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Border rank of ternary trilinear forms and the j-invariant

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We first describe how one associates a cubic curve to a given ternary trilinear form $\phi \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. We explore relations between the rank and border rank of the tensor ϕ and the geometry of the corresponding cubic curve. When the curve is smooth, we show there is no relation. When the curve is singular, normal forms are available, and we review the explicit correspondence between the normal forms, rank and border rank.

1. Introduction

Given a multilinear map, i.e., a tensor[1], how hard is it to evaluate? Two ways mathematicians have chosen to quantify "hard" are the notions of rank and border rank. We say a tensor $\phi \in V_1 \otimes \cdots \otimes V_n$ is of *rank 1* if it is of the form $v_1 \otimes \cdots \otimes v_n$, where each $v_i \in V_i$.

Definition 1.1. Let $\phi \in V_1 \otimes \cdots \otimes V_n$. The rank of ϕ, denoted $\mathbf{R}(\phi)$ is the smallest natural number r such that $\phi = \sum_{j=1}^{r} \phi_j$, where each $\phi_j \in V_1 \otimes \cdots \otimes V_n$ is of rank 1.

To better understand this concept, consider the reduction to linear algebra, in which $\phi \in V_1 \otimes V_2$ may be considered as a linear map $V_1^* \to V_2$. Recall that every linear map on finite dimensional vector spaces can be written as a matrix, after choosing bases, and that the rank of a matrix M is the number of rank 1 matrices M_i needed to write $M = \sum_i M_i$. In this special case, the above definition is natural.[2]

But rank doesn't give us the whole picture when $n > 2$. To illustrate this, consider the following classical example.

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[1]Throughout the paper, we will assume the reader is familiar with the tensor product of vector spaces. For a quick review, see the Appendix.

[2]However, it is worth mentioning that *rank* as it is defined here is one of several generalizations of the rank of a linear map (e.g., multilinear rank).

The tensor

$$\phi = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$$

is of rank at most 3 since

$$\phi = a_1 \otimes b_1 \otimes (c_1 + c_2) + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1,$$

and it is not of rank 2 by explicit computation. However, notice that ϕ is the limit as $\epsilon \to 0$ of the following sequence of rank 2 tensors:

$$\phi(\epsilon) = \frac{1}{\epsilon}\big((\epsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \epsilon a_2) \otimes (b_1 + \epsilon b_2) \otimes (c_1 + \epsilon c_2)\big).$$

So the rank of the tensor is 3, but we can approximate it as closely as we like with rank 2 tensors. We say ϕ has *border rank* 2, and we have the following definition.

Definition 1.2. A tensor $\phi \in V_1 \otimes \cdots \otimes V_n$ is said to be of border rank r, denoted $\underline{\mathbf{R}}(\phi) = r$, if it is the limit of tensors of rank r but not of tensors of rank s for any $s < r$.

One way to approach the difficult general problem of understanding the border rank of tensors is to reduce multilinear algebra to linear algebra. Below is one such reduction, in which we consider $\phi \in A \otimes B \otimes C = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ as a linear map $A^* \to B \otimes C$ and then represent the image in $B \otimes C$ as a matrix. We then take the determinant of this representation to find an associated cubic curve to ϕ.

Choose bases $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ for A, B, C, respectively, with $\{a_i^*\}$, $\{b_i^*\}$, $\{c_i^*\}$ the dual bases. Now let

$$\phi = \sum_{i,j,k} \phi_{ijk}\, a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C,$$

where $\phi_{ijk} \in \mathbb{C}$ are constants and let

$$a^* = x a_i^* + y a_2^* + z a_3^*, \quad x, y, z \in \mathbb{C},$$

be an arbitrary element of $A^* = (\mathbb{C}^3)^*$. Then, the matrix representation of ϕ parametrized by a^*, denoted $[\phi \lrcorner\, a^*]$, has (j,k)-th entry

$$[\phi \lrcorner\, a^*]_{j,k} = \phi_{1jk}\, x + \phi_{2jk}\, y + \phi_{3jk}\, z.$$

In the same way, we can find matrix representations $[\phi \lrcorner\, b^*]$ and $[\phi \lrcorner\, c^*]$ parametrized by $b^* \in B^*$ and $c^* \in C^*$. For the tensors we study in this paper, all of these representations turn out to be equal, so we work with $[\phi \lrcorner\, a^*]$ without loss of generality.

Let's look at an example. If

$$\phi = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_1 \otimes c_2 + a_3 \otimes b_3 \otimes c_3,$$

then,

$$\phi_{111} = \phi_{222} = \phi_{312} = \phi_{333} = 1,$$

and $\phi_{ijk} = 0$ otherwise. Thus,

$$[\phi \lrcorner a^*] = \begin{pmatrix} x & z & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}.$$

Now take the determinant to find the determinantal cubic associated to ϕ,

$$xyz = 0.$$

It has been known since as early as 1938 (see e.g., [Thrall and Chanler 1938]) that any cubic curve in three variables is projectively equivalent to one of the following:

(1) triple line $x^3 = 0$

(2) double line and a line $x^2y = 0$

(3) 3 lines intersecting at a point $xy(x-y) = 0$

(4) 3 lines in general position $xyz = 0$

(5) a conic and a tangent line $z(x^2 + yz) = 0$

(6) a conic and a transverse line $x(x^2 + yz) = 0$

(7) cuspidal cubic $x^3 - y^2z = 0$

(8) node $x^3 + y^3 - xyz = 0$

(9) a smooth cubic: the general case

(10) a cubic identically zero

The tensors to which these other singular cases correspond are dealt with in [Thrall and Chanler 1938] and later in more modern language in [Ng 1995]. In particular, normal forms are given, and in [Allums 2011], the border rank of each of these singular tensors is calculated.

Since the singular cases have been dealt with, the next question is: how is border rank related to the intrinsic geometry of the determinantal cubic in the general case? That is, how does the border rank vary in the open set of *smooth* cubics? To answer this, we need to introduce the classical invariants S, T and J, which are rational functions in the coefficients of a cubic.

Under the action of $\mathrm{SL}(\mathbb{C}, 3)$ on the cubic, there is a unique (up to scale) degree 4 invariant S and a unique (up to scale) degree 6 invariant T [Sturmfels 1993]. These generate the ring of invariants of a cubic of which

$$J := \frac{S^3}{T^2 - 64S^3},$$

the j-invariant, is a member. The invariants S and T are extrinsic invariants of the curve, while J is an intrinsic invariant[3]. Here this means S and T classify the curve up to change of coordinates while J classifies smooth cubics up to isomorphism as abelian varieties, i.e., as groups and as algebraic varieties. One goal of this paper is to find out what relationship, if any, exists between the border rank of $\phi \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and the geometry of its determinantal cubic curve. Equivalently, we want to describe the relationship between border rank and S, T and thus J.

The maximum possible border rank of $\phi \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is 5 [Landsberg 2012], and since a tensor of border rank 5 depends on twelve parameters, we start with a smaller case and consider tensors of border rank 4, which we show depend on only three parameters in Proposition 3.1. We take such a tensor and calculate the invariants S and T of its determinantal cubic, summarizing our analysis in Proposition 3.2. In particular, we conclude that there is no meaningful relationship between the border rank of ϕ and S or T, and thus no meaningful relationship between border rank and J, if the cubic is smooth.

2. Background

Some background material is given in the appendix. We present the rest here, with most of it coming from [Landsberg 2012].

There exists a geometric interpretation of border rank as follows. Let V be a finite dimensional complex vector space and let $X \subset \mathbb{P}V$ be a variety. For any point q not on X, we define the *join of q and X* to be the set of all secant lines containing q and some point of X, denoted $J(q, X)$. If $q = x \in X$, we do the same thing, but we also allow tangent lines at x since a tangent line is a limit of secant lines. The *secant variety* of X is

$$\sigma(X) := \overline{\bigcup_{x \in X} J(x, X)},$$

where the bar denotes Zariski closure. The notation $J(X, X) = \sigma(X)$ is also used. We can also define the join of two distinct varieties $Y, Z \subset \mathbb{P}V$ by

$$J(Y, Z) = \overline{\bigcup_{q \in Y} J(q, Z)},$$

where $J(q, Z)$ is the set of all secant lines containing $q \in Y$ and some point of Z.

Definition 2.1 [Landsberg 2012]. The join of k varieties $X_1, \dots, X_k \subset \mathbb{P}V$ is the closure of the union of the corresponding secant $(k-1)$-planes, or by induction,

[3]Consider the difference between "extrinsic" and "intrinsic" in surface theory: mean curvature is extrinsic (invariant under Euclidean motion) but Gauss curvature is intrinsic (invariant under isometry).

$J(X_1, \dots, X_k) = J(X_1, J(X_2, \dots, X_k))$. Define the k-th secant variety of X to be $\sigma_k(X) = J(X, \dots, X)$, the join of k copies of X.

We move on to another crucial concept: the Segre variety.

Definition 2.2. The n-factor Segre variety is the image of the map

$$\mathrm{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n \to \mathbb{P}(V_1 \otimes \cdots \otimes V_n),$$
$$([v_1], \dots, [v_n]) \mapsto [v_1 \otimes \cdots \otimes v_n].$$

Note that for fixed $n \in \mathbb{N}$, the image of the Segre map is the projectivization of the rank 1 n-tensors.

A tensor $\phi \in V_1 \otimes \cdots \otimes V_n$ may be interpreted as a linear map

$$V_1^* \to V_2 \otimes \cdots \otimes V_n, \dots, V_n^* \to V_1 \otimes \cdots \otimes V_{n-1}.$$

Recall a matrix is rank 1 if and only if all its 2×2 minors are 0. The set of rank 1 tensors in $V_1 \otimes \cdots \otimes V_n$ is exactly the set of tensors such that each of the previous linear maps has rank 1 [Landsberg 2012]. The collection of these 2×2 minors are homogeneous polynomials called flattenings. Thus, using Definition 5.3, the set of tensors of rank 1 is an algebraic variety.

Tensors of border rank r are described as limits of tensors of rank r, so the set of tensors of border rank at most r is the closure of the set of tensors of rank r, where a tensor of rank r is contained in the linear span of r points of the set of tensors of rank 1. Since in this case the Zariski and Euclidean closures coincide (see [Mumford 1976, Theorem 2.33]), the (projectivization of the) set of tensors of border rank at most r is thus exactly $\sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n))$, and so we now have an entirely geometric interpretation of border rank with which to work. In particular, we can now restate some of the introduction in more modern language.

For $A \otimes B \otimes C = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, the representation of ϕ as a matrix defines a vector space of matrices in $\phi(A^*) \subset B \otimes C$ of dimension 3 parametrized by $a^* \in A^*$. When we move into projective space, it becomes a copy of $\mathbb{P}^2 \subset \mathbb{P}(B \otimes C)$. By requiring that its determinant vanish, we are demanding that the matrix be of rank at most 2. That is, we want the matrix to be contained in $\sigma_2(\mathrm{Seg}(\mathbb{P}B \times \mathbb{P}C))$. Our goal is then to see how border rank varies in the intersection

$$\{\mathbb{P}(\phi(A^*)) \mid \phi \in A \otimes B \otimes C\} \cap \sigma_2(\mathrm{Seg}(\mathbb{P}B \times \mathbb{P}C)).$$

3. Primary results

First, we show that a general point in $\sigma_4 := \sigma_4(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, i.e., a tensor of border rank 4, depends on only three parameters.

Proposition 3.1. *A general point in σ_4, up to the action of* $\mathrm{GL}(\mathbb{C}, 3)$, *depends on exactly three parameters.*

Proof. Let $\zeta_i, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ be constants and choose bases $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ for A, B, C. We first show that $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3$ is a general point in σ_3 by beginning with an arbitrary general point in σ_3. To do this, define

$$\begin{aligned} u_i &= \alpha_{i1} a_1 + \alpha_{i2} a_2 + \alpha_{i3} a_3, \\ v_j &= \beta_{j1} b_1 + \beta_{j2} b_2 + \beta_{j3} b_3, \\ w_k &= \gamma_{k1} c_1 + \gamma_{k2} c_2 + \gamma_{k3} c_3, \end{aligned}$$

where $\alpha_{ip}, \beta_{jp}, \gamma_{kp}$ are constants such that each set $\{u_i\}, \{v_j\}, \{w_k\}$ is linearly independent, which can be done in any open set; so this is a sufficiently arbitrary choice of elements. Let

$$u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes w_2 + u_3 \otimes v_3 \otimes w_3$$

be a general point in σ_3. Since our group of normalizations, $\mathrm{GL}(\mathbb{C}, 3)$, is 9-dimensional, we can send each $u_i \mapsto a_1$, $v_j \mapsto b_j$ and $w_k \mapsto c_k$, totaling nine transformations. We then have

$$a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3, \tag{11}$$

as desired. A general point in σ_4 is obtained by taking an arbitrary point in $\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ and adding it to (11) to obtain a point on an honest secant line:

$$\begin{aligned} &a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3 \\ &\qquad + (\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) \otimes (\beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3) \otimes (\gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3). \end{aligned}$$

Since $\mathrm{GL}(\mathbb{C}, 3)$ is 9-dimensional, we may make six dimensions worth of changes by sending $\alpha_i a_i \mapsto a_i$ and $\beta_j b_j \mapsto b_j$, with three dimensions worth of changes left over. However, these transformations add additional constants to the first three summands; we end up with

$$\sum_{i=1}^{3} \frac{1}{\alpha_i \beta_i} a_i \otimes b_i \otimes c_i + (a_1 + a_2 + a_3) \otimes (b_1 + b_2 + b_3) \otimes (\gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3).$$

Using our last three dimensions to send

$$\frac{1}{\alpha_i \beta_i} c_i \mapsto c_i$$

gives

$$\sum_{i=1}^{3} a_i \otimes b_i \otimes c_i + (a_1 + a_2 + a_3) \otimes (b_1 + b_2 + b_3) \otimes (\alpha_1 \beta_1 \gamma_1 c_1 + \alpha_2 \beta_2 \gamma_2 c_2 + \alpha_3 \beta_3 \gamma_3 c_3).$$

Finally, for the sake of notation, relabel

$$\lambda_i = \alpha_i \beta_i \gamma_i .$$

Thus, a general point in σ_4,

$$a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3 \\ + (a_1 + a_2 + a_3) \otimes (b_1 + b_2 + b_3) \otimes (\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3),$$

depends on only the three parameters $\lambda_1, \lambda_2, \lambda_3$. □

Note that the action of $\mathrm{GL}(\mathbb{C}, 3)$ on σ_4 does not change S or T as these are invariant under changes of coordinates. Now represent this tensor as a matrix, as described in the introduction:

$$\begin{pmatrix} x + \lambda_1(x+y+z) & \lambda_2(x+y+z) & \lambda_3(x+y+z) \\ \lambda_1(x+y+z) & y + \lambda_2(x+y+z) & \lambda_3(x+y+z) \\ \lambda_1(x+y+z) & \lambda_2(x+y+z) & z + \lambda_3(x+y+z) \end{pmatrix}.$$

Take the determinant to find the determinantal cubic curve, which is

$$(1 + \gamma_1 + \gamma_2 + \gamma_3)xyz + \gamma_1 y^2 z + \gamma_1 y z^2 + \gamma_2 x^2 z + \gamma_2 x z^2 + \gamma_3 x^2 y + \gamma_3 x y^2. \quad (12)$$

From here, one uses the formulae for S and T found in [Sturmfels 1993].

Proposition 3.2. *The border rank of $\phi \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is not related to the projective geometry of its determinantal cubic curve, if it is smooth.*

Proof. The polynomials S and T are in the ten coefficients of a cubic in general, but as shown in Proposition 3.1, the coefficients of our curve depends only on three parameters $\gamma_1, \gamma_2, \gamma_3$, so here S and T are in three variables. Now fix $\gamma_1 = \gamma_2 = 1$. Then S and T become nonconstant polynomials in the single complex variable γ_3:

$$S = \tfrac{1}{16}\gamma_3^4 - \tfrac{5}{12}\gamma_3^3 + \tfrac{7}{8}\gamma_3^2 + \tfrac{43}{108}\gamma_3 + \tfrac{169}{1296},$$
$$T = -\tfrac{1}{8}\gamma_3^6 + \tfrac{5}{4}\gamma_3^5 - \tfrac{113}{24}\gamma_3^4 + \tfrac{283}{54}\gamma_3^3 + \tfrac{691}{216}\gamma_3^2 - \tfrac{559}{324}\gamma_3 - \tfrac{2197}{5832}.$$

By Picard's theorem, S and T each either attain every value in $\mathbb{C}$ or attain all but one value in $\mathbb{C}$. However, if there was some $w \in \mathbb{C}$ not hit by S or T, then $S = w$ would have no solution. But since $\mathbb{C}$ is algebraically closed, $S - w = 0$ does have a root. Thus, S and T are onto, so we may obtain any value for them by suitable choices of $\gamma_1, \gamma_2, \gamma_3$. □

4. On the 24 singular cases

Define

$$\Delta := T^2 - 64S^3$$

to be the *discriminant* of a cubic curve. Since a cubic is singular if and only if $\Delta = 0$, one expects each of the determinantal cubics associated to the normal forms in [Ng 1995] to have $\Delta = 0$. The determinantal cubics are:

$$
\begin{aligned}
xyz &= 0 && \{1, 2, 3, 5, 6, 8\}\\
xyz - x^3 &= 0 && \{4, 9, 10\}\\
(\lambda-1)xyz &= 0 && \{7\}\\
y^2z + yz^2 &= 0 && \{11\}\\
x^2y + xy^2 &= 0 && \{12\}\\
x^2y - xz^2 &= 0 && \{13, 14\}\\
(\lambda-1)(\lambda z^3 + xyz) &= 0 && \{15\}\\
xyz - \lambda z^3 + y^3 &= 0 && \{16\}\\
xyz + \lambda x^3 &= 0 && \{17, 18\}\\
z^2y - zy^2 - xy^2 &= 0 && \{19\}\\
xz^2 + y^3 + \mu zy^2 &= 0 && \{20\}\\
-\mu x^2y - xy^2 + x^2z &= 0 && \{21, 22\}\\
(\lambda_3\lambda_5)z^3 + (\lambda_1\lambda_5 + \lambda_4\lambda_6)xz^2 & && \\
+ (\lambda_2\lambda_6)y^2z + (\lambda_2\lambda_5 + \lambda_3\lambda_6)yz^2 & && \\
- (\lambda_4\lambda_6 + \lambda_1\lambda_5)xy^2 + (\lambda_1\lambda_6)xyz &= 0 && \{23\}\\
-\mu z^3 - 2\mu^3y^2z + 3\mu^2yz^2 + 3\mu xy^2 &= 0 && \{24\}
\end{aligned}
$$

The set of numbers to the right are the normal forms to which the curve corresponds and

$$
\begin{aligned}
&\lambda_1 = (\lambda-1), \quad \lambda_2 = (\lambda-1)^2(\lambda^2+\lambda+1), \quad \lambda_3 = (\lambda^2-1)(\lambda^2+\lambda+1),\\
&\lambda_4 = (\lambda+1), \quad \lambda_5 = (\lambda^2+1), \quad \lambda_6 = (\lambda^2-1),
\end{aligned}
$$

where $\lambda \neq 0, 1$ for $\{7, 15\}$; $\lambda \neq 0$ for $\{16, 17, 18\}$; $\lambda \neq 0, \omega$ for $\{23\}$ (where $\omega^3 = 1$); $\mu = 0, 1$ for $\{20, 21, 22\}$; and $\mu \neq 0$ for $\{24\}$. Using the formulae in [Sturmfels 1993], we find $\Delta = 0$ for each of these cubics.

Notice that some of these cubics are projectively equivalent. Some of these equivalences are immediate[4], such as

$$
\begin{aligned}
\{1, 2, 3, 5, 6, 8\}, \{7\} &\sim (4),\\
\{4, 9, 10\}, \{15\}, \{17, 18\} &\sim (6),\\
\{11\}, \{12\} &\sim (3),\\
\{16\} &\sim (8),
\end{aligned}
$$

[4]Explanation of notation by example: The cubics $\{1, 2, 3, 5, 6, 8\}$ in [Ng 1995] correspond to $xyz = 0$ above, and this corresponds to three lines in general position, which is case (4) in [Thrall and Chanler 1938]. Additionally, $\{7\}$ corresponds to $(\lambda-1)xyz = 0$, which is projectively equivalent to $xyz = 0$ and so (4) as well. Thus we write $\{1, 2, 3, 5, 6, 8\}, \{7\} \sim (4)$.

where the numbers to the right come from the classification in the introduction. To find the others, we find the singular points and expand in a Taylor series about that point. We then look at the second order term: if it is of rank 1, then the singularity is a cusp, and if it is of rank 2, the singularity is a node. As an example, let's examine $f(x, y, z) = x^2y - xz^2$, which is the cubic corresponding to $\{13, 14\}$. The curve is singular at a point p if and only if the differential, D, vanishes at p. In this case,

$$D = (2xy - z^2, x^2, -2xz).$$

Since $D(p) = 0$ if and only if $p = [x : y : z] = [0 : 1 : 0]$, this is our singular point. Expand in a Taylor series about this point:

$$f(x, y, z) = f(p) + xf_x(p) + yf_y(p) + zf_z(p) + \tfrac{1}{2}x^2 f_{xx}(p) + \cdots .$$

The only nonzero term of second order is $\frac{1}{2}x^2 f_{xx}(p) = x^2$, which is of rank 1. Thus, our curve has a cusp and corresponds to case (7).

The classification of the remaining cases is a simple exercise in calculus, and we end up with

$$\begin{aligned} \{13, 14\}, \{19\}, \{20\}, \{21, 22\}, \{24\} &\sim (7), \\ \{23\} &\sim (8). \end{aligned}$$

5. Appendix

We begin with the definition of the tensor product of vector spaces. Although the tensor product is typically defined by its universal property, those familiar with it will have no trouble relating the following definition, which is sufficient for our purposes, to the standard one. In all cases, $\otimes = \otimes_{\mathbb{C}}$ and recall that for a vector space V, we denote by V^* the dual space to V, which is the space of all linear maps $V \to \mathbb{C}$.

Definition 5.1. Let $V_1, \dots, V_n, W$ be finite-dimensional vector spaces. A map $f : V_1 \times \cdots \times V_n \to W$ is said to be n-linear if it is linear in each factor. The tensor product of these spaces is

$$V_1 \otimes \cdots \otimes V_n \otimes W = \{f : V_1^* \times \cdots \times V_n^* \to W \mid f \text{ is } n\text{-linear}\}.$$

Note that when $W = \mathbb{C}$, we have that

$$V_1 \otimes \cdots \otimes V_n \otimes W = V_1 \otimes \cdots \otimes V_n \otimes \mathbb{C} \simeq V_1 \otimes \cdots \otimes V_n.$$

This is a standard result, whose statement in full generality can be seen in, e.g., Theorem 5.7 in [Hungerford 1980]. It is a straightforward exercise to show that $V \otimes W$ is the space of linear maps $V^* \to W$, the space of linear maps $W^* \to V$, the space of bilinear maps $V^* \times W^* \to \mathbb{C}$, etc. Inductively, we have many different equivalent ways to realize $V_1 \otimes \cdots \otimes V_n \otimes W$. The tensor product of vector spaces is again a vector space, whose elements are called tensors.

Next, since our work is done in complex projective space, we need a definition; n-dimensional complex projective space is the space of all one-dimensional subspaces (lines) in $\mathbb{C}^{n+1}$ [Harris 1995]:

Definition 5.2. Define n-dimensional complex projective space to be

$$\mathbb{P}^n = \mathbb{P}\mathbb{C}^n := (\mathbb{C}^{n+1}\backslash\{0\})/\sim,$$

where $\sim$ is the equivalence relation given by $\mathbb{C}^n \ni (v_1, \ldots, v_n) \sim (\lambda v_1, \ldots, \lambda v_n)$ for some nonzero scalar λ.

For a complex vector space V of finite dimension, denote the set of equivalence classes of some $v \in V$ by $[v] \in \mathbb{P}V$. Let

$$\begin{aligned} \pi : V\backslash\{0\} &\to \mathbb{P}V, \\ v &\mapsto [v] \end{aligned}$$

denote the projection. For a subset $Z \subset \mathbb{P}V$, let $\hat{Z} := \pi^{-1}(Z)$ denote the cone over Z. Call the image of such a cone in projective space its projectivization. We need a final crucial definition from [Harris 1995]:

Definition 5.3. A projective variety is the projectivization of the set of common zeros of some collection of homogeneous polynomials on V.

Should the reader want to read more relevant background material, see the sections on the tensor product in [Landsberg 2012; Hungerford 1980; Dummit and Foote 2004] and the sections on basic algebraic geometry in [Landsberg 2012; Harris 1995].

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