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A new partial ordering of knots

Arazelle Mendoza, Tara Sargent, John Travis Shrontz and Paul Drube

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Our research concerns how knots behave under crossing changes. In particular, we investigate a partial ordering of alternating knots that results from performing crossing changes. A similar ordering was originally introduced by Kouki Taniyama in the paper "A partial order of knots". We amend Taniyama's partial ordering and present theorems about the structure of our ordering for more complicated knots. Our approach is largely graph theoretic, as we translate each knot diagram into one of two planar graphs by checkerboard coloring the plane. Of particular interest are the class of knots known as pretzel knots, as well as knots that have only one direct minor in the partial ordering.

1. Introduction

Basic knot theory. A *knot* K is a smooth embedding of a circle S^1 in $\mathbb{R}^3$. Some of our results generalize to links. A *link* L is a smooth embedding of multiple disjoint copies of S^1 in $\mathbb{R}^3$. Knot theorists generally do not want to work with 3-dimensional objects, which is why it is common to use knot diagrams. A *knot diagram* D of the knot K is a way of projecting K onto $\mathbb{R}^2$. This projection is one-to-one everywhere except a finite number of points called *crossings* where it is two-to-one. At every crossing there is an unbroken line for the overstrand and a broken line for the understrand. The overstrand corresponds to the arc that was initially closer to the viewer in $\mathbb{R}^3$.

A leading problem in knot theory is that one knot K may have many different diagrams that don't look remotely similar. We then need methods to determine when two knot diagrams represent the same knot.

The required machinery to deal with this problem are the *Reidemeister moves*, which are a set of three moves that connect diagrams of the same knot. The Reidemeister moves are shown in Figure 1. These moves are local, meaning the knot is unchanged outside of the exhibited region. The fundamental result about Reidemeister moves is the following:

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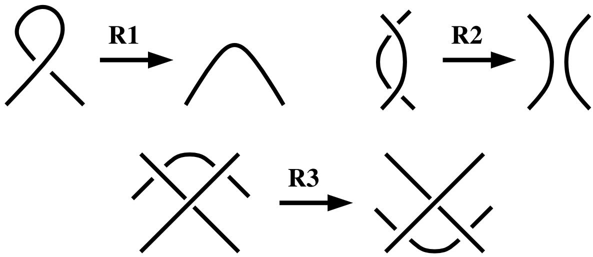


Figure 1. Reidemeister moves.

Theorem 1.1 (Reidemeister). *Two diagrams D_1 and D_2 represent the same knot K if and only if they may be connected by a finite number of Reidemeister moves.*

Proof. See [Reidemeister 1927] for a proof of this standard result. □

The *crossing number* $c(K)$ of a knot K is the minimum number of crossings over all diagrams K. A *minimal knot diagram* is a diagram D where the number of crossings equals $c(K)$. The standard way to denote knots takes the form N_n, where N denotes the crossing number of the knot and the subscript n is a traditional ordering (which depends upon an invariant known as the determinant).

Our research concerns how knots behave under crossing changes. A *crossing change* is a local operation that flips the role of the overstrand and the understrand at a single crossing in a knot diagram. The most important thing to note here is that a crossing change may change the underlying knot. An example of a crossing change is shown in Figure 2. As with our images for the Reidemeister moves, it is assumed that the link is unchanged outside of the region shown.

We focus on the class of knots known as prime alternating knots since they have many nice properties that allow for stronger results. An *alternating knot* is a knot with an alternating diagram, which is a knot diagram that alternates between overstrands and understrands as one travels around the diagram in a fixed direction. A *prime knot* is a knot that cannot be drawn as a connect sum of two nontrivial knots (i.e., it doesn't look like two or more nontrivial knots that have been strung together). Figure 3 shows the granny knot, which is a connect sum of two trefoil knots 3_1.

The following two theorems are important results that make prime alternating knots especially nice to work with.

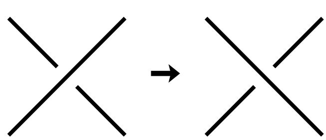


Figure 2. Crossing change.

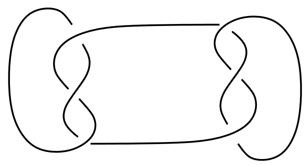


Figure 3. A nonprime knot.

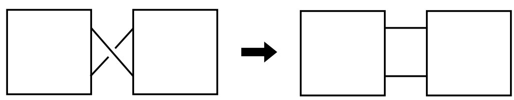


Figure 4. A nugatory crossing and untwisting that crossing.

Theorem 1.2 (Kauffman, Murasugi, and Thistlethwaite). *Let K be a prime alternating knot with diagram D. Then D is a minimal diagram for K if and only if D is a reduced alternating diagram.*

Proof. See [Adams 2004] for a proof of this foundational result. □

The term reduced above means that the diagram contains no nugatory crossings. A crossing in a diagram D is a *nugatory* (*removable*) *crossing* if removing a neighborhood of that crossing splits the knot diagram into two separate pieces. These are the crossings that can obviously be eliminated (via a 180-degree twist) to lower the crossing number of D without changing the underlying knot. See Figure 4.

Theorem 1.3 (Tait's flyping conjecture, Menasco & Thistlethwaite). *Let D_1 and D_2 be two minimal diagrams of the same prime alternating knot K in S^2. Then D_1 can be transformed into D_2 via a series of flypes.*

Proof. See [Menasco and Thistlethwaite 1993] for the (surprisingly complex) proof of this result, which had eluded knot theorists for a century. □

An example of a *flype* is shown in Figure 5. This operation involves a 180-degree twist of the portion of the knot denoted by T (known as a tangle), effectively moving a single half-twist from one side of that tangle to the other side. A flype is usually a complex combination of Reidemeister moves, but just like the basic Reidemeister moves, it does not change the underlying knot.

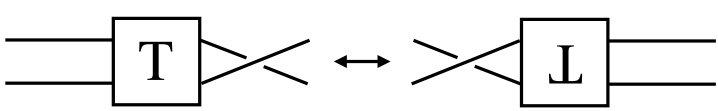


Figure 5. Flype operation.

A partial ordering of knots. The starting point for our research was the partial ordering on knots defined by Kouki Taniyama [1989]. To distinguish Taniyama's ordering from our own, we will henceforth refer to this partial ordering as the T-order:

Definition 1.4. Let K_1 and K_2 be knots. The *T-order* defines $K_1 \leq K_2$ if every diagram of K_2 can be transformed into some diagram of K_1 via some number of simultaneous crossing changes.

The number of simultaneous crossing changes required above depends upon the diagram of K_2 chosen, and there may not be a systematic way to determine the required crossing changes in a given diagram of K_2.

We present a modified version of Taniyama's T-ordering that was also influenced by the distinct partial ordering of Ernst, Diao, and Stasiak [Diao et al. 2009]. We will call our ordering the V-order, in honor of Valparaiso University (the site of the REU where we conducted this research).

Definition 1.5. Let K_1 and K_2 be prime alternating knots. The *V-order* defines K_1 to be a *V-minor* of K_2 if there exists a minimal diagram of K_2 that can be transformed into some diagram of K_1 via simultaneous crossing changes. We then define $(K_n, K_{n-1}, \ldots, K_2, K_1)$ to be a *proper sequence* of knots if K_i is a V-minor of K_{i+1} for all i, and $K_1 \leq K_2$ if there exists a proper sequence containing both K_1 and K_2, where K_1 appears to the right of K_2.

In this partial ordering (as was the case in Taniyama's original partial ordering), we do not differentiate between a knot, its reflection, and its reverse.

The reason that we present such a complicated definition involving proper sequences is to ensure that the resulting relation is transitive. One can quickly verify that the V-order defines a partial order of alternating knots, meaning that

(1) $K \leq K$ for all K;

(2) if $K_1 \leq K_2$ and $K_2 \leq K_3$, then $K_1 \leq K_3$;

(3) if $K_1 \leq K_2$ and $K_2 \leq K_1$, then $K_1 = K_2$.

It is the third condition in the partial ordering definition above that requires us to restrict our attention solely to prime alternating knots. There exist nonalternating knots such that $K_1 \leq K_2$ and $K_2 \leq K_1$, yet $K_1 \neq K_2$ (see Theorem 2.3 for more details).

We represent the V-order with a Hasse diagram, which is a graphical way to represent the relationships in the partial ordering. If two knots K_1 and K_2 are connected by a series of edges on the Hasse diagram, and if K_1 lies below K_2 on the edge, then $K_1 \leq K_2$. We manually verified that the V-order is identical to the T-order for the first eight nontrivial prime alternating knots (through 7_1), yielding the Hasse diagram in Figure 6.

Note that our ordering requires that we check only one minimal diagram of K_2 to verify $K_1 \leq K_2$, while Taniyama's ordering requires that we check all diagrams

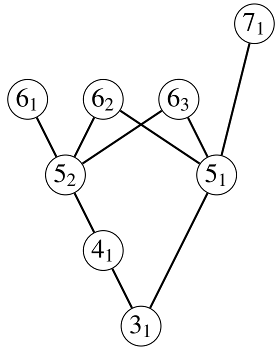


Figure 6. Partial ordering for the first eight prime knots.

of K_2. Also notice that if $K_1 \leq K_2$ in the T-order, then $K_1 \leq K_2$ in the V-order. The converse is not necessarily true a priori, although we conjecture that it is true for prime alternating knots (see Conjecture 4.1). The V-order relates to Ernst, Diao, and Stasiak's work [Diao et al. 2009] in that their ordering allows for only one crossing change, while ours allows for multiple simultaneous crossing changes. This seemingly simple modification actually makes our ordering significantly more complicated, yet also helps our ordering maintain a closer relationship with Taniyama's original ordering (which also allows for multiple simultaneous crossing changes).

In future sections, we will be especially interested in direct V-minors:

Definition 1.6. K_1 is a *direct V-minor* of K_3 if $K_1 \leq K_3$ and there does not exist K_2 ($K_2 \neq K_1, K_3$) such that $K_1 \leq K_2 \leq K_3$.

Definition 1.7. K_1 is a *remote V-minor* of K_3 if $K_1 \leq K_3$ and there exists K_2 ($K_2 \neq K_1, K_3$) such that $K_1 \leq K_2 \leq K_3$.

For example, as easily read from our Hasse diagram in Figure 6, the knot 3_1 is a remote V-minor of 7_1 because $3_1 \leq 5_1 \leq 7_1$. However, 3_1 is a direct V-minor of 5_1 since there does not exist a distinct knot K such that $3_1 \leq K \leq 5_1$.

Graph theoretical methods in knot theory. By Theorem 1.1, we know that any two diagrams of one knot may be connected via a series of Reidemeister moves, but it is tedious to constantly redraw the diagram every time we perform a Reidemeister move. To make calculations easier, we convert knot diagrams to a specific type of signed planar graph that contains all of the same information. The procedure for converting a knot diagram to a graph is as follows:

(1) Checkboard color the regions of the plane in the complement of the knot diagram so that around each crossing there are two white regions and two gray regions. Then mark each crossing by dropping a line segment connecting the two regions that lie counterclockwise from the overstrand.

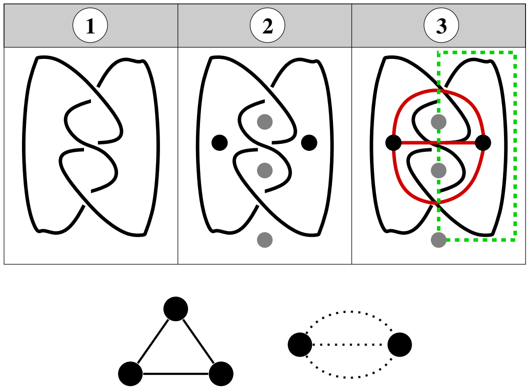


Figure 7. Checkerboard graphs of 3_1.

(2) Pick one of the two colors. Place a vertex inside each region of this fixed color.

(3) If two of the chosen regions share a crossing, add an edge between the corresponding vertices in the graph. This edge is solid if the marking associated with that crossing falls within the chosen regions and is dotted if the marking falls within the regions of the other color.

Since we had two choices in (2) above, we get two distinct graphs for any knot diagram. These graphs are always signed duals of one another. We illustrate this entire procedure for the trefoil knot 3_1 in Figure 7, showing both of the resulting planar graphs in the second row.

Our next challenge is to determine how the Reidemeister moves for knot diagrams translate to checkerboard graphs, as we need a reliable way of determining when two graphs represent the same underlying knot. It is important to note that every Reidemeister move for knot diagrams actually corresponds to two graph Reidemeister moves that are duals of one another. We illustrate all of these graph Reidemeister moves in Figure 8. In this figure, each diagram represents a local piece of the entire checkerboard graph. E and F represent nodes in the graph that may or may not have other edges entering them, while the small black vertices are adjacent only to the edges shown. In the second R2 move, $E \cup F$ denotes that the central node is now adjacent to all edges that were formerly incident upon either the E node or the F node.

One more important thing to note is that both graph representations of an alternating diagram only have one type of edge (one of them has all solid edges, while the signed dual has all dotted edges). This makes alternating diagrams especially easy to identify from checkerboard graphs: you no longer have to trace along the entire diagram to see if the knot alternates between overstrands and understrands!

Since our research deals with how knots behave under crossing changes, we need to determine how a crossing change effects a knot diagram's associated graph.

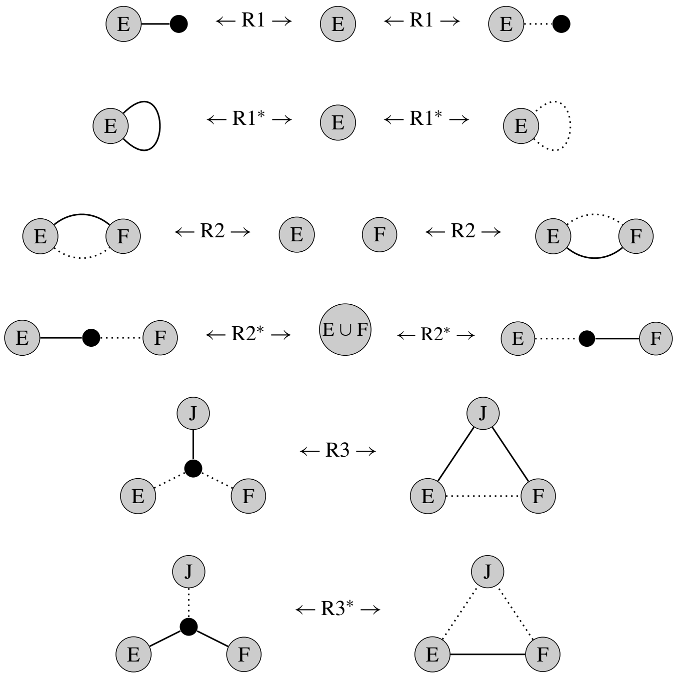


Figure 8. Reidemeister moves for graphs.

Crossing changes switch the roles of the overstrand and understrand at a single crossing. In either checkerboard graph for the diagram, this changes the marking on the associated edge and hence flips the type of edge that appears in the graph (dotted to solid, or solid to dotted).

Finally, we need to know how flypes effect our graphs. Figure 9 shows the graph representations of flypes. Just as with the Reidemeister moves, a flype has two different graph representations that are (signed) duals of one another.

Notice that, in the first flype equivalence, we are rearranging edges that separate the same two regions in our graph. In the second equivalence, we are rearranging edges that connect the same two vertices.

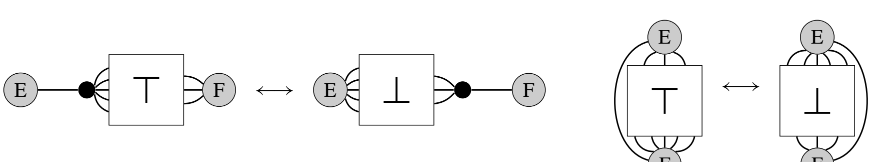


Figure 9. Graph equivalents of a flype.

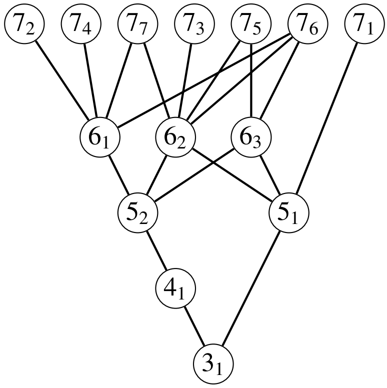


Figure 10. The V-order for prime alternating knots through 7_7.

2. Our partial ordering

Now we will investigate our V-order. Recall the definition:

Definition 2.1. Let K_1 and K_2 be prime alternating knots. The *V-order* defines K_1 to be a *V-minor* of K_2 if there exists a minimal diagram of K_2 that can be transformed into some diagram of K_1 via simultaneous crossing changes. We then define $(K_n, K_{n-1}, \ldots, K_2, K_1)$ to be a *proper sequence* of knots if K_i is a V-minor of K_{i+1} for all i, and $K_1 \leq K_2$ if there exists a proper sequence containing both K_1 and K_2, where K_1 appears to the right of K_2.

Our first goal was to directly expand the Hasse diagram of Section 1 up through 7-crossing prime alternating knots. If Conjecture 4.1 proves to be true, these results will translate into a direct extension of Taniyama's original T-order.

In order to directly determine which knots were V-minors of a particular knot K, we exhaustively checked all possible ways to make simultaneous crossing changes on the graph for a fixed minimal diagram D of K. We checked all of the (combinatorially distinct) ways to make one crossing change at a time, and then two crossing changes at a time, etc., up to half of the crossing number of K. We did not need to change more than half of the crossings at a time because we do not distinguish between a knot and its reflection: if changing some set of crossings yields a diagram of K, then changing the complement of that set gives a diagram of the reflection of K.

Our updated Hasse diagram is shown in Figure 10. See the Appendix for the calculations that yielded this Hasse diagram.

Invariants and the V-order. The problem with the direct technique above is that there are an extremely large number of cases to check for each knot. In order to quickly eliminate many possible relationships in the V-order, we prove several

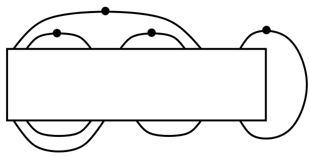


Figure 11. A knot diagram with $br(D) = 4$.

results about the ordering that involve knot invariants. A *knot invariant* is a function $i : \kappa \to \alpha$ from the set of all knots κ to some algebraic structure α. Distinct diagrams of the same knot must get sent to the same value by the invariant, so if an invariant gives different values for two diagrams, they cannot represent the same knot.

The knot invariants we work with are crossing number $c(K)$, bridge index $br(K)$, and braid index $b(K)$. It should be noted that some of our proofs in this section are similar to those presented in [Endo et al. 2010], where Endo, Itah, and Taniyama relate an entirely distinct partial ordering of links to common link invariants.

Theorem 2.2. *Let K_1, K_2 be distinct knots with $K_1 \leq K_2$, then $c(K_1) \leq c(K_2)$.*

Proof. Let K_1 and K_2 be knots, where $K_1 \leq K_2$ and $c(K_2) = n$. Then there exists a minimal diagram D_2 of K_2 that can be transformed into a diagram D_1 of K_1 via some number of simultaneous crossing changes. Now, D_1 has n crossings, and thus the crossing number of K_1 can be at most n. □

The following theorem, which was originally proven by Taniyama [1989], is more specific to our research since our V-order is restricted to alternating knots.

Theorem 2.3. *Let K_1, K_2 be alternating knots with $K_1 \leq K_2$, then $c(K_1) < c(K_2)$.*

Proof. Let K_1 and K_2 be alternating knots, where $K_1 \leq K_2$ and $c(K_2) = n$. Then there exists a minimal diagram D_2 of K_2 that can be transformed into a diagram D_1 of K_1 by simultaneously changing some but not all of the crossings in D_2. Now, D_1 has n crossings, so by Theorem 1.2, D_1 cannot be a minimal diagram of K_1. Thus $c(K_1) < n$. □

The second invariant we work with is the bridge number. The bridge number of a knot diagram D of K is the number of local maximums in D with respect to the y-coordinate in $\mathbb{R}^2$ (the number of "top points" in the diagram). The *bridge index* $br(K)$ of a knot K is the minimal bridge number over all diagrams of K. Note that, for every diagram D of K, there is one local minimum for every local maximum, so the bridge number could have been defined using local minimums.

An example of a knot diagram D with $br(D) = 4$ is shown in Figure 11. Here the box represents some (possibly complex) part of the knot diagram that contains no local maxima or minima.

Theorem 2.4. *If $K_1 \leq K_2$, then $br(K_1) \leq br(K_2)$.*

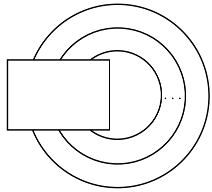


Figure 12. A knot diagram with $b(K) = 3$.

Proof. Let K_1 and K_2 be knots, where $K_1 \leq K_2$ and $br(K_2) = n$. Then there exists a minimal bridge diagram D_2 of K_2 that can be transformed into a diagram D_1 of K_1 via some number of simultaneous crossing changes. Since D_1 has n local maxes, the bridge number of K_1 can be at most n. □

The last invariant we work with is the braid index. The *braid index* $b(K)$ is the minimal number of strands over all braid representations of a knot.

An example of a braid representation is shown in Figure 12. As with our figure for bridge number, the box represents some (possibly complex) part of the knot diagram that contains no local maxima or minima.

Theorem 2.5. *If $K_1 \leq K_2$, then $b(K_1) \leq b(K_2)$.*

Proof. Let K_1 and K_2 be knots where $K_1 \leq K_2$ and $b(K_2) = n$. Then there exists a minimal braid diagram D_2 of K_2 (with n braid strands) that can be transformed into a diagram D_1 of K_1 via some number of simultaneous crossing changes. Since D_1 has n braid strands, the braid index of K_1 can be at most n. □

Direct V-minors. We now turn our attention to finding direct V-minors. Recall that K_1 is a direct V-minor of K_3 if $K_1 \leq K_3$ and there does not exist a distinct K_2 such that $K_1 \leq K_2 \leq K_3$. As we are restricting ourselves to prime alternating knots, we will search for direct minors by finding alternating knots $K_1 \leq K_3$ such that $c(K_1) = c(K_3) - 1$. Theorem 2.3 ensures that all pairs of knots with this property yield a direct V-minor. Although this strategy won't find all direct V-minors, it will locate most of them (as you can tell from our expanded Hasse diagram, the vast majority of edges connect knots that differ by a crossing number of one).

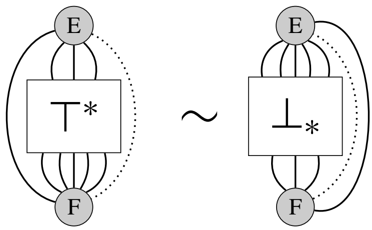
Our primary tool in applying this strategy is the following theorem, which vastly limits the number of cases where $c(K_1) = c(K_3) - 1$ is possible.

Theorem 2.6. *Let K_1 and K_2 be alternating knots with $K_1 \leq K_2$, and let G_2 be any minimal graph of K_2.*

(1) *In G_2, if we switch some but not all of the edges connecting two vertices, then $c(K_1) \leq c(K_2) - 2$.*

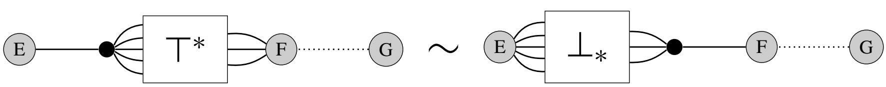
(2) *In G_2, if we switch some but not all of the edges separating two regions, then $c(K_1) \leq c(K_2) - 2$.*

Proof. We are given that K_1 and K_2 are alternating knots with $K_1 \leq K_2$. Let G_2 be an alternating graph of K_2. Switch some but not all of the edges connecting two vertices, so that those two vertices have at least one dotted edge and one solid edge between them. In general these edges need not be directly adjacent. If they are not directly adjacent, we can perform the flype below to make them adjacent:



After performing this flype we can always perform an R2 move, which will produce a graph with two edges less than the original G_2. Thus, K_1 has at most $c(K_2) - 2$ crossings and $c(K_1) \leq c(K_2) - 2$.

The proof for the case of switching some but not all of the edges separating two regions is similar to above. Now the relevant flype that yields an R2 move takes the form of the diagram below:

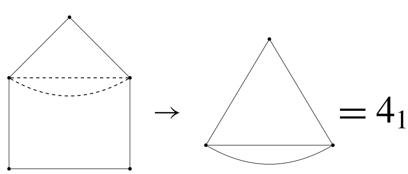


□

When searching for direct V-minors, we restrict our attention to the combinatorial cases that involve changing all crossings that connect a fixed pair of vertices or all crossings that separate a fixed pair of regions (or a multiple number of such cases). Using terminology from the literature, these cases correspond to changing all crossings in fixed number of twist boxes. These guidelines directly guided the calculations that we performed in the Appendix.

It should be noted that the conditions from Theorem 2.6, although necessary for obtaining a direct V-minor with $c(K_1) = c(K_2) - 1$, are not sufficient to guarantee that $c(K_1) = c(K_2) - 1$. Below is an example where we follow the conditions of Theorem 2.6 but still end up with a knot such that $c(K_1) \leq c(K_2) - 2$.

Example 2.7. If we change both of the middle edges of the graph of 7_5, we drop to the graph of 4_1, which has $c(4_1) = c(7_5) - 3$.



3. Pretzel links and our partial ordering

Basic properties. A particularly simple class of links that behave nicely with respect to our partial ordering are pretzel links. A link is a *pretzel link* if it has a diagram that takes the form on the left side of Figure 13. Here the boxes represent twist boxes full of half-twists in either direction. Since it is sometimes difficult to tell whether a pretzel link is a one-component knot or a multiple-component link, all of our theorems in this subsection have been extended to alternating links.

If we take the gray regions from our checkerboard coloring on the left, we see that a pretzel link always has a graph of the form on the right side of Figure 13. Here the half-twists in the link diagram translate into parallel edges between adjacent vertices. We refer to graphs of this type as *polygonal graphs*. We denote the pretzel link of Figure 13 by $P_v(x_1, x_2, x_3, \dots, x_v)$, where v is the number of twist boxes in the link diagram (or the number of vertices in the associated polygonal graph) and x_i is an integer corresponding to the number of half-twists in each twist box (or the number of edges connecting the consecutive vertices v_i and v_{i+1}). We define v_v to precede v_1. By convention, x_i will be negative if all of the edges in the given twist box are dotted, and positive if all of the edges are solid (if there are solid and dotted edges between two fixed vertices, we immediately eliminate them with an R2 move).

For example, in Figure 14 we have $P_3(3, 3, 2) = 8_5$. Notice that $P_3(3, 3, 2) = P_3(3, 2, 3) = P(2, 3, 3)$.

Pretzel links and our partial order. The reason that pretzel links are extremely nice in relation to our partial ordering is that many of them have only one or two direct V-minors (and almost all knots with only one or two direct V-minors appear to be pretzel knots; see Section 4). Here we present several theorems characterizing the role of several classes of pretzel links in our partial ordering.

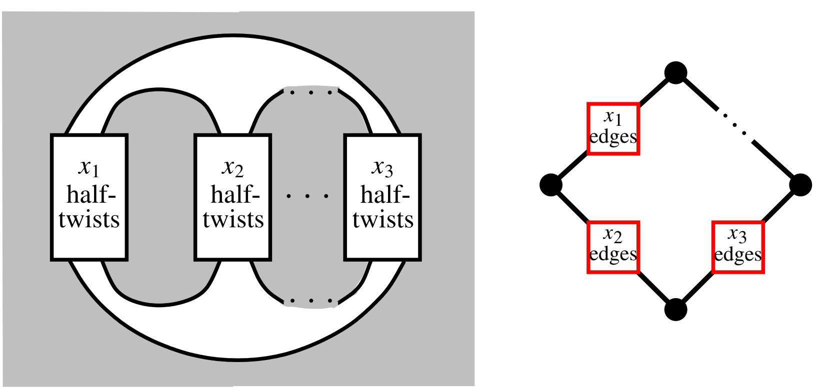


Figure 13. Pretzel knot diagram and its graph.

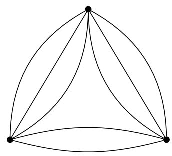


Figure 14. $8_5 = P_3(3, 3, 2)$.

The simplest class of pretzel links are $(p, 2)$-torus links. A $(p, 2)$-*torus link* is a link with only a single twist box, where p is the total number of half-twists in the twist box. They are so named because they fit upon the surface of a torus in $\mathbb{R}^3$ and wrap around the torus p times in the meridian direction for every two times that they wrap around the torus in the longitudinal direction. If p is odd then the $(p, 2)$-torus link is a knot; if p is even then the $(p, 2)$-torus link is a two-component link. In terms of our pretzel link notation, the $(p, 2)$-torus link is $P_p(1, 1, \ldots, 1)$. Figure 15 shows the general form for the checkerboard graph of a torus knot.

Theorem 3.1. *Every V-minor of the* $(p, 2)$*-torus link is a* $(q, 2)$*-torus link with* $q < p$*. Furthermore, the* $(p, 2)$*-torus link has a single direct V-minor in the* $(p-2, 2)$*-torus link.*

Proof. Consider the graph $P_p(1, \ldots, 1)$ of the $(p, 2)$-torus link. If we change $m < p$ crossings in the polygonal graph's sole twist box, there will be a solid edge next to a dotted edge. This means that we can always perform an R2 move, removing edges in pairs until the edges are all solid or all dotted. Every time we perform an R2 move, we lose two edges. The resulting graph will always be of the form $P_{p-2k}(1, \ldots, 1)$, where k is the minimum between the number of dotted edges and the number of solid edges that we start with. □

This theorem supports what we already found for the torus knots $3_1, 5_1, 7_1$ in our Hasse diagram: the $(p, 2)$-torus knots line up in our Hasse diagram and have the smaller $(p, 2)$-torus knots below them in a line. Note that many non-$(p, 2)$-torus knots may have a $(p, 2)$-torus knot as their V-minor: our theorem doesn't work in the other direction.

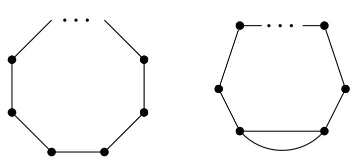


Figure 15. Left: $(p, 2)$-torus knot checkerboard graph. Right: twist knot checkerboard graph.

Another basic class of pretzel links are twist links, which are always one-component knots. A *twist knot* is a pretzel link whose checkerboard graph is of the form shown in Figure 15. Its two polygonal graphs are always of the form $P_{c(K)-1}(2, 1, 1, \ldots, 1)$ and $P_3(c(K)-2, 1, 1)$. The smallest nontrivial twist knots are $3_1 = P_3(1, 1, 1)$, $4_1 = P_3(3, 1, 1)$, $5_1 = P_3(4, 1, 1)$, and $6_1 = P_3(4, 1, 1)$. Notice that 3_1 is both a twist knot and a $(p, 2)$-torus knot.

Theorem 3.2. *Every V-minor of the twist knot* $P_3(n, 1, 1)$ *is a twist knot* $P_3(m, 1, 1)$ *with* $m < n$. *Furthermore, the twist knot* $P_3(n, 1, 1)$ *has a single direct V-minor in* $P_3(n-1, 1, 1)$.

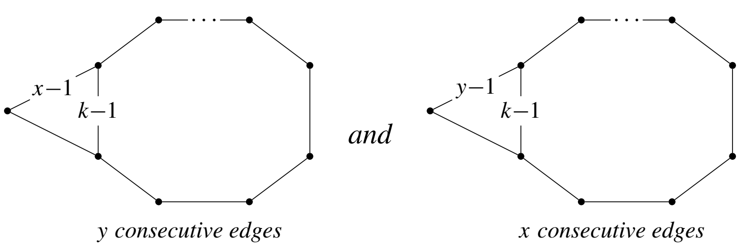
Proof. Changing $m < n$ crossings in the big twist box always allows for R2 moves, similarly to Theorem 3.1. The result is always a twist knot of the form $P_3(n-2k, 1, 1)$ for some integer $k > 0$. Changing one but not both of the remaining two crossings always results in the unknot (technically a twist knot), as an R2 move on the bottom allows us to completely untwist the knot. Changing both of the remaining crossings results in the direct V-minor $P_3(n-1, 1, 1)$; see the proof of Theorem 3.3 for a more general demonstration of this fact. Changing both of the remaining two crossings and some number of crossings in the big twist box results in the same knot as changing the complement of these crossings, which falls into the same case as above. In every case, we are left with a twist knot. □

As with Theorem 3.1, the implication of Theorem 3.2 is easily seen in our Hasse diagram: the twist knots 3_1, 4_1, 5_1, etc. line up along the left side of the diagram and only have other twist knots underneath them.

Theorems 3.1 and 3.2 are actually special cases of the theorem below, which gives a very broad class of pretzel links with only one or two direct V-minors:

Theorem 3.3. *Consider the pretzel link* $L = P_{k+2}(x, y, 1, 1, 1, \ldots)$, *where* $k > 1$.

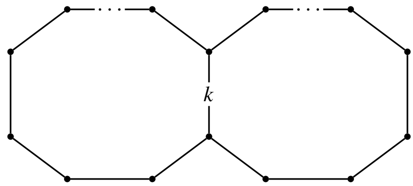
(1) *If* $x, y \neq 1$, *then* L *has two direct V-minors, each of which has crossing number* $c(L)-1$. *These two V-minors, which are equivalent if* $x = y$, *have (possibly nonpolygonal) graphs of the form*



Here the $x-1$, $y-1$, *and* $k-1$ *refer to that number of parallel strands.*

(2) *If* $x = 1$, *then* L *has one direct V-minor of the form* $P_3(k, y-1, 1)$. *Equivalently, if* $y = 1$, *then* L *has only one direct V-minor of the form* $P_3(k, x-1, 1)$.

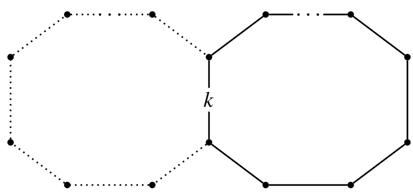
Proof. Given L as defined above, the dual graph of $P_{k+2}(x, y, 1, 1, 1, \ldots)$ is



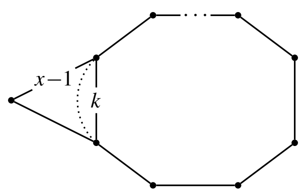
Here we have k parallel strands in the middle, a string of x consecutive strands of the left, and a string of y consecutive strands on the right. We choose to perform our possible crossing changes on this dual graph.

From Theorem 2.6, we know that we can only achieve a direct V-minor L' with $c(L') = c(L) - 1$ if we perform crossing changes on entire twist boxes. From the diagram above, we clearly have three twist boxes: one on the left, one on the right, and one with the k parallel strands down the middle. We then have three cases to check, corresponding to changing all of the crossings in each twist box (notice that, up to reflection, changing all crossings in two twist boxes yields the same knot as changing all of the crossings in the remaining twist box).

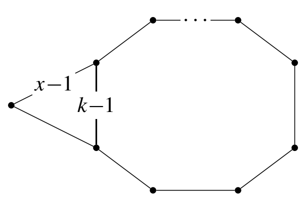
First we change all crossings on the left side, giving



After adding a free solid edge on the left side (corresponding to an R1 move), a series of R3 moves reduces the graph to

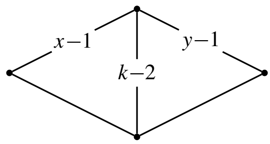


Notice that this graph has $c(L) + 1$ edges. After performing an R2 move in the middle, we are left with the following graph with $c(L) - 1$ edges, corresponding to the first direct V-minor from the theorem statement:



Changing all of the crossings on the right side of the original graph is equivalent to the above, and results in the second direct V-minor from the theorem statement.

Lastly, we consider changing all crossings in the middle twist box. This is equivalent (up to reflection) to changing all of the crossings on the left and on the right, which allows us to perform the procedure above two consecutive times to arrive at



This graph has $c(L)-2$ edges and is actually a direct V-minor of the two $c(L)-1$ crossing knots derived above. Hence it is a remote V-minor of our original link. Thus our link has only the two direct V-minors stated in the theorem.

Part (2) of the theorem is a special case of part (1). When $x = 1$, the string of consecutive edges in the right graph from the theorem statement is a single edge that adds to the twist box in the middle (which now has k parallel edges instead of $k-1$ parallel edges). The argument for $y = 1$ is similar. □

4. Future work

Our work revealed several questions that we hope to address in future papers. The biggest open question that lay behind much of our research was what we referred to as the minimal conjecture.

Conjecture 4.1 (The minimal conjecture). *Let K_2 be a prime alternating knot* (*link*) *and let K_1 be any knot* (*link*). *If there exists a minimal diagram of K_2 that can be transformed into a diagram of K_1 via some number of simultaneous crossing changes*, *then every diagram of K_2 can be transformed into K_1 via some number of simultaneous crossing changes.*

As noted earlier in the paper, if Conjecture 4.1 is true, it implies that the V-order and T-order are equivalent for prime alternating knots. This means that our work would be a direct refinement of Taniyama's original methods. Unfortunately, this conjecture seems to resist all direct methods of proof that we attempted.

In Section 3, we produced many knots with only one direct V-minor. For knots with low crossing number, the only knots we found that had only one direct V-minor were pretzel knots. This begs the following conjecture.

Conjecture 4.2. *Pretzel knots are the only prime alternating knots with one direct V-minor.*

Below are a few additional general avenues of research that we may address in future research.

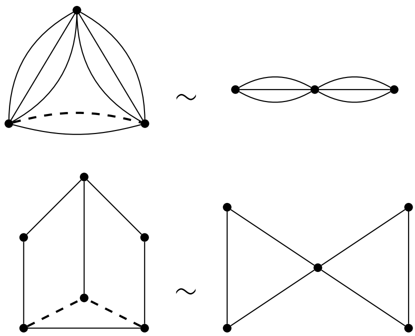


Figure 16. Top: $8_5 \geq 3_1\#3_1$. Bottom: $8_{16} \geq 3_1\#3_1$.

Future Topic 4.3. All $(p, 2)$-torus knots K lack direct V-minors K' with $c(K') = c(K)-1$. Most other knots seem to have at least one V-minor with $c(K') = c(K)-1$, but there are still examples of non-$(p, 2)$-torus knots that fail in this regard. The knots 8_5 and 8_{16} are non-$(p, 2)$-torus knots K that have no direct V-minors K' with $c(K') = c(K) - 1$. Is there something special about these knots that we can generalize? Notice that these problematic eight-crossing knots are also the eight-crossing alternating knots with nonprime V-minors; see Figure 16.

Is it possible to expand our work to nonprime or nonalternating links? At the very least, is it possible to fully categorize which prime alternating knots have nonprime or nonalternating knots directly beneath them in our ordering?

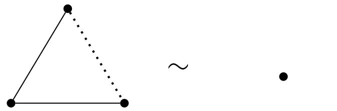
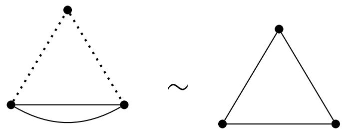
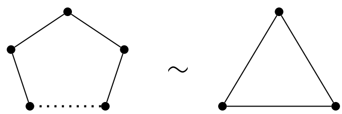
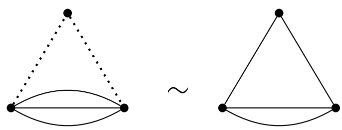
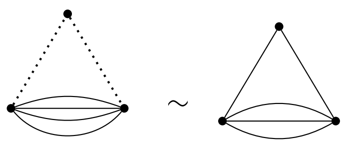
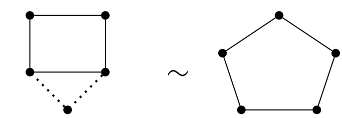
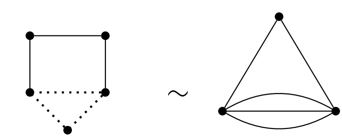
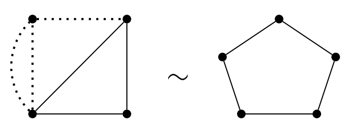
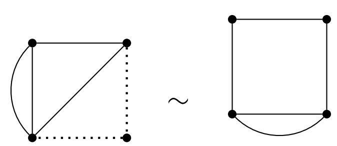
Future Topic 4.4. In relation to this final topic, we already have one result about the placement of nonalternating knots within the V-order:

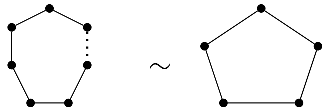
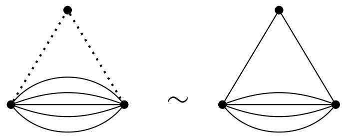
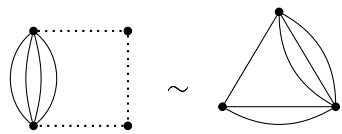
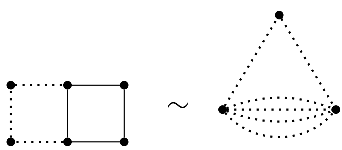
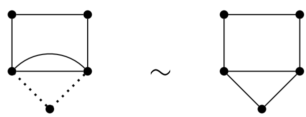
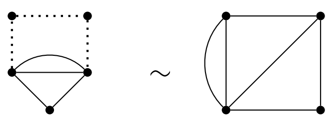
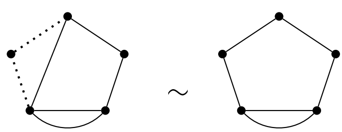
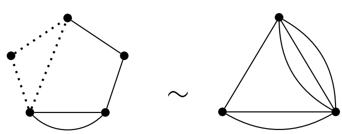
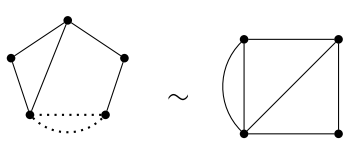
Theorem 4.5. *Let L_1 be a nonalternating link with $c(L_1) = n$. Then there exists an alternating link L_2, where $c(L_2) = n$, such that $L_1 \leq L_2$.*

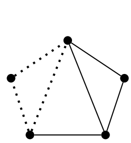
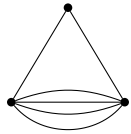
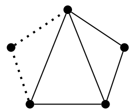
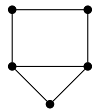
Proof. If L_1 is a nonalternating link with $c(L_1) = n$, the minimal graph for L_1 will have both dotted and solid edges with n edges total. If we change all the dotted edges to solid, we now have a graph of a link L_2 with all solid edges. Since this projection is reduced alternating, Theorem 1.2 implies that this graph of L_2 is minimal. So we have a minimal graph of L_2 with crossing number n. We also can see that $L_1 \leq L_2$ since we are able to transform a minimal diagram of L_2 into L_1 via crossing changes. □

Appendix: Expansion of the Hasse diagram

Here we exhibit the calculations that yielded our expansion of the Hasse diagram in Section 2. For each edge in the diagram, which corresponds to $K_1 \leq K_2$, we show a minimal diagram of K_2 with the crossing changes needed to produce the direct V-minor K_1.

K_2		K_1
3_1		0_1
4_1		3_1
5_1		3_1
5_2		4_1
6_1		5_2
6_2		5_1
6_2		5_2
6_3		5_1
6_3		5_2

K_2		K_1
7_1		5_1
7_2		6_1
7_3		6_2
7_4		6_1
7_5		6_2
7_5		6_3
7_6		6_1
7_6		6_2
7_6		6_3

K_2				K_1
7_7		$\sim$		6_1
7_7		$\sim$		6_2

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arazelle.mendoza.09@cnu.edu *Christopher Newport University, Newport News, VA 23606, United States*

tara.sargent@clarke.edu *Clarke University, Dubuque, IA 52001, United States*

jts0012@uah.edu *University of Alabama in Huntsville, Huntsville, AL 35816, United States*

paul.drube@valpo.edu *Department of Mathematics and Computer Science, Valparaiso University, 1900 Chapel Drive, Valparaiso, IN 46383, United States*

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