermost k of these to include $n + 1$, looping as k goes from 1 to m.

For example, consider the convex topology τ on [1, 8] having a base of minimal neighborhoods $\mathcal{B} = \{\{1\}, [2, 8], [3, 4], \{5\}, [5, 8], \{8\}\}$, as shown in Figure 1.

We may add 9 to this topology without changing any of the minimal neighborhoods of j for $j \in [1, 8]$, or since MN(2), MN(6) = MN(7), and MN(8) include the right endpoint 8, they may be extended to include the added point 9. Since $8 \in \text{MN}(8) \subset \text{MN}(7) = \text{MN}(6) \subset \text{MN}(2)$, we note that MN(6) is expanded if and only if MN(7) is expanded, so we do not need to treat MN(6) and MN(7) as distinct basis elements and we may effectively ignore the duplicate MN(7). Also, if MN(6) is expanded, then MN(6) $\subset$ MN(2) implies that MN(2) would also have to be expanded. Repeating this idea, we may expand nothing except the outermost (i.e., longest) minimal neighborhood containing 8, namely MN(2), the outermost two minimal neighborhoods containing 8, namely MN(2) and MN(6), or the outermost three, MN(2), MN(6), and MN(8). See Figure 2.

Step 2: Defining the minimal neighborhood of the added point. Having determined the expansion of minimal neighborhoods of $j \in [1, n]$, it remains to define the minimal neighborhood MN$(n+1)$ of $n+1$. Clearly we must have $n+1 \in \text{MN}(n+1)$. The convexity condition and our need to retain the original topology τ on $[1, n]$ as a subspace imply that MN$(n + 1)$ must be of form $\{n+1\} \cup I$, where I is increasing and open in τ. The final condition is the minimality of the neighborhood MN$(n+1)$. In Step 1, we may have expanded some neighborhoods of n to contain $n + 1$ and, if so, the minimal neighborhood of $n+1$ must be contained in each of these previously defined neighborhoods of $n + 1$. Thus, MN$(n + 1)$ must be of the form $\{n + 1\} \cup I$,
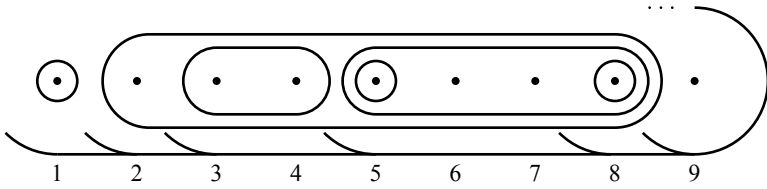
Figure 3. Possible choices for MN(9) if no minimal neighborhoods MN(j) are expanded for $j \in [1, 8]$.
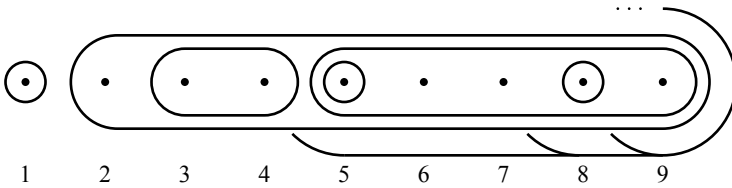
Figure 4. Possible choices for MN(9) if MN(2) and MN(6) are expanded to include 9.

where I is increasing and τ-open, and I is contained in the innermost (shortest) neighborhood MN(j) which was expanded in Step 1.

Continuing the example presented above, we may expand none of the original minimal neighborhoods of $j \in [1, 8]$ to include 9, and then the minimal neighborhood MN(9) of 9 may be defined as $\{9\} \cup I$, where I is an increasing τ-open set in any of the six ways suggested in Figure 3.

Figure 4 shows the three possible choices for the minimal neighborhood MN(9) if the outermost two minimal neighborhoods containing 8, namely MN(2) and MN(6), have been expanded to include 9.

A computer implementation of this algorithm yields the values for $T_{\text{cvx}}(n)$ shown in Table 2 below. With the $T_{\text{cvx}}(2) = 4$ convex topologies on $[1, 2]$ as input, the computer implementation loops through all the topologies τ on $[1, n]$, adds $n + 1$, determines the number m of distinct minimal neighborhoods of $j \in [1, n]$ containing n, expands the outermost k of these to contain $n + 1$ (as k goes from 0 to m), determines the increasing τ-open sets, defines the minimal neighborhood MN($n + 1$) of $n + 1$ as $\{n + 1\} \cup I$, where I is one of the increasing τ-open sets contained in the smallest MN(j) previously expanded to include $n + 1$, and, at each selection of an option above, increments the $T_{\text{cvx}}(n + 1)$ counter and records the data for this new topology on $[1, n + 1]$ required for the next iteration.

The efficiency of this algorithm can be improved by eliminating duplication of computations. For example, if p is the largest integer with MN(p) = X for two topologies s and t which agree to the right of p, then the computation for s duplicates that for t, as noted by a helpful referee.

n	$T_{\text{Nest}}(n)$	$T_{\text{cvx}}(n)$	$T(n)$
1	1	1	1
2	3	4	4
3	10	21	29
4	34	129	355
5	116	876	6,942
6	396	6,376	209,527
7	1,352	48,829	9,535,241
8	4,616	388,771	642,779,354
9	15,760	3,191,849	63,260,289,423
10	53,808	26,864,936	8,977,053,873,043

Table 2. The numbers $T_{\text{Nest}}(n)$ and $T_{\text{cvx}}(n)$ of nested convex topologies and convex topologies on an n-point totally ordered set, and the number $T(n)$ of topologies on an n-point set.

The numbers $T_{\text{cvx}}(n)$ in Table 2 were also verified for $n \leq 8$ without the algorithm using an exhaustive generation scheme. For comparison, we also include the number $T_{\text{Nest}}(n)$ of nested convex topologies and the number $T(n)$ of topologies on n points in the table.

References

[Alexandroff 1937] P. Alexandroff, "Diskrete Räume", *Mat. Sb.* (*N.S.*) **2**:3 (1937), 501–518.

[Clark and Richmond 2010] T. Clark and T. Richmond, "Collections of mutually disjoint convex subsets of a totally ordered set", *Fibonacci Quart.* **48**:1 (2010), 77–79. Currently also at http://people.wku.edu/tom.richmond/Papers/FibQuarterly.pdf. MR 2011e:11029 Zbl 1211.05017

[Erné and Stege 1991] M. Erné and K. Stege, "Counting finite posets and topologies", *Order* **8**:3 (1991), 247–265. MR 93b:06004 Zbl 0752.05002

[Hwang and Mallows 1995] F. K. Hwang and C. L. Mallows, "Enumerating nested and consecutive partitions", *J. Combin. Theory Ser. A* **70**:2 (1995), 323–333. MR 96e:05014 Zbl 0819.05005

[Mallows 2014] C. Mallows, "A056242: Triangle read by rows", entry A056242 in *The on-line encyclopedia of integer sequences* (http://oeis.org), 2014.

[Mallows et al. 2014] C. Mallows, N. J. A. Sloane, and S. Plouffe, "A007052: Number of order-consecutive partitions of n", entry A007052 in *The on-line encyclopedia of integer sequences* (http://oeis.org), 2014.

[Nachbin 1965] L. Nachbin, *Topology and order*, Van Nostrand Mathematical Studies **4**, D. Van Nostrand, Princeton, N.J., 1965. MR 36 #2125 Zbl 0131.37903

[Richmond 1998] T. A. Richmond, "Quasiorders, principal topologies, and partially ordered partitions", *Internat. J. Math. Math. Sci.* **21**:2 (1998), 221–234. MR 99h:06003 Zbl 0898.54005

[Sloane 2014] N. J. A. Sloane, "A000670: Fubini numbers", entry A000670 in *The on-line encyclopedia of integer sequences* (http://oeis.org), 2014.

[Stephen 1968] D. Stephen, "Topology on finite sets", *Amer. Math. Monthly* **75** (1968), 739–741. MR 38 #2725 Zbl 0191.20701

thomas.clark973@topper.wku.edu   *Department of Mathematics, Western Kentucky University,*
1906 College Heights Boulevard,
Bowling Green, KY 42101, United States

tom.richmond@wku.edu   *Department of Mathematics, Western Kentucky University,*
1906 College Heights Boulevard,
Bowling Green, KY 42101, United States

involve

msp.org/involve

involve

2015　　vol. 8　　no. 1