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Stephanie Perez and Robert Styer

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Persistence: a digit problem

Stephanie Perez and Robert Styer

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We examine the persistence of a number, defined as the number of iterations of the function which multiplies the digits of a number until one reaches a single digit number. We give numerical evidence supporting Sloane's 1973 conjecture that there exists a maximum persistence for every base. In particular, we give evidence that the maximum persistence in each base 2 through 12 is 1, 3, 3, 6, 5, 8, 6, 7, 11, 13, 7, respectively.

1. Introduction

Neil J. A. Sloane [1973] considered the function that multiplies the digits of a number and formally conjectured that the number of iterates needed to reach a fixed point is bounded. In particular, in base 10, he conjectured that one needs at most 11 iterates to reach a single digit. The problem did arise earlier; see [Gottlieb 1969, Problems 28–28; Beeler et al. 1972].

Definition 1. Let $n = \sum_{j=0}^{r} d_j B^j$, with $0 \le d_j < B$ for each d_j, be the base B expansion of n. We define the digital product function as $f(n) = \prod_{j=0}^{r} d_j$.

The persistence of a number n is defined as the minimum number k of iterates $f^k(n) = d$ needed to reach a single digit d.

Theorem 1. *If $n \ge B$, then $n > f(n)$. If $0 \le n < B$, then $f(n) = n$ is a fixed point. Thus, every n has a finite persistence.*

Proof. Let $n = \sum_{j=0}^{r} d_j B^j$, with $0 \le d_j < B$ for each d_j and $r > 0$. Since $r > 0$,

$$n \ge d_r B^r > d_r \prod_{j=0}^{r-1} d_j = f(n).$$

If $n < B$, then clearly $f(n) = n$. So, by induction on n one can show that every n has a finite persistence. □

For the remainder of this section, assume the base B equals 10.

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Keywords: persistence, digit problem, multiplicative persistence, iterated digit functions.

persistence	least n with given persistence	$\ln\ln n$
2	25	1.1690
3	39	1.2984
4	77	1.4688
5	679	1.8750
6	6788	2.1774
7	68889	2.4106
8	2677889	2.6947
9	26888999	2.8395
10	3778888999	3.0934
11	277777788888899	3.5043

Table 1. Smallest number with a given persistence.

Example. Let $n = 23487$. Then

$$f(23487) = 2 \cdot 3 \cdot 4 \cdot 8 \cdot 7 = 1344,$$
$$f(1344) = 1 \cdot 3 \cdot 4 \cdot 4 = 48,$$
$$f(48) = 4 \cdot 8 = 32,$$

and finally, $f(32) = 3 \cdot 2 = 6$. In other words, $f^4(23487) = 6$, so 23487 has persistence 4.

One easily sees that $n = 23114871$, $n = 642227$ and $n = 78432$ also have persistence 4 since each of these has $f(n) = 1344$. Thus, adding or removing the digit 1 does not change the persistence, nor does rearranging the digits or replacing digits that are products of smaller digits by these smaller digits.

In particular, since 288888899777777 has persistence 11, so do

1288888899777777, 11288888899777777 and 111288888899777777,

etc. Hence, there are an infinite number of integers with persistence 11.

We note some other immediate observations.

Let $n = 543210$. Then $f(n) = 0$, so it has persistence 1. More generally, any number with a 0 digit has persistence 1.

Let $n = 54321$. Then $f(54321) = 120$, so $f^2(54321) = 0$. More generally, in base 10, any number with a 5 digit, with an even digit, and with no 0 digit, has persistence 2.

Some preliminary calculations suggest that persistence depends on the size of the number. We list the smallest number with a given persistence (avoiding the contentious issue of defining the persistence of single digit numbers) in Table 1.

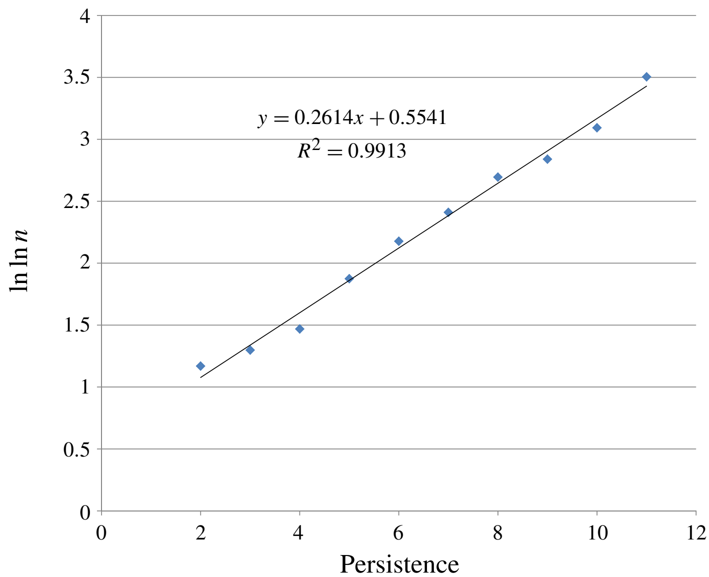

Figure 1. The double logarithm of the smallest number with persistence p versus p seems linear.

Table 1 and Figure 1 might suggest that the persistence grows roughly as the double logarithm of the number; using a linear fit to the log-log of the data, one might expect to find a number of size about $3 \cdot 10^{17}$ with persistence 12. Sloane [1973] showed, however, that no number less than 10^{50} has persistence 12; this was extended by Carmody [2001] to 10^{233}, and Diamond [2010] extended it to 10^{333}, while we extend it to 10^{1500}.

This paper has grown out of the senior research paper of the first author, intrigued by the mention of the problem in [Guy 2004, Problem F25].

2. Results

This section summarizes some results which give bounds for the persistence in various bases. We used Maple to calculate these results.

Since a large random number almost always has a 0 digit, we can prove the following theorem.

Theorem 2. *In any base B, the density of positive integers up to N with persistence greater than 1 approaches zero as N approaches infinity.*

Proof. Assume $B > 2$; the next theorem deals with base $B = 2$.

Consider all numbers with k digits in base B, that is, all integers N with $B^{k-1} \leq N < B^k$. There are precisely $(B-1)^k$ integers in this range without a 0 digit. Thus, considering all integers in the range $0 < N < B^k$, there are

$$\sum_{j=1}^{k}(B-1)^j = \frac{(B-1)((B-1)^k-1)}{B-2}$$

integers without a 0 digit. Thus, the density of integers with persistence greater than 1 up to B^k is

$$\frac{(B-1)((B-1)^k-1)}{(B-2)B^k} = \frac{B-1}{B-2}\left(\left(1-\frac{1}{B}\right)^k - \frac{1}{B^k}\right) < 2\left(1-\frac{1}{B}\right)^k.$$

As k approaches infinity, this last term goes to zero, proving the asymptotic density goes to zero. □

We now prove the well-known result that every number in base $B = 2$ has persistence 1 (some authors define the persistence of a single digit to be 0, so we only consider numbers with two or more digits).

Theorem 3. *In base* 2, *each number* $n > 2$ *has persistence* 1.

Proof. Either n has all digits equal to 1, in which case $f(n) = 1$, or n has at least one 0 digit, in which case $f(n) = 0$. □

Base 2 is the only base where we can prove Sloane's conjecture, but we can support his conjecture in other bases. In particular, Beeler and Gosper [1972, Item 57] showed that any number in base 3 with persistence greater than 3 must have more than 30739014 digits. We extend this to 10^9 digits.

Theorem 4. *In base* 3, *if* $n < 3^{10^9}$, *then* n *has persistence at most* 3, *and if* $n < 3^{10^9}$ *has persistence* 3, *then* $f(n) = 2^3$ *or* 2^{15}.

Proof. As noted above, if n has a digit of 0, then it has persistence 1, and if n has a digit of 1, then the persistence is unchanged if we remove all 1 digits. Thus, we may assume n has every digit equal to 2, so $f(n) = 2^k$ for some k. One can verify that the powers of 2 below 87 have persistence 1 except 2^3 and 2^{15}, which have persistence 2. Beeler and Gosper showed that each power of 2 between 2^{87} and $2^{30739014}$ contains a 0 in its base 3 expansion, and hence has persistence 1. With today's faster computers, we easily extend this to all powers of 2 up to 10^9. □

Theorem 5. *In base* 4, *if* $n < 4^{10^9}$, *then* n *has persistence at most* 3. *If* $n < 4^{10^9}$ *has persistence* 3, *then* $f(n) = 2^a 3^b$, *where* $(a, b) = (0, 3), (1, 3), (1, 5), (0, 6), (0, 10)$, *or* $(1, 11)$.

Proof. We have already noted that we need not consider any n with a digit of 0 or 1. Further, if n in base 4 has the digit 2 at least twice, then $f(n)$ has low-order digit 0, so $f(f(n)) = 0$. Thus, we may assume n has at most one digit 2 and the rest of the digits are 3; in other words, $f(n) = 2^a 3^b$ with $a \in \{0, 1\}$. We now calculate the persistence of 3^b and of $2 \cdot 3^b$ for all $b \le 10^9$ and note that none have persistence greater than 1 except for the listed values. For $b > 1000$, we do not actually calculate the persistence; we merely verify that there is a 0 digit in the last 64 digits. $\square$

Theorem 6. *In base* 5, *if* $n < 5^{10000}$, *then* n *has persistence at most* 6. *If* $n < 5^{10000}$ *has persistence* 6, *then* $f(n) = 2^{40}3^2$.

Proof. As before, we need not consider any n with a digit of 0 or 1. If n has a digit of 4, we may replace it by two digits 2. Thus, we may assume n has all digits equal to 2 or 3, in other words, $f(n) = 2^a 3^b$ for $a \ge 0$ and $b \ge 0$. We now calculate the persistence of $2^a 3^b$ for a and b with $\lceil a/2 \rceil + b \le 1000$; the factor of $1/2$ arises because each digit 4 is replaced by two digits 2. For large $a + b$, we merely verify that there is a 0 digit in the last 64 digits. The calculations show that each such $2^a 3^b$ has persistence less than 5 except for $2^{40}3^2$, which has persistence 5; hence, n has persistence at most 6 for all $n < 5^{10000}$. $\square$

Theorem 7. *In base* 6, *if* $n < 6^{10000}$, *then* n *has persistence at most* 5. *If* $n < 6^{10000}$ *has persistence* 5, *then* $f(n) = 2^a 5^b$, *where* $(a, b) = (7, 1)$, $(1, 4)$, $(0, 5)$, $(7, 2)$, $(4, 4)$, $(9, 3)$, $(7, 4)$, $(0, 8)$, *or* $(17, 2)$.

Proof. As before, we eliminate digits of 0 or 1, and replace digits of 4 by two digits 2. If n has a digit of 3 and an even digit, then $f(f(n)) = 0$, so we may assume n either has all digits equal to 2 or 5, or else n has all digits equal to 3 or 5. In other words, $f(n) = 2^a 5^b$ or $3^a 5^b$ for $a \ge 0$ and $b \ge 0$. We now calculate the persistence of $2^a 5^b$ for a and b with $\lceil a/2 \rceil + b \le 10000$ (the factor of $1/2$ covers the case where each digit 4 is replaced by two digits 2), and also calculate the persistence of $3^a 5^b$ where $a + b \le 10000$. The calculations show that all such expressions have persistence less than 4 except for the listed values, which have persistence 4; hence, n has persistence at most 5 for all $n < 6^{10000}$. $\square$

Theorem 8. *In base* 7, *if* $n < 7^{1000}$, *then* n *has persistence at most* 8. *If* $n < 7^{1000}$ *has persistence* 8, *then* $f(n) = 2^a 3^b 5^c$, *where* $(a, b, c) = (9, 3, 12)$, $(9, 17, 4)$, $(11, 8, 10)$, $(10, 20, 5)$, $(10, 8, 16)$, $(19, 25, 1)$, $(1, 44, 0)$, $(27, 0, 20)$, $(39, 24, 1)$, *or* $(11, 39, 3)$.

Proof. As before, we eliminate digits of 0 or 1, replace digits of 4 by two digits 2, and now also replace digits 6 by digits 2 and 3. So, we may assume n has all digits equal to 2, 3 or 5. In other words, $f(n) = 2^a 3^b 5^c$ for $a \ge 0$, $b \ge 0$, and $c \ge 0$. We now calculate the persistence of $2^a 3^b 5^c$; since we replaced digits of 4 by $2 \cdot 2$ and digits of 6 by $2 \cdot 3$, we must consider a, b, c with

$$a + b + c - \min(a, b) - \left\lfloor \frac{a - \min(a, b)}{2} \right\rfloor \le 1000.$$

We calculate the persistence of each such $2^a3^b5^c$ to find that all such expressions have persistence less than 6 except for the listed values, which have persistence 6; hence, n has persistence at most 7 for all $n < 7^{1000}$. □

Theorem 9. *In base* 8, *if* $n < 8^{1000}$, *then* n *has persistence at most* 6. *If* $n < 8^{1000}$ *has persistence* 6, *then* $f(n) = 3^3 5^4 7^2$.

Proof. As before, we eliminate digits of 0 or 1, replace digits of 4 by two digits 2, and now also replace digits 6 by digits 2 and 3. So, we may assume n has all digits equal to 2, 3, 5 or 7. If there are three or more digits 2, then $f(f(n)) = 0$. Therefore,

$$f(n) = 2^d 3^a 5^b 7^c \quad \text{for } a \geq 0,\ b \geq 0,\ c \geq 0, \text{ and } d \in \{0, 1, 2\}.$$

We consider a, b, c with $a + b + c \leq 1000$ to guarantee we are considering up to 1000 digits. We calculate the persistence of each such $2^d3^a5^b7^c$ to find that all such expressions have persistence less than 5 except for $3^35^47^2$, which has persistence 5; hence, n has persistence at most 6 for all $n < 8^{1000}$. □

Theorem 10. *In base* 9, *if* $n < 9^{1000}$, *then* n *has persistence at most* 7. *If* $n < 9^{1000}$ *has persistence* 7, *then* $f(n) = 2^a5^b7^c$, *where* $(a, b, c) = (1, 1, 5)$, $(3, 3, 4)$, $(24, 1, 1)$, $(4, 6, 4)$, $(11, 5, 3)$, *or* $(16, 7, 1)$.

Proof. As before, we eliminate digits of 0 or 1, replace digits of 4 by two digits 2, replace digits 6 by digits 2 and 3, and now also replace 8 by three digits 2. So, we may assume n has all digits equal to 2, 3, 5 or 7. If there are two or more digits 3, then $f(f(n)) = 0$, so we may assume $f(n) = 2^a5^b7^c$ or $f(n) = 3 \cdot 2^a5^b7^c$ for $a \geq 0$, $b \geq 0$, and $c \geq 0$. We now calculate the persistence of $3^d2^a5^b7^c$ for $d = 0$ or 1; in order to guarantee that we consider all numbers up to 1000 digits, we must consider a, b, c with $\lceil a/3 \rceil + b + c \leq 1000$. We calculate the persistence of each such $3^d2^a5^b7^c$ to find that all such expressions have persistence less than 6 except for the listed values (all having $d = 0$), which have persistence 6; hence, n has persistence at most 7 for all $n < 9^{1000}$. □

We now deal with base 10. Diamond [2010] calculated the persistence of all numbers $2^a3^b7^c$ and $3^a5^b7^c$ with $a \leq 1000$, $b \leq 1000$ and $c \leq 1000$. We verify his calculations and extend them to cover all numbers up to 1500 digits.

Theorem 11. *In base* 10, *if* $n < 10^{1500}$, *then* n *has persistence at most* 11. *If* $n < 10^{1500}$ *has persistence* 11, *then* $f(n) = 2^4 3^{20} 7^5$ *or* $2^{19} 3^4 7^6$.

Proof. As before, we eliminate digits of 0 or 1, replace digits of 4 by two digits 2, replace digits 6 by digits 2 and 3, replace the digit 8 by three digits 2, and now also replace 9 by two digits 3. In base 10, if we have both a digit 2 and a digit 5, then $f(f(n)) = 0$. So, we may assume $f(n) = 2^a3^b7^c$ or $f(n) = 3^a5^b7^c$ for $a \geq 0$, $b \geq 0$, and $c \geq 0$. To consider all n with less than 1500 digits, we only need to consider $f(n) = 2^a 3^b 7^c$ with $\lfloor a/3 \rfloor + \lfloor b/2 \rfloor + c \le 1500$, as well as $f(n) = 3^a 5^b 7^c$ with $\lceil a/2 \rceil + b + c \le 1500$. We find that all such expressions have persistence at most 9, except for the listed exceptions which have persistence 10; hence, n has persistence at most 11 for all $n < 10^{1500}$. □

Theorem 12. *In base* 11, *if* $n < 11^{250}$, *then* n *has persistence at most* 13. *If* $n < 11^{250}$ *has persistence* 13, *then* $f(n) = 2^{42}3^{13}5^{20}7^{17}$, $2^{91}3^{37}5^{7}7^{6}$, *or* $2^{32}3^{3}5^{35}7^{18}$.

Proof. As before, we eliminate digits of 0 or 1, replace digits of 4 by two digits 2, replace digits 6 by digits 2 and 3, replace the digit 8 by three digits 2, and now also replace 9 by two digits 3. We may assume $f(n) = 2^a 3^b 5^c 7^d$ for $a, b, c, d \ge 0$. To consider all n with less than 250 digits, we only need to consider $f(n) = 2^a 3^b 5^c 7^d$ with $\lfloor a/3 \rfloor + \lfloor b/2 \rfloor + c + d \le 250$. We find that all such expressions have persistence at most 11, except for the listed exceptions which have persistence 12; hence, n has persistence at most 13 for all $n < 11^{250}$. □

Theorem 13. *In base* 12, *if* $n < 12^{250}$, *then* n *has persistence at most* 7. *If* $n < 12^{250}$ *has persistence* 7, *then* $f(n) = 2^5 5^8 11^9$ *or* $3^5 5^1 7^6$.

Proof. As before, we eliminate digits of 0 or 1, replace digits of 4 by two digits 2, replace digits 6 by digits 2 and 3, replace the digit 8 by three digits 2, and now also replace 9 by two digits 3. We may assume $f(n) = 2^a 5^b 7^c 11^d$ or $3^a 5^b 7^c 11^d$ or $6 \cdot 3^a 5^b 7^c 11^d$ for $a, b, c, d \ge 0$. To consider all n with less than 250 digits, we only need to consider $f(n) = 2^a 5^b 7^c 11^d$ with $\lfloor a/3 \rfloor + b + c + d \le 250$, and for $f(n) = 3^a 5^b 7^c 11^d$ or $6 \cdot 3^a 5^b 7^c 11^d$, we consider $\lfloor a/2 \rfloor + b + c + d \le 250$. We find that all such expressions have persistence at most 5, except for the listed exceptions which have persistence 6; hence, n has persistence at most 7 for all $n < 12^{250}$. □

3. Conclusion

These calculations support Sloane's conjecture that the persistence is bounded for a given base. This makes sense since when a product of powers like $2^a 3^b 7^c$ has many digits, one expects to find a 0 digit among them. For instance, in base 10, we saw that $2^4 3^{20} 7^5 = 937638166841712$ has persistence 10, but

$$2^3 3^{20} 7^5 = 468819083420856, \quad 2^4 3^{19} 7^5 = 312546055613904,$$
$$2^4 3^{20} 7^4 = 133948309548816$$

all have a digit of 0. In general, almost all such powers will have a persistence of 1.

We used simple Maple programs, so the calculations for each theorem above took several hours to a few days to run on a laptop.

The first author tried to develop a method to work backwards, in order to answer questions such as which numbers iterate to the digit 1. We can devise many such interesting questions. Paul Erdős [Weisstein] asked what would happen if one multiplies only the nonzero digits (i.e., ignore the zero digits). Presumably this Erdős multiplicative persistence is no longer bounded, and the question of which numbers iterate to the digit 1 becomes more interesting. See [Wagstaff 1981] for another fascinating variation. We hope this paper inspires others to pursue the many fascinating problems related to multiplicative persistence.

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sperez03@villanova.edu *Department of Mathematics and Statistics, Villanova University, 800 Lancaster Avenue, Villanova, PA 19085-1699, United States*

robert.styer@villanova.edu *Department of Mathematics and Statistics, Villanova University, 800 Lancaster Avenue, Villanova, PA 19085-1699, United States*

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