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Counting set classes with Burnside's lemma

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Mathematical tools from combinatorics and abstract algebra have been used to study a variety of musical structures. One question asked by mathematicians and musicians is: how many d-note set classes exist in a c-note chromatic universe? In the music theory literature, this question is answered with the use of Pólya's enumeration theorem. We solve the problem using simpler techniques, including only Burnside's lemma and basic results from combinatorics and abstract algebra. We use interval arrays that are associated with pitch class sets as a tool for counting.

1. Introduction

For the past three decades, mathematical tools from combinatorics and abstract algebra have been used to study a variety of musical structures. The elements of a c-note chromatic universe are typically labeled $0, 1, 2, \dots, c-1$ and are considered elements of Z_c, the group of integers modulo c. In the traditional 12-note chromatic universe, C is labeled 0. Following the language of [Clough and Myerson 1985], a *d-note pitch class set* in a c-note chromatic universe is a subset of $\{0, 1, \dots, c-1\}$ of size d. As explained in [Reiner 1985; Hook 2007], two pitch class sets are considered equivalent if one can be obtained from the other either by rotation or reflection. A *d-note set class* contains all equivalent d-note pitch class sets. One question asked by musicians and music theorists is: how many d-note set classes exist in a c-note chromatic universe? Figure 1 shows a way to visualize the case where $c = 12$ and $d = 7$.

Let n be a positive integer. The *Euler φ-function*, $\varphi(n)$, is the number of positive integers that are less than or equal to n that are also relatively prime to n.

Theorem 1.1 [Reiner 1985; Hook 2007]. *The number of d-note set classes in a c-note chromatic universe is*

$$\frac{1}{2c}T(c, d) + \frac{1}{2}I(c, d), \tag{1-1}$$

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where

$$T(c,d) = \sum_{j \mid \gcd(c,d)} \varphi(j) \binom{c/j}{d/j}$$

and

$$I(c,d) = \begin{cases} \binom{c/2-1}{\lfloor d/2 \rfloor} & \text{if } c \text{ is even and } d \text{ is odd}, \\ \binom{\lfloor c/2 \rfloor}{\lfloor d/2 \rfloor} & \text{otherwise}. \end{cases}$$

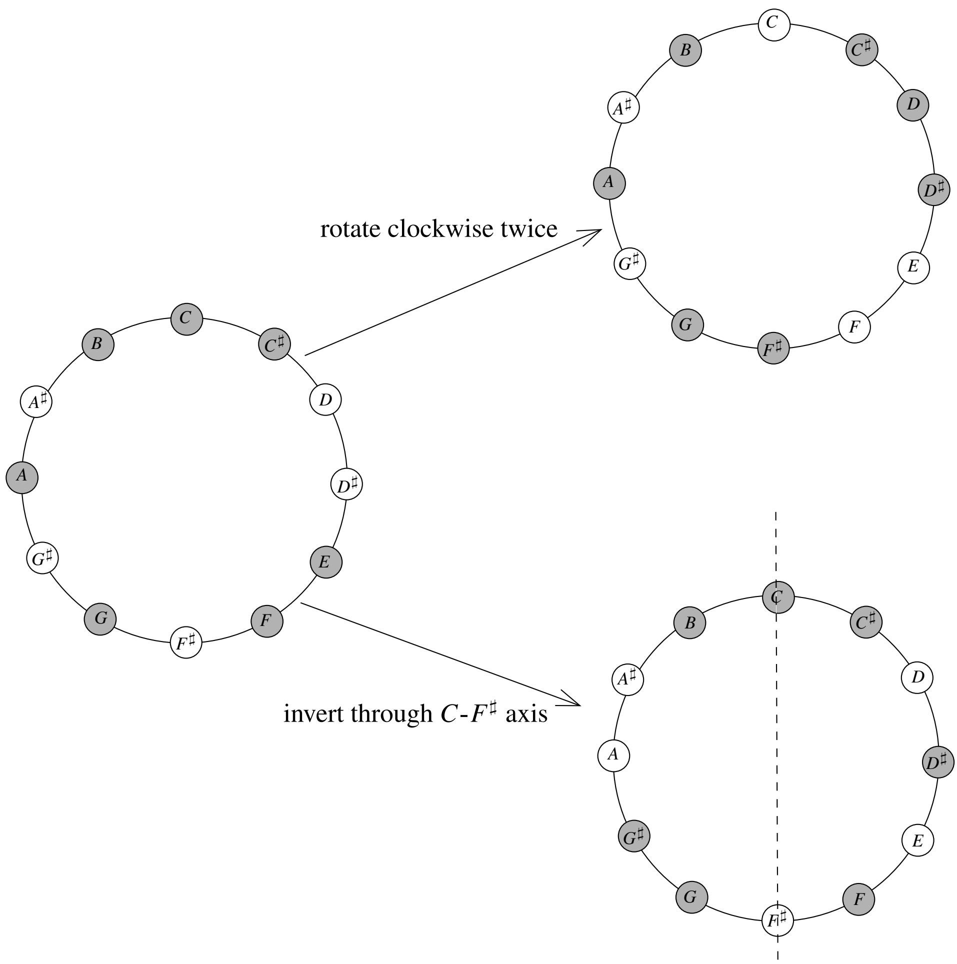

Figure 1. Visualizing a 7-note pitch class set in a 12-note chromatic universe. The three pitch class sets {C, C$^\sharp$, E, F, G, A, B}, {C$^\sharp$, D, D$^\sharp$, F$^\sharp$, G, A, B}, and {C, C$^\sharp$, D$^\sharp$, F, G, G$^\sharp$, B} are equivalent and are therefore all part of the same set class.

In the music theory literature, Theorem 1.1 is proved using an advanced combinatorial theorem, namely Pólya's enumeration theorem (the final theorem stated in [Brualdi 2010]). Our contribution is that we make Theorem 1.1 more accessible by using only tools that would be seen in introductory classes in combinatorics and abstract algebra. The most advanced concept is Burnside's lemma, which appears in [Reiner 1985; Hook 2007] as a general tool for counting the number of equivalence classes generated by a group action, but is abandoned in the proof of Theorem 1.1 in favor of Pólya's result. In [Graham et al. 2008], the application of Burnside's lemma to our problem is discussed, but only specific examples, and not a general result, are reported. An additional contribution is that we use the structure of *interval arrays* (see Section 2), which were introduced in [Clough and Myerson 1985] and developed in [Fripertinger 1992], but have not been connected to this theorem.

2. Equivalent pitch class sets

The *dihedral group of order* $2n$, D_{2n}, is the set of symmetries of a regular n-gon. There are n rotations and n reflections. Musically, rotations are known as transpositions and reflections are known as inversions.

Mathematically speaking, the number of d-note set classes in a c-note chromatic universe is the number of equivalence classes when D_{2c} acts on the set of d-note pitch class sets. In Figure 1, all 7-note pitch class sets that are equivalent to {C, C$^\sharp$, E, F, G, A, B} can be found by inverting and transposing the left-most figure in all 24 possible ways. Consult [Hook 2007] for more details about group actions in this context.

Let $\{i_1, i_2, \ldots, i_d\}$ be a d-note pitch class set. Without loss of generality, let $i_1 < i_2 < \cdots < i_d$. The *interval array* associated with this d-note pitch class set is

$$\langle i_2 - i_1, i_3 - i_2, \ldots, i_d - i_{d-1}, i_1 - i_d \rangle,$$

where all subtraction is done modulo d [Fripertinger 1992, Definition 2.5]. Note that $\langle j_1, j_2, \ldots, j_d \rangle$ is the interval array of a d-note pitch class set in a c-note chromatic universe if and only if $j_1 + j_2 + \cdots + j_d = c$ [Fripertinger 1992, Remark 2.4]. See Table 1.

Instead of counting the number of equivalence classes when D_{2c} acts on the set of d-note pitch class sets, we will count the number of equivalence classes when

7-note pitch class set	pitch class set in Z_c	interval array
{C, C$^\sharp$, E, F, G, A, B}	$\{0, 1, 4, 5, 7, 9, 11\}$	$\langle 1, 3, 1, 2, 2, 2, 1 \rangle$
{C$^\sharp$, D, D$^\sharp$, F$^\sharp$, G, A, B}	$\{1, 2, 3, 6, 7, 9, 11\}$	$\langle 1, 1, 3, 1, 2, 2, 2 \rangle$
{C, C$^\sharp$, D$^\sharp$, F, G, G$^\sharp$, B}	$\{0, 1, 3, 5, 7, 8, 11\}$	$\langle 1, 2, 2, 2, 1, 3, 1 \rangle$

Table 1. The interval arrays for the pitch class sets in Figure 1.

D_{2d} acts on $\{\langle j_1, j_2, \ldots, j_d\rangle \mid j_1 + j_2 + \cdots + j_d = c\}$, the set of interval arrays. In Theorem 2.3 of the same work, Fripertinger proves that the number of equivalence classes is the same in both situations.

3. Algebraic and combinatorial tools

Below are the theorems from introductory combinatorics [Brualdi 2010] and abstract algebra [Dummit and Foote 2004] that we will apply.

Theorem 3.1. *Let n and k be positive integers. Then*

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Theorem 3.2. *The equation $x_1 + x_2 + \cdots + x_k = n$ has $\binom{n-1}{k-1}$ positive-integral solutions.*

Theorem 3.3 (hockey stick theorem). *If m and n are nonnegative integers, then*

$$\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}.$$

Theorem 3.4. *Let j, k, and n be integers such that $0 \le j \le k \le n$. Then*

$$\sum_{m=j}^{n-k+j}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}.$$

Theorem 3.5. *In a group, assume that element a has order d. Then*

$$\langle a^j\rangle = \langle a^{\gcd(d,j)}\rangle \quad \textit{and} \quad |\langle a^j\rangle| = \frac{d}{\gcd(d,j)}.$$

Theorem 3.6. *If m is a positive divisor of d, then the number of elements of order m in a cyclic group of order d is $\varphi(m)$.*

Theorem 3.7 (Burnside's lemma). *Let G be a group acting on a set S. The number of equivalence classes is*

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{Fix}(g),$$

where $\mathrm{Fix}(g)$ *is the number of elements of S that are fixed by g.*

4. The main theorem proved with Burnside's lemma

Theorem 4.1. *The number of d-note set classes in a c-note chromatic universe is*

$$\frac{1}{2d}T_B(c,d) + \frac{1}{2}I(c,d), \tag{4-1}$$

where

$$T_B(c,d) = \sum_{m|d \text{ and } d|cm} \varphi(d/m)\binom{cm/d-1}{m-1},$$

and $I(c,d)$ is defined as in Theorem 1.1.

Proof. Instead of visualizing a regular c-gon and counting the number of equivalence classes when D_{2c} acts on the set of d-note pitch class sets, as is typically done, we visualize a regular d-gon and count the number of equivalence classes when D_{2d} acts on the set of interval arrays $\{\langle j_1, j_2, \ldots, j_d\rangle \mid j_1 + j_2 + \cdots + j_d = c\}$. According to Burnside's lemma, we must count the number of interval arrays that are fixed by elements of D_{2d}.

First, we consider the d inversions. Assume that c and d are both odd. We have a regular d-gon whose vertices are labeled $j_1, j_2, \ldots, j_d$. Every possible axis of inversion passes through a single vertex. Let A be the value of that vertex, and let $B = (c - A)/2$. See Figure 2. Once the value of A is chosen, Theorem 3.2 says there are

$$\binom{\frac{c-A}{2} - 1}{\frac{d-1}{2} - 1}$$

ways to assign values to the vertices that add up to B. Also note that A must be odd, and it ranges from 1 to $c - (d - 1)$. Thus the number of interval arrays fixed by this inversion is

$$\sum_{\substack{A=1\\ A \text{ odd}}}^{c-(d-1)} \binom{\frac{c-A}{2} - 1}{\frac{d-1}{2} - 1},$$

which equals $\binom{(c-1)/2}{(d-1)/2}$ by the hockey stick theorem. Since there are d inversions, the sum of the number of interval arrays fixed by an inversion is $d\binom{\lfloor c/2\rfloor}{\lfloor d/2\rfloor}$.

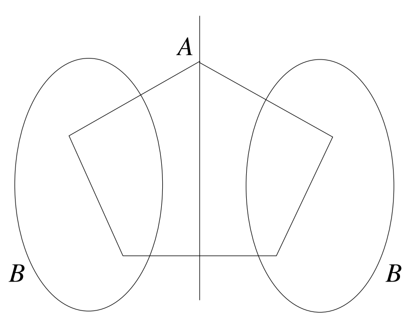

Figure 2. The inversion when d is odd.

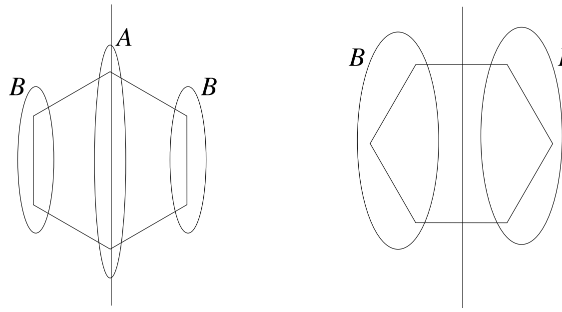

Figure 3. Two inversions when d is even.

When c is even and d is odd, repeat the previous argument, except that A must be even and it ranges from 2 to $c-(d-1)$. The hockey stick theorem yields

$$\binom{\frac{c-2}{2}}{\frac{d-1}{2}},$$

and the sum of the number of interval arrays fixed by an inversion is $d\binom{c/2-1}{\lfloor d/2 \rfloor}$.

Now assume that c and d are both even. When d is even, there are two types of inversions: $d/2$ of each type in Figure 3. For an inversion through opposite edges, Theorem 3.2 says there are $\binom{c/2-1}{d/2-1}$ ways to assign values to the $d/2$ vertices that add up to $B = c/2$. For an inversion through a pair of vertices, A is chosen and then $B = (c-A)/2$. Note that A must be even and ranges from 2 to $c-(d-2)$. The number of interval arrays fixed by this inversion is

$$\sum_{\substack{A=2\\ A \text{ even}}}^{c-(d-2)} \binom{A-1}{1}\binom{\frac{c-A}{2}-1}{\frac{d-2}{2}-1} = \sum_{\substack{A=2\\ A \text{ even}}}^{c-(d-2)} \binom{A}{1}\binom{\frac{c-A}{2}-1}{\frac{d-2}{2}-1} - \sum_{\substack{A=2\\ A \text{ even}}}^{c-(d-2)} \binom{\frac{c-A}{2}-1}{\frac{d-2}{2}-1}$$
$$= 2\binom{\frac{c}{2}}{\frac{d}{2}} - \binom{\frac{c}{2}-1}{\frac{d}{2}-1},$$

where the first term simplifies by Theorem 3.4 and the second term simplifies by Theorem 3.3. The sum of the number of interval arrays fixed by the d inversions is

$$\frac{d}{2}\binom{\frac{c}{2}-1}{\frac{d}{2}-1} + \frac{d}{2}\left(2\binom{\frac{c}{2}}{\frac{d}{2}} - \binom{\frac{c}{2}-1}{\frac{d}{2}-1}\right) = d\binom{\frac{c}{2}}{\frac{d}{2}}.$$

The argument when c is odd and d is even is identical.

Second, we consider the d transpositions $R^1, R^2, \ldots, R^d$, where R^1 is a single transposition clockwise which generates the cyclic group of order d. Let m be a divisor of d. According to Theorem 3.5, each R^j with $\gcd(d, j) = m$ generates the same subgroup, and this subgroup has order d/m. If an interval array can be fixed

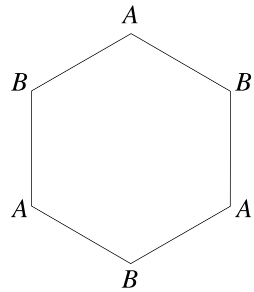

Figure 4. If $d = 6$, rotating the hexagon 120° is acting on the interval arrays with R^2, an element of order 3. If an interval array is fixed, then the values A and B must each be repeated twice.

by a transposition of order d/m, it is necessary that $(d/m) \mid c$ or, equivalently, that $d \mid cm$. Thus, if $m \mid d$ and $d \mid cm$, the number of interval arrays fixed by an element of order d/m is the number of ordered partitions of

$$\frac{c}{d/m} = \frac{cm}{d}$$

into m parts. According to Theorem 3.2, this can be done $\binom{cm/d-1}{m-1}$ ways. Moreover, Theorem 3.6 says that $\varphi(d/m)$ transpositions have order d/m. Thus the sum of all $\mathrm{Fix}(R^j)$ is

$$\sum_{m|d \text{ and } d|cm} \varphi(d/m)\binom{cm/d-1}{m-1}.$$

See Figure 4 for an example. Applying Burnside's lemma completes the proof. □

Theorem 4.2. *Expressions* (1-1) *and* (4-1) *are equal.*

Proof. Since these expressions both count the number of d-note set classes in a c-note chromatic universe, they are equal. However, we provide a different proof, outside the context of music theory.

We must show that

$$\frac{1}{c}\sum_{j|\gcd(c,d)} \varphi(j)\binom{c/j}{d/j} = \frac{1}{d}\sum_{m|d \text{ and } d|cm} \varphi(d/m)\binom{cm/d-1}{m-1}. \tag{4-2}$$

We start with the right-hand side and reindex, letting $j = d/m$. Then

$$\frac{1}{d}\sum_{m|d \text{ and } d|cm} \varphi(d/m)\binom{cm/d-1}{m-1} = \frac{1}{d}\sum_{d/j|d \text{ and } d|\frac{cd}{j}} \varphi(j)\binom{c/j-1}{d/j-1}$$

$$= \frac{1}{d}\sum_{j|\gcd(c,d)} \varphi(j)\binom{c/j-1}{d/j-1}.$$

The last equality is valid because

$$\{j : j \mid \gcd(c, d)\} = \{j : (d/j) \mid d \text{ and } d \mid (cd/j)\}.$$

The equality of (4-2) follows from termwise equality, as a result of Theorem 3.1. □

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