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# Nonreal zero decreasing operators related to orthogonal polynomials

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Laguerre's theorem regarding the number of nonreal zeros of a polynomial and its image under certain linear operators is generalized. This generalization is then used to (1) exhibit a number of previously undiscovered complex zero decreasing sequences for the Jacobi, ultraspherical, Legendre, Chebyshev, and generalized Laguerre polynomial bases and (2) simultaneously generate a basis $B$ and a corresponding complex zero decreasing sequence for the basis $B$. An extension to transcendental entire functions in the Laguerre–Pólya class is given, which, in turn, gives a new and short proof of a previously known result due to Piotrowski. The paper concludes with several open questions.

## 1. Introduction

For a function $f : \mathbb{C} \to \mathbb{C}$ which is not the identically zero function, denote the number (counted according to multiplicity) of real and nonreal zeros of $f$ by $Z_R(f)$ and $Z_C(f)$, respectively. For the identically zero function, define $Z_R(0) = 0$ and $Z_C(0) = 0$. Let $L : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. If $L$ has the property that

$$Z_C(L(p)) \leq Z_C(p) \tag{1}$$

for every real polynomial $p$, then $L$ is called a *complex zero decreasing operator*, or a CZDO. Such an operator $L$ is diagonal with respect to a basis $B = \{b_k\}_{k=0}^{\infty}$ for $\mathbb{R}[x]$ if and only if there are real constants $\{\gamma_k\}_{k=0}^{\infty}$ for which

$$L(b_k(x)) = \gamma_k b_k(x) \quad (k = 0, 1, 2, \ldots). \tag{2}$$

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In this case, the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *complex zero decreasing sequence for the basis $B$*, or a $B$-CZDS.

A theorem of Laguerre demonstrates the existence of CZDSs for the standard basis. We give two versions of his theorem here, the first of which can be found in [Obreschkoff 1963, p. 6] and [Craven and Csordas 2004, p. 23].

**Theorem 1** (Laguerre's Theorem). *Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary real polynomial of degree $n$. If $\alpha$ lies outside the interval $(-n, 0)$, then*

$$Z_C\left(\sum_{k=0}^{n}(k+\alpha)a_k x^k\right) \le Z_C\left(\sum_{k=0}^{n} a_k x^k\right).$$

*In particular, if $\alpha \ge 0$, the sequence $\{k+\alpha\}_{k=0}^{\infty}$ is a CZDS for the standard basis.*

With notation as in Theorem 1,

$$xp'(x) + \alpha p(x) = \sum_{k=0}^{n}(k+\alpha)a_k x^k,$$

and Laguerre's theorem may be restated accordingly.

**Theorem 2** (Laguerre's Theorem; Differential Operator Version). *Let $p(x)$ be an arbitrary real polynomial of degree $n$. If $\alpha$ lies outside the interval $(-n, 0)$, then*

$$Z_C(xp'(x) + \alpha p(x)) \le Z_C(p(x)).$$

*In particular, if $\alpha \ge 0$, then the differential operator $xD + \alpha I$ is a CZDO.*

**Remark 3.** The differentiation operator $D$ defined by $D(p) = p'$ is a CZDO. This is included in Laguerre's theorem as the special case $\alpha = 0$. Indeed, this choice gives

$$Z_C(p'(x)) = Z_C(xp'(x)) \le Z_C(p(x)).$$

Alternatively, the fact that $D$ is a CZDO can be proved via Rolle's theorem from elementary calculus (see, for example, [Obreschkoff 1963, p. 2–3]).

Laguerre's theorem is easily extended by iteration to sequences of the form $\{h(k)\}_{k=0}^{\infty}$, where $h$ is a real polynomial having only real nonpositive zeros. This, in turn, leads to a further extension via Hurwitz's theorem to sequences of the form $\{\varphi(k)\}_{k=0}^{\infty}$, where $\varphi$ is an entire function which is the uniform limit on compact subsets of $\mathbb{C}$ of polynomials having only real nonpositive zeros (see, for example, [Craven and Csordas 1995, Theorem 1.4], [Obreschkoff 1963, p. 6], [Pólya 1929]). We have opted to state Laguerre's theorem in its simplest form to ease the comparison of this theorem with some of its generalizations demonstrated below.

In 2007, Piotrowski gave a generalization of Laguerre's theorem to obtain a class of $H$-CZDSs, where $H$ denotes the set of Hermite polynomials defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad (n = 0, 1, 2, \dots).$$

**Theorem 4** [Piotrowski 2007, p. 57, Proposition 68]. *Suppose $p(x)$ is an arbitrary real polynomial of degree $n$. If $\alpha$, $\beta$, $c$, $d$ are real numbers such that $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + cn \geq 0$, then*

$$Z_C\bigl(-\beta p''(x) + (cx+d)p'(x) + \alpha p(x)\bigr) \leq Z_C(p(x)).$$

*In particular, if $\alpha$, $\beta$, and $c$ are all nonnegative, then $-\beta D^2 + (cx+d)D + \alpha I$ is a CZDO.*

Since the Hermite polynomials satisfy the differential equation

$$nH_n(x) = -\tfrac{1}{2}H_n''(x) + xH_n'(x) \quad (n = 0, 1, 2, \dots)$$

(see, for example, [Rainville 1960, p. 188]), the previous theorem gives, as a special case, the existence of $H$-CZDSs which can be interpolated by linear polynomials.

**Theorem 5** [Piotrowski 2007, p. 87, Theorem 101]. *Let $p(x) = \sum_{k=0}^{n} a_k H_k(x)$ be an arbitrary real polynomial of degree $n$. If $\alpha$ lies outside the interval $(-n, 0)$, then*

$$Z_C\left(\sum_{k=0}^{n}(k+\alpha)a_k H_k(x)\right) \leq Z_C\left(\sum_{k=0}^{n} a_k H_k(x)\right).$$

*In particular, if $\alpha \geq 0$, then the sequence $\{k+\alpha\}_{k=0}^{\infty}$ is an $H$-CZDS.*

While no complete characterization of CZDSs is currently known for any basis, the characterization of CZDSs which can be interpolated by polynomials has been achieved for both the standard basis and the Hermite basis.

**Theorem 6** [Craven and Csordas 1995, p. 13]. *Let $h(x)$ be a real polynomial. Then $\{h(k)\}_{k=0}^{\infty}$ is a CZDS for the standard basis if and only if either*

(1) *$h(0) \neq 0$ and $h(x)$ has only real negative zeros, or*

(2) *$h(0) = 0$ and $h(x)$ is of the form*

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{k=1}^{p}(x-b_k), \tag{3}$$

*where $m \geq 1$ and $p \geq 0$ are integers and $b_k < m$ for $k = 1, 2, 3, \dots, p$.*

The previous theorem remains valid mutatis mutandis if "CZDS for the standard basis" is replaced by "$H$-CZDS" (see [Piotrowski 2007, p. 95, Theorem 111]).

The main results of this paper include a generalization of Laguerre's theorem (Theorem 8), the demonstration of classes of CZDSs for the Jacobi, ultraspherical, Legendre, Chebyshev, and generalized Laguerre polynomial bases (Proposition 10, Theorem 14, Corollaries 15 and 16, and Theorem 24), a method for simultaneously generating a basis $B$ and a corresponding $B$-CZDS (Section 4), and the extension of these results to transcendental entire functions in the Laguerre–Pólya class (Section 5.1).

## 2. A class of complex zero decreasing operators

This section contains two theorems which generalize Laguerre's theorem.

**Theorem 7.** *Let $p$ and $q$ be real polynomials, each with degree at least one, and let $\alpha \geq 0$. Then*

$$Z_R(f(x)) \geq Z_R(p(x)) + Z_R(q(x)) - 1,$$

*where*

$$f(x) = q(x)p'(x) + \alpha q'(x)p(x).$$

*Proof.* When $\alpha = 0$, we have

$$Z_R(q(x)p'(x)) = Z_R(q(x)) + Z_R(p'(x)) \geq Z_R(p(x)) + Z_R(q(x)) - 1,$$

where the last inequality is a consequence of Rolle's theorem.

We will now suppose $\alpha > 0$ for the remainder of the proof. Suppose $x_0$ is a zero of $p(x) \cdot q(x)$ and write

$$\begin{aligned} p(x) &= (x - x_0)^m h_1(x) \quad (h_1(x_0) \neq 0), \\ q(x) &= (x - x_0)^w h_2(x) \quad (h_2(x_0) \neq 0). \end{aligned}$$

Then

$$f(x) = (x - x_0)^{m+w-1} h_3(x),$$

where

$$h_3(x_0) = (m + \alpha w) h_1(x_0) h_2(x_0) \neq 0.$$

That is to say, if $x_0$ is a zero of $p \cdot q$ of multiplicity $m + w$, then $x_0$ is a zero of $f$ of multiplicity $m + w - 1$. We will now complete the proof by demonstrating that $f$ must vanish between consecutive real zeros of $p \cdot q$. Define

$$g(x) = \begin{cases} [q(x)]^\alpha & \text{if } q(x) \geq 0, \\ -[-q(x)]^\alpha & \text{if } q(x) < 0, \end{cases}$$

so that

$$\left|q(x)\right|^{1-\alpha} \frac{d}{dx}[g(x)p(x)] = q(x)p'(x) + \alpha q'(x)p(x) \quad (x \notin \{z \mid q(z) = 0\}).$$

Let $x_1, x_2$ be consecutive zeros of $p \cdot q$ with $x_1 < x_2$. Then they are also consecutive zeros of $g \cdot p$, which is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$. By

Rolle's theorem, $(g \cdot p)'$, and therefore $q(x)p'(x)+\alpha q'(x)p(x)$ has a zero in the interval $(x_1, x_2)$ and the conclusion of the theorem holds. □

We note that Theorem 7 is best possible in the sense that the conclusion does not necessarily hold for any $\alpha < 0$. For example, if $\alpha < 0$, $p(x) = x^n(x^2+\alpha)$, and $q(x) = x$, then $f(x) = x^n\big((\alpha+n+2)x^2+\alpha(\alpha+n)\big)$. Choosing

$$n = \max\{m \in \mathbb{Z} \mid m \geq 0 \text{ and } \alpha+m < 0\}$$

yields $Z_R(f) = n < n+2 = Z_R(p)+Z_R(q)-1$.

**Theorem 8.** *Let $p$ and $q$ be real polynomials and $\alpha \geq 0$. Then*

$$Z_C\big(q(x)p'(x)+\alpha q'(x)p(x)\big) \leq Z_C(p(x))+Z_C(q(x)).$$

*In particular, if $q$ has only real zeros, then $q(x)D+\alpha q'(x)I$ is a CZDO.*

*Proof.* First note that the result is trivial when the function $q(x)p'(x)+\alpha q'(x)p(x)$ is identically zero. Furthermore, if either $p$ or $q$ is a nonzero constant function, then the result follows from Rolle's theorem as was noted in Remark 3 above. We may, therefore, assume that $p$ and $q$ each have degree at least one. Suppose

$$p(x) = \sum_{k=0}^{n} a_k x^k \quad \text{and} \quad q(x) = \sum_{k=0}^{m} b_k x^k.$$

Then the leading term of

$$f(x) = q(x)p'(x)+\alpha q'(x)p(x)$$

is $(n+\alpha m)a_n b_m x^{n+m-1}$, so $f$ has degree $n+m-1$. Applying Theorem 7, we have

$$\begin{aligned} Z_C(f) &= n+m-1-Z_R(f) \\ &\leq n+m-1-(Z_R(p)+Z_R(q)-1) \\ &= n+m-1-(n-Z_C(p)+m-Z_C(q)-1) \\ &= Z_C(p)+Z_C(q). \end{aligned}$$

Therefore, $Z_C\big(q(x)p'(x)+\alpha q'(x)p(x)\big) \leq Z_C(p(x))+Z_C(q(x))$. □

Note that part of Laguerre's theorem (Theorem 2) is obtained when we set $q(x) = x$ in Theorem 8.

**Remark 9.** The two theorems in this section can be extended to any finite number of constants and functions. For example, using the same techniques as above, one can show that

$$Z_C(pqr'+\alpha p'qr+\beta pq'r) \leq Z_C(p)+Z_C(q)+Z_C(r),$$

where $\alpha$ and $\beta$ are nonnegative real numbers and $p$, $q$, and $r$ are polynomials.

## 3. CZDSs for the Jacobi polynomial basis

**3.1.** ***The Jacobi polynomials.*** We now apply the results of the previous section to demonstrate the existence of CZDSs for the Jacobi polynomial basis. Following [Rainville 1960, p. 257], we define the Jacobi polynomials with parameters $\alpha > -1$ and $\beta > -1$ by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}].$$

For each nonnegative integer $n$, the Jacobi polynomials satisfy the differential equation

$$\big((x^2-1)D^2+[(2+\alpha+\beta)x+\alpha-\beta]D\big)P_n^{(\alpha,\beta)}(x)=n(n+1+\alpha+\beta)P_n^{(\alpha,\beta)}(x) \quad (4)$$

(see [Rainville 1960, p. 258]).

**Proposition 10.** *The sequence* $\{k(k+1+\alpha+\beta)\}_{k=0}^{\infty}$ *is a* $P^{(\alpha,\beta)}$*-CZDS.*

*Proof.* Define the linear operator $L : \mathbb{R}[x] \to \mathbb{R}[x]$ by

$$L(P_k^{(\alpha,\beta)}(x)) = k(k+1+\alpha+\beta)P_k^{(\alpha,\beta)}(x) \quad (k = 0, 1, 2, \dots),$$

so that, by linearity,

$$L\bigg(\sum_{k=0}^{n} a_k P_k^{(\alpha,\beta)}(x)\bigg) = \sum_{k=0}^{n} a_k L(P_k^{(\alpha,\beta)}(x)) = \sum_{k=0}^{n} a_k k(k+1+\alpha+\beta)P_k^{(\alpha,\beta)}(x).$$

Our goal, then, is to show that $L$ is a CZDO. From the differential equation (4), the linear operator $L$ is equal to the differential operator

$$L = \big((x^2-1)D+[(2+\alpha+\beta)x+\alpha-\beta]I\big)D.$$

If, in Remark 9, we take $p(x) = x-1$, $q(x) = x+1$, and replace $\alpha$ and $\beta$ by $\alpha+1$ and $\beta+1$, respectively, then we see that

$$(x^2-1)D+[(2+\alpha+\beta)x+\alpha-\beta]I \quad (\alpha, \beta > -1)$$

is a complex zero decreasing operator. Thus, $L$ is the composition of two CZDOs (recall that D is a CZDO as discussed in Remark 3) and so it is a CZDO itself. □

**3.2.** ***Operator identities.*** In order to extend the preceding result, we will develop a number of operator identities. We consider two operators $L_1$ and $L_2$ on $\mathbb{R}[x]$ to be equal if $L_1(p) = L_2(p)$ for every real polynomial $p$. For example, as a consequence of the product rule for differentiation, $(Dx)p(x) = xp'(x)+p(x)$, and thus we obtain the equality

$$Dx = xD + I. \quad (5)$$

**Proposition 11.** *Suppose that* $\{g_k(x)\}_{k=0}^m$ *is a sequence of polynomials satisfying* $\deg(g_k) \le k$ *for all* $k$. *Then*

$$D^n \sum_{k=0}^m g_k(x) D^k = \left( \sum_{j=0}^m \sum_{k=j}^m \binom{n}{k-j} g_k^{(k-j)}(x) D^j \right) D^n.$$

*Proof.* We first note that we are following the convention that $\binom{n}{k} = 0$ whenever $k > n$.

Using the fact that the derivative operator is linear, applying Leibniz's formula for the $n$-th derivative of a product, and noting our assumption on the degree of the polynomials $g_k$, we have

$$D^n g_k(x) D^k = \sum_{i=0}^k \binom{n}{i} g_k^{(i)}(x) D^{k+n-i} = \left( \sum_{i=0}^k \binom{n}{i} g_k^{(i)}(x) D^{k-i} \right) D^n.$$

Making the substitution $j = k - i$ and then switching the order of summation gives

$$\begin{aligned} D^n \sum_{k=0}^m g_k(x) D^k &= \left( \sum_{k=0}^m \sum_{j=0}^k \binom{n}{k-j} g_k^{(k-j)}(x) D^j \right) D^n \\ &= \left( \sum_{j=0}^m \sum_{k=j}^m \binom{n}{k-j} g_k^{(k-j)}(x) D^j \right) D^n. \qquad \square \end{aligned}$$

In what follows, we will make frequent use of Proposition 11 with $m = 2$, which asserts that if

$$L = D^n \big( g_2(x) D^2 + g_1(x) D + g_0(x) I \big), \tag{6}$$

then

$$L = \left( g_2(x) D^2 + (n g_2'(x) + g_1(x)) D + \left( \binom{n}{2} g_2''(x) + n g_1'(x) + g_0(x) \right) I \right) D^n, \tag{7}$$

provided $\deg(g_k) \le k$ for all $k$.

**3.3.** ***Ultraspherical polynomials.*** We now focus on the Jacobi polynomials for which $\alpha = \lambda = \beta$, which are called the ultraspherical polynomials (see, e.g., [Rainville 1960, p. 143]). To ease notation, we define

$$P_n^{(\lambda)}(x) = P_n^{(\lambda,\lambda)}(x) \quad (\lambda > -1;\, n = 0, 1, 2, \dots).$$

With this choice, the differential equation (4) takes on the form

$$[(x^2 - 1) D^2 + (1 + \lambda) 2x D] P_n^{(\lambda)}(x) = n(n + 1 + 2\lambda) P_n^{(\lambda)}(x). \tag{8}$$

Due to the frequent use of the operator involved in the previous equation we define, for any $a \in \mathbb{R}$,

$$\Phi_a = (x^2 - 1) D + (1 + a) 2x I. \tag{9}$$

**Lemma 12.** *Suppose $\lambda > -1$. Then, for all nonnegative integers $n$,*

$$D^n\big(\Phi_\lambda D - n(n+1+2\lambda)I\big) = (\Phi_{\lambda+n})D^{n+1},$$

*where $\Phi_a$ is defined in (9).*

*Proof.* This is an immediate application of (6) and (7). □

We now use a product notation for composition of operators. Since differential operators need not commute, care is required in using this notation. For a collection of operators $L_1, L_2, \ldots, L_n$ on $\mathbb{R}[x]$, we define

$$\left(\prod_{k=1}^{n} L_k\right)p = (L_1L_2\cdots L_n)p = L_1(L_2(\cdots(L_n(p)))) \quad (p \in \mathbb{R}[x]).$$

**Proposition 13.** *Let $w$ be a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$. Then*

$$\prod_{k=0}^{w-1}\big(\Phi_\lambda D - k(k+1+2\lambda)I\big)^{m_k} = \left(\prod_{k=0}^{w-1}[(\Phi_{\lambda+k}D)^{m_k-1}\Phi_{\lambda+k}]\right)D^w,$$

*where $\Phi_a$ is defined by (9).*

*Proof.* We will argue by mathematical induction. The case $w = 1$ is clear. Now suppose that the result is true for some integer $w \geq 1$ and fix natural numbers $m_0, m_1, \ldots, m_w$. Then

$$\prod_{k=0}^{w}\big(\Phi_\lambda D - k(k+1+2\lambda)I\big)^{m_k} = \Theta D^w\big(\Phi_\lambda D - w(w+1+2\lambda)I\big)^{m_w}, \tag{10}$$

where

$$\Theta = \prod_{k=0}^{w-1}[(\Phi_{\lambda+k}D)^{m_k-1}(\Phi_{\lambda+k})]. \tag{11}$$

Applying Lemma 12 a total of $m_w$ times, we see that

$$D^w\big(\Phi_\lambda D - w(w+1+2\lambda)I\big)^{m_w} = (\Phi_{\lambda+w}D)^{m_w}D^w. \tag{12}$$

Together, (10), (11), and (12) show that

$$\prod_{k=0}^{w}\big(\Phi_\lambda D - k(k+1+2\lambda)I\big)^{m_k} = \left(\prod_{k=0}^{w}\big((\Phi_{\lambda+k}D)^{m_k-1}(\Phi_{\lambda+k})\big)\right)D^{w+1}. \qquad □$$

We are now in a position to demonstrate the existence of several $P^{(\lambda)}$-CZDSs for any fixed $\lambda > -1$.

**Theorem 14.** *If $\lambda > -1$, $w$ is a positive integer, and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then the sequence*

$$\left\{\prod_{k=0}^{w-1}\bigl(n(n+1+2\lambda)-k(k+1+2\lambda)\bigr)^{m_k}\right\}_{n=0}^{\infty} \tag{13}$$

*is a $P^{(\lambda)}$-CZDS, where $P^{(\lambda)}$ is the set of ultraspherical polynomials.*

*Proof.* Let the linear operator $L : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by

$$L(P_n^{(\lambda)}(x)) = \left(\prod_{k=0}^{w-1}\bigl(n(n+1+2\lambda)-k(k+1+2\lambda)\bigr)^{m_k}\right)P_n^{(\lambda)}(x).$$

From the differential equation (8), we have

$$L = \prod_{k=0}^{w-1}\bigl((x^2-1)D^2+(1+\lambda)2xD-k(k+1+2\lambda)I\bigr)^{m_k},$$

or, using the notation in (9) and applying Proposition 13,

$$L = \prod_{k=0}^{w-1}\bigl(\Phi_\lambda D-k(k+1+2\lambda)I\bigr)^{m_k} = \left(\prod_{k=0}^{w-1}\bigl((\Phi_{\lambda+k}D)^{m_k-1}\Phi_{\lambda+k}\bigr)\right)D^w.$$

The operator $L$ is, therefore, a composition of individual operators, each of which is a CZDO. This can be seen by appealing to Theorem 8, which shows that $\Phi_a$ is a CZDO whenever $a > -1$. □

**3.4. *CZDSs for Legendre basis.*** The polynomials

$$P_n(x) = P_n^{(0)}(x) = P_n^{(0,0)}(x) \quad (n = 0, 1, 2, \dots)$$

are known as the Legendre polynomials (see [Rainville 1960, p. 254]).

In [Blakeman et al. 2012], Open Question (4) conjectures that a certain type of falling factorial sequence is a multiplier sequence for the Legendre basis, or a $P$-MS. Since every $P$-CZDS is a $P$-MS, we can apply the results of the previous section to settle a variation of this question.

**Corollary 15.** *If $w$ is a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then the sequence*

$$\left\{\prod_{k=0}^{w-1}\bigl(n(n+1)-k(k+1)\bigr)^{m_k}\right\}_{n=0}^{\infty} = \left\{\prod_{k=0}^{w-1}\bigl((n+k+1)(n-k)\bigr)^{m_k}\right\}_{n=0}^{\infty} \tag{14}$$

*is a CZDS for the Legendre basis.*

*Proof.* Apply Theorem 14 with $\lambda = 0$. □

Corollary 15 strengthens and extends some of the results obtained in [Blakeman et al. 2012] by showing that $\{k^2+k\}_{k=0}^{\infty}$ is a $P$-CZDS and by demonstrating the existence of $P$-CZDSs (and hence $P$-multiplier sequences) which are not products of quadratic $P$-multiplier sequences.

**3.5.** ***CZDS for the Chebyshev basis.*** The Chebyshev polynomials $\mathcal{T} = \{T_n(x)\}$ and $\mathcal{U} = \{U_n(x)\}$ of the first and second kind, respectively, can be defined by

$$T_n(x) := \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{(-1/2)}(x) \qquad (n = 0, 1, 2, \dots),$$

$$U_n(x) := \frac{(n+1)!}{\left(\frac{3}{2}\right)_n} P_n^{(1/2)}(x) \quad (n = 0, 1, 2, \dots),$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ is the rising factorial (see [Rainville 1960, p. 301]). In [Piotrowski 2007, Lemma 156] it is shown that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS for a simple set $Q = \{q_k(x)\}_{k=0}^{\infty}$ if and only if it is a $\widehat{Q}$-CZDS, where $\widehat{Q}$ consists of the polynomials

$$\hat{q}_n(x) = c_n q_n(\alpha x + \beta) \quad (\beta \in \mathbb{R};\, \alpha, c_n \in \mathbb{R} \setminus \{0\}).$$

Combining this with Theorem 14, we arrive at the following corollary.

**Corollary 16.** *If $w$ is a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then*

(1) *the sequence* $\left\{\prod_{k=0}^{w-1}(n^2-k^2)^{m_k}\right\}_{n=0}^{\infty}$ *is a $\mathcal{T}$-CZDS, and*

(2) *the sequence* $\left\{\prod_{k=0}^{w-1}\bigl(n(n+2)-k(k+2)\bigr)^{m_k}\right\}_{n=0}^{\infty}$ *is a $\mathcal{U}$-CZDS.*

*Proof.* Apply Theorem 14 with $\lambda = -1/2$ and again with $\lambda = 1/2$. □

## 4. Simultaneous generation of a basis $B$ and a class of $B$-CZDSs

Given a basis $B$ and a sequence $\{\gamma_k\}_{k=0}^{\infty}$, a typical strategy in showing that $\{\gamma_k\}_{k=0}^{\infty}$ is a $B$-CZDS is to find a differential operator representation for the diagonal operator which is a CZDO. In this section, we begin with a known CZDO and use it to demonstrate the existence of a basis $B$ and a corresponding $B$-CZDS. Our results focus on bases which are *simple sets*, i.e., those for which $\deg(b_k) = k$ for all $k$. In the product notation that follows, we adopt the convention that

$$\prod_{k=0}^{n} a_k = 1$$

whenever $n < 0$.

**Theorem 17.** *Let $\alpha \geq 0$ and let*

$$q(x) = c_0 + c_1 x + \cdots + c_r x^r \quad (r \geq 1,\, c_r \neq 0)$$

*be a real polynomial with only real zeros. Then there is a simple set of polynomials* $B = \{b_n(x)\}_{n=0}^{\infty}$ *which satisfy the differential equation*

$$q(x)b_n^{(r)}(x) + \alpha q'(x)b_n^{(r-1)}(x) = \gamma_n b_n(x) \quad (n = 0, 1, 2, \dots), \tag{15}$$

*where*

$$\gamma_n = c_r\bigl(n + (\alpha - 1)r + 1\bigr)\prod_{k=0}^{r-2}(n-k) \quad (n = 0, 1, 2, \dots).$$

*Consequently, the sequence* $\{\gamma_n\}_{n=0}^{\infty}$ *is a* $B$*-CZDS.*

**Remark 18.** We note that, for the case where $r = 1$ and $\alpha \neq 0$, the explicit form of the sequence and the existence of the basis $B$ follow from results contained in the beginning of Section 2 of [Azad et al. 2011] and the beginning of Section II of [Krall and Sheffer 1964]. The proof of the general case is similar, yet different enough to warrant its inclusion here.

*Proof of Theorem 17.* Consider the differential operator

$$L = q(x)D^r + \alpha q'(x)D^{r-1}.$$

With this notation, the differential equation (15) becomes $L(b_n(x)) = \gamma_n b_n(x)$ and our goal is to find the eigenvalues $\gamma_n$ of $L$ and show there is a simple set of polynomials consisting of eigenfunctions $b_n$ of $L$. The matrix representation of $L$ with respect to the standard basis is upper triangular, with eigenvalues on the main diagonal given by the coefficient of $x^n$ in $L(x^n)$. Since

$$L(x^n) = \left(c_r\prod_{k=0}^{r-1}(n-k) + \alpha r c_r\prod_{k=0}^{r-2}(n-k)\right)x^n + h(x),$$

where $h$ is a polynomial of degree less than or equal to $n-1$, the eigenvalue sequence is given by $\gamma_n = p(n)$ for all $n$, where

$$p(x) = c_r\bigl(x + (\alpha - 1)r + 1\bigr)\prod_{k=0}^{r-2}(x-k).$$

Since $p$ has only real zeros, each of which lies in the interval $(-\infty, r-1]$, we either have

$$0 = \gamma_0 = \gamma_1 = \cdots = \gamma_{m-1} < \gamma_m < \gamma_{m+1} < \cdots$$

or

$$0 = \gamma_0 = \gamma_1 = \cdots = \gamma_{m-1} > \gamma_m > \gamma_{m+1} > \cdots,$$

where

$$m = \begin{cases} r-1 & \text{if } \alpha \neq 0, \\ r & \text{if } \alpha = 0. \end{cases}$$

In either case, all the nonzero eigenvalues must be distinct. Furthermore,

$$L(x^n) \equiv 0 \quad (n = 0, 1, \ldots, m-1),$$

so $L$ has the form

$$L = \left[\begin{array}{c|c} 0_{m\times m} & A \\ \hline 0_{\infty\times m} & T \end{array}\right],$$

where $T$ is an upper triangular matrix with distinct nonzero eigenvalues on the main diagonal.

We now show that there is a simple set $B$ consisting of eigenfunctions of the operator $L$. Indeed, let $L_n$ denote the $n \times n$ truncation of the matrix $L$. Since $L_m = 0_{m\times m}$, we have complete freedom in choosing our first $m$ eigenfunctions, say

$$b_n(x) = x^n \quad (n = 0, 1, 2, \ldots m-1).$$

For $L_{m+1}$, there is an eigenfunction corresponding to the (nonzero) eigenvalue $\gamma_m$. This eigenfunction is linearly independent from those corresponding to the eigenvalue 0, thus it must be of degree $m$. Continuing in this fashion, we can construct a simple set $B$ consisting of eigenfunctions of $L$ as desired.

To show that $\{\gamma_n\}_{n=0}^{\infty}$ is a $B$-CZDS, suppose

$$g(x) = \sum_{k=0}^{j} d_k b_k(x) \quad (d_j \neq 0)$$

is a real polynomial. Then

$$Z_C(g(x)) \geq Z_C(g^{(r-1)}(x)) \geq Z_C\big(q(x)g^{(r)}(x) + \alpha q'(x) g^{(r-1)}(x)\big),$$

where we have made use of Remark 3 and Theorem 8. Since

$$\begin{aligned} q(x)g^{(r)}(x) + \alpha q'(x)g^{(r-1)}(x) &= \sum_{k=0}^{j} d_k\big(q(x)b_k^{(r)}(x) + \alpha q'(x) b_k^{(r-1)}(x)\big) \\ &= \sum_{k=0}^{j} \gamma_k d_k b_k(x), \end{aligned}$$

the desired result is obtained. □

As an example, if we choose $q(x) = (x+1)^3$ and $\alpha = 1$, then the corresponding sequence would be $\gamma_n = (n+1)n(n-1)$, and we would need to find a simple set $B = \{b_n(x)\}_{n=0}^{\infty}$ which solves the differential equation

$$(n+1)n(n-1)b_n(x) = (x+1)^3 b_n'''(x) + 3(x+1)^2 b_n''(x) \quad (n = 0, 1, 2, \ldots). \tag{16}$$

With some effort, one finds that sets $B$ which solve (16) have the form

$$\begin{aligned} b_0(x) &= r, \\ b_1(x) &= sx+t, \\ b_n(x) &= c_n(x+1)^n \quad (n=2,3,4,\dots), \end{aligned}$$

where $t \in \mathbb{R}$ and $r, s, c_2, c_3, \dots$ are any (fixed) nonzero real numbers. Thus, the sequence

$$\{(n+1)n(n-1)\}_{n=0}^{\infty}$$

is a $B$-CZDS for any such basis $B$.

## 5. An extension to certain transcendental entire functions

**5.1.** ***The Laguerre–Pólya class.*** A real entire function $\varphi$ is said to belong to the *Laguerre–Pólya class*, denoted $\varphi \in \mathcal{L}\text{–}\mathcal{P}$, if it can be written in the form

$$\varphi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega}\Big(1+\frac{x}{x_k}\Big)e^{-x/x_k}, \tag{17}$$

where $b, c, x_k \in \mathbb{R}$, $m$ is a nonnegative integer, $a \geq 0$, $0 \leq \omega \leq \infty$, and $\sum_{k=1}^{\omega} x_k^{-2} < \infty$.

An alternate characterization of this class is as follows: $\varphi \in \mathcal{L}\text{–}\mathcal{P}$ if and only if $\varphi$ is the uniform limit on compact subsets of $\mathbb{C}$ of real polynomials having only real zeros (see, for example, [Levin 1964, Chapter VIII ] or [Obreschkoff 1963, Satz 9.2]). This point of view, together with Hurwitz's theorem (see [Marden 1949, p. 4]), allows us to obtain some useful extensions of results in Section 2.

**Theorem 19.** *Suppose $\varphi$ belongs to the class $\mathcal{L}\text{–}\mathcal{P}$, $p$ and $q$ are real polynomials, and $\alpha \geq 0$. Then*

$$Z_C\big(\varphi q p' + \alpha(\varphi q)' p\big) \leq Z_C(p) + Z_C(q).$$

*Proof.* Suppose $\{f_k\}_{k=0}^{\infty}$ is a sequence of real polynomials with only real zeros which converge uniformly on compact subsets of $\mathbb{C}$ to $\varphi$. By Theorem 8,

$$Z_C\big(f_k q p' + \alpha(f_k q)' p\big) \leq Z_C(p) + Z_C(q) \quad (k=0,1,2,\dots).$$

Taking into account that $f_k q p' + \alpha(f_k q)' p$ converges uniformly on compact subsets of $\mathbb{C}$ to $\alpha(\varphi q)' p + \varphi q p'$, Hurwitz's theorem gives the desired result. □

In order to prove an extension of Laguerre's theorem related to $H$-CZDSs (Theorem 4), Piotrowski first proved a special case as a lemma. We now show how to obtain a new proof of this lemma using Theorem 19.

**Corollary 20** [Piotrowski 2007, p. 55, Lemma 67]. *Suppose that $p(x)$ is a real polynomial of degree $n$. If $c, d, \beta$ are real numbers such that $c \geq 0$ and $\beta \geq 0$, then*

$$Z_C\big((cx+d)p(x) - \beta p'(x)\big) \leq Z_C(p(x)).$$

*Proof.* If $\beta = 0$, the result clearly holds. If $\beta > 0$, we may appeal to Theorem 19 with $\alpha = \beta^{-1}$, $q(x) = 1$, and

$$\varphi(x) = -\exp\Big(-\frac{c}{2}x^2 - dx\Big) \quad (c \geq 0, d \in \mathbb{R})$$

to obtain the desired result. □

**5.2.** ***CZDSs for the generalized Laguerre polynomial basis.*** In this section, we combine the results of the previous section with the methods of Section 3.3 to obtain a class of CZDSs for the generalized Laguerre polynomial basis, defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \quad (\alpha > -1; n = 0, 1, 2, \ldots).$$

The generalized Laguerre polynomials satisfy the differential equation

$$-x\frac{d^2}{dx^2}L_n^{(\alpha)}(x) + (x - (\alpha+1))\frac{d}{dx}L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) \tag{18}$$

(see, e.g., [Rainville 1960, p. 204]). Just as with the Jacobi basis, we will develop a number of operator identities in order to arrive at a collection of $L^{(\alpha)}$-CZDSs. We begin by defining, for any $a \in \mathbb{R}$,

$$\Psi_a = -xD + (x - (a+1))I. \tag{19}$$

**Lemma 21.** *Suppose $\alpha \in \mathbb{R}$. Then, for all nonnegative integers $n$,*

$$D^n(\Psi_\alpha D - nI) = \Psi_{\alpha+n}D^{n+1}.$$

*Proof.* This is an immediate application of (6) and (7). □

**Proposition 22.** *Let $w$ be a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$. Then*

$$\prod_{k=0}^{w-1}(\Psi_\alpha D - kI)^{m_k} = \Bigg(\prod_{k=0}^{w-1}[(\Psi_{\alpha+k}D)^{m_k-1}\Psi_{\alpha+k}]\Bigg)D^w,$$

*where $\Psi_a$ is defined in (19).*

*Proof.* We will argue by mathematical induction. The case $w = 1$ is clear. Now suppose that the result is true for some integer $w \geq 1$ and fix natural numbers $m_0, m_1, \ldots, m_w$. Then

$$\prod_{k=0}^{w}(\Psi_\alpha D - kI)^{m_k} = \prod_{k=0}^{w-1}[(\Psi_{\alpha+k}D)^{m_k-1}\Psi_{\alpha+k}]D^w(\Psi_\alpha D - wI)^{m_w}. \tag{20}$$

Applying Lemma 21 a total of $m_w$ times, we see that

$$D^w(\Psi_\alpha D - wI)^{m_w} = (\Psi_{\alpha+w} D)^{m_w} D^w = (\Psi_{\alpha+w} D)^{m_w-1}\Psi_{\alpha+w} D^{w+1}. \tag{21}$$

Together, (20) and (21) give the desired result. $\square$

In order to use the operator identities above to find a collection of $L^{(\alpha)}$-CZDSs for any $\alpha > -1$, we will use the result of Section 5.1.

**Lemma 23.** *For any $a > -1$, the operator*

$$\Psi_a = -xD + (x - (a+1))I$$

*is a CZDO.*

*Proof.* Suppose $a > -1$ and set $c = a+1$. By Theorem 19, for any real polynomial $p$,

$$Z_C\Big(c\frac{d}{dx}(-x\exp(-x/c))\, p(x) + (-x\exp(-x/c))p(x)\Big) \le Z_C(p(x)).$$

The smaller quantity above simplifies to

$$Z_C\big((-xp'(x) + (x-c)p(x))\exp(-x/c)\big).$$

Since the exponential function never vanishes, we have shown that

$$Z_C(\Psi_a p(x)) = Z_C\big(-xp'(x) + (x-c)p(x)\big) \le Z_C(p(x)). \qquad \square$$

We now arrive at the main theorem of this section.

**Theorem 24.** *Fix $\alpha > -1$. If $w$ is a positive integer and $\{m_k\}_{k=0}^{w-1} \subset \mathbb{N}$, then the sequence*

$$\left\{\prod_{k=0}^{w-1}(n-k)^{m_k}\right\}_{n=0}^{\infty} \tag{22}$$

*is an $L^{(\alpha)}$-CZDS.*

*Proof.* Let the linear operator $\Theta : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by

$$\Theta(L_n^{(\alpha)}(x)) = \left(\prod_{k=0}^{w-1}(n-k)^{m_k}\right) L_n^{(\alpha)}(x).$$

Combining the differential equation (18), the notation in (19), and Proposition 22, we have

$$\Theta = \prod_{k=0}^{w-1}(\Psi_\alpha D - kI)^{m_k} = \left(\prod_{k=0}^{w-1}[(\Psi_{\alpha+k} D)^{m_k-1}\Psi_{\alpha+b}]\right) D^w.$$

The operator $\Theta$ is, therefore, a composition of individual operators, each of which is a CZDO. This can be seen by appealing to Lemma 23. $\square$

Theorem 24 is a significant generalization and extension of a theorem due to Forgács and Piotrowski [2013, Theorem 4.4] and a stronger result on a narrower class of sequences than those characterized by Brändén and Ottergren [2014].

## 6. Open questions

Any sequence of the form

$$\{k(k-1)\cdots(k-(m-1))\}_{k=0}^{\infty}$$

(the "falling-factorial sequence") is a CZDS for the standard basis. By Corollary 16, any sequence of the form

$$\{k^2(k^2-1)\cdots(k^2-(m-1)^2)\}_{k=0}^{\infty}$$

is a $T$-CZDS. The similarity of these results leads us to wonder if an analog of Theorem 6 could be obtained for the Chebyshev basis.

**Problem 25.** Find a complete characterization of polynomials $h$ for which $\{h(k)\}_{k=0}^{\infty}$ is a $T$-CZDS, where $T$ denotes the Chebyshev basis.

We note that the characterization will be different from that of the standard basis since the sequence $\{k\}_{k=0}^{\infty}$ is not a $T$-CZDS.

The results on ultraspherical and Laguerre CZDSs also have a falling factorial nature which leads us to consider the more general problem.

**Problem 26.** For any basis $B$, find a complete characterization of polynomials $h$ for which $\{h(k)\}_{k=0}^{\infty}$ is a $B$-CZDS.

Recall that this problem has been solved when the basis is taken to be either the standard basis or the Hermite basis. The result [Piotrowski 2007, Lemma 157] solves the problem for any affine transformation of the standard basis or the Hermite basis. To date, Problem 26 remains unsolved for any other choice of the basis $B$.

As it was mentioned earlier, no complete characterization of CZDSs for the standard basis is known. In particular, it is not known whether or not every rapidly decreasing sequence (such as $\{\exp(-k^3)\}_{k=0}^{\infty}$) is a CZDS for the standard basis (see [Craven and Csordas 2004, Problem 4.8] for more details). A theorem of Piotrowski gives a connection between these and CZDSs for other bases.

**Theorem 27** [Piotrowski 2007, Theorem 159]. *Let $B = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of polynomials. If the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a $B$-CZDS, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS for the standard basis.*

This prompts us to state a weaker version of Problem 4.8(a) of [Craven and Csordas 2004], which may be easier to settle.

**Problem 28.** Is there a simple set $B$ for which $\{\exp(-k^3)\}_{k=0}^{\infty}$ is a $B$-CZDS?

We mention that our methods of simultaneously generating a basis and a CZDS may apply. However, the original operator will have to be modified as all of our methods generated sequences which can be interpolated by polynomials.

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