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Maximization of the size of monic orthogonal polynomials on the unit circle corresponding to the measures in the Steklov class

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We investigate the size of monic, orthogonal polynomials defined on the unit circle corresponding to a finite positive measure. We find an upper bound for the L_∞ growth of these polynomials. Then we show, by example, that this upper bound can be achieved. Throughout these proofs, we use a method developed by Rahmanov to compute the polynomials in question. Finally, we find an explicit formula for a subsequence of the Verblunsky coefficients of the polynomials.

1. Introduction

Let $V = C(\mathbb{T};\mathbb{C})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We define an inner product on V by

$$\langle f, g\rangle_{d\mu} = \int_{\mathbb{T}} f(z)\overline{g(z)}\, d\mu,$$

where $d\mu$ is of the form

$$d\mu = p(\theta)\, d\theta + \sum_{j=1}^{n} m_j\, \delta(\theta - \theta_j),$$

where $p(\theta)$ is a continuous function, δ is the Dirac delta function, and the m_j are masses placed at the θ_j satisfying $m_j \geq 0$. We will confine our analysis to measures in the restricted Steklov class of order δ, denoted S_δ, which consists of measures with the properties

$$p(\theta) > \delta, \quad m_j \geq 0, \quad \langle 1, 1\rangle_{d\mu} = 2. \tag{1-1}$$

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This inner product gives a norm $\|\cdot\|$ defined as

$$\|f(z)\|_{d\mu} = \sqrt{\langle f, f\rangle_{d\mu}}.$$

Given a measure $d\mu \in S_\delta$, there exists a unique set of monic orthogonal polynomials $\{\phi_n(z; d\mu)\}$ [Simon 2005]. We will adopt the convention that $\phi_n(z; d\mu)$ is the polynomial of degree n in this set. When there is no ambiguity about what measure is being used, we will simply write these polynomials as $\phi_n(z)$. Corresponding to the set $\{\phi_n(z; d\mu)\}$ is the set $\{\varphi_n(z)\}$ of orthonormal polynomials, defined by

$$\varphi_n(z) = \frac{\phi_n(z)}{\|\phi_n(z)\|}.$$

These polynomials form an orthonormal set. Uniqueness of this set follows the from uniqueness of $\{\phi_n(z)\}$.[1]

A conjecture of Steklov stated that the sequence

$$M_{n,\delta} = \sup_{d\mu\in S_\delta} \max_{z\in\mathbb{T}} |\varphi_n(z; d\mu)|$$

is bounded in n. This was disproven by Rahmanov [1979]. In particular, Rahmanov proved the existence of a probability measure $d\eta = \phi(\theta)\, d\theta + \sum_{j=1}^{n} m_j\, \delta(\theta - \theta_j)$ such that

$$|\varphi_n(1, d\eta)| \geq C \ln(n) + B$$

for some constants B, C. The hard part in making such estimates is that, in general, there are few tools available to compute $\varphi_n(z)$ other than the Gram–Schmidt process. To establish his result, Rahmanov found a formula for computing the $\phi_n(z; d\eta)$, where $d\eta = d\mu + \sum_{k=0}^{n} m_j\, \delta(\theta-\theta_j)$ in terms of $\phi_n(z; d\mu)$, meaning that $d\eta$ differs from $d\mu$ only in its masses. This formula uses the Christoffel–Darboux kernel

$$K_n(z, \xi) = \sum_{j=0}^{n} \overline{\varphi_j(z)}\varphi_j(\xi). \tag{1-2}$$

The roots of the Christoffel–Darboux kernel are those ξ_j satisfying

$$K_n(\xi_j, \xi_i) \begin{cases} = 0 & \text{for } i \neq j, \\ \neq 0 & \text{for } i = j. \end{cases} \tag{1-3}$$

Rahmanov's formula, in light of these definitions, is

$$\phi_n(z; d\eta) = \phi_n(z; d\mu) - \sum_{j=1}^{n} \frac{m_j \phi_n(\xi_j; d\mu)}{1 + m_j K_{n-1}(\xi_j, \xi_j)} K_{n-1}(z, \xi_j). \tag{1-4}$$

[1] Originally, the last condition given for the Steklov class is stated as $\langle 1, 1\rangle_{d\mu} = 1$. This is a minor modification though, because $\varphi_n(z)/\sqrt{2}$ is in the Steklov class $S_{\delta/\sqrt{2}}$, given the original definition.

We now outline our results. In Theorem 2.3, we use Rahmanov's method for computing $\phi_n(\theta)$ for the measure $d\eta = d\theta/2\pi + \sum_{j=1}^{\lfloor n/4 \rfloor} m_j\,\delta(\theta-\theta_j)$, with $\theta_j = (2\pi j - \pi)/n$, $m_j = 4/n$ and show that the corresponding polynomials $\phi_n(z; d\mu)$ are uniformly bounded above by $8/(5\pi^2)\log(\lfloor n/4 \rfloor - 1) + C$, where C is a constant. Our next main result is Theorem 4.1, where we construct a family of measures $d\mu_n$ such that $\phi_n(1; d\mu_n) > 1/\pi \log n + c$, c a bounded constant. Finally, in Theorem 5.1, we show that, given the measure $d\mu = d\theta/2\pi + \sum_{j=1}^{n} m_j\,\delta(\theta-\theta_j)$, with $\theta_j = 2\pi j/n - \theta_0$, the subsequence $\{\alpha_{nk-1}\}_{k=1}^{\infty}$ of the Verblunsky coefficients $\alpha_j(d\mu)$ satisfies

$$\alpha_{nk-1} = e^{ink\theta_0} \sum_{j=1}^{n} \frac{m_j}{1+m_j nk}.$$

The reader may notice that all of our results are stated in terms of $\phi_n(z)$, while Steklov's conjecture is stated in terms of $\varphi_n(z)$. We will end this introduction with a lemma, proven by Rahmanov [1979], that shows why bounds on $\phi_n(z)$ imply bounds on $\varphi_n(z)$, and thus why it is sufficient to evaluate $\{\phi_n(z)\}$.

Lemma 1.1. *Given a measure $d\mu \in S_\delta$, $\delta > 0$, there exists a constant C such that*

$$\frac{1}{C}\|\phi_n(z, d\mu)\| \le \|\varphi_n(z, d\mu)\| \le C\|\phi_n(z, d\mu)\|$$

for all $n \ge 0$.

Proof. Since

$$|\varphi_n(z)| = \frac{|\phi_n(z)|}{\|\phi_n(z)\|_{d\mu}},$$

it suffices to find constant upper and lower bounds on $\|\phi_n(z)\|_{d\mu}$.

To find an upper bound, we first claim that $\phi_n(z)$ minimizes the integral

$$\int_{\mathbb{T}} |P(z)|^2\,d\mu,$$

where $P(z)$ is any monic polynomial of degree n. Let $q(z)$ be an arbitrary polynomial of degree less than n. Then, since $\phi_n(z)$ is orthogonal to all polynomials of degree less than n under the measure $d\mu$ and the inner product of a polynomial with itself is nonnegative, we have

$$\begin{aligned}
&\langle \phi_n(z)+q(z), \phi_n(z)+q(z)\rangle_{d\mu} \\
&\qquad = \langle \phi_n(z), \phi_n(z)\rangle_{d\mu} + \langle \phi_n(z), q(z)\rangle_{d\mu} + \langle q(z), \phi_n(z)\rangle_{d\mu} + \langle q(z), q(z)\rangle_{d\mu} \\
&\qquad = \langle \phi_n(z), \phi_n(z)\rangle_{d\mu} + \langle q(z), q(z)\rangle_{d\mu} \\
&\qquad \ge \langle \phi_n(z), \phi_n(z)\rangle_{d\mu}.
\end{aligned}$$

Hence $\phi_n(z)$ minimizes the integral $\int_{\mathbb{T}} |P(z)|^2\, d\mu$. In particular this gives us

$$\|\phi_n(z)\|_{d\mu}^2 = \int_0^{2\pi} |\phi_n(z)|^2\, d\mu \le \int_0^{2\pi} |z^n|^2\, d\mu = \int_0^{2\pi} 1\, d\mu = 2. \tag{1-5}$$

We can derive a lower bound using the fact that $d\mu \in S_\delta$ and, in particular, that $d\mu$ satisfies (1-1), which gives

$$\begin{aligned}\|\phi_n(z)\|_{d\mu}^2 &= \frac{1}{2\pi}\int_0^{2\pi} |\phi_n(e^{i\theta})|^2 p(\theta)\, d\theta + \sum_{j=1}^{l} m_j |\phi_n(e^{i\theta_j})|^2 \\ &\ge \frac{\delta}{2\pi}\int_0^{2\pi} |\phi_n(e^{i\theta})|^2\, d\theta.\end{aligned}$$

Let the coefficient of the z^k term of $\phi_n(z)$ be a_k. In particular, $a_n = 1$. Using that the integral of $e^{ik\theta}$ over the unit circle is 0 for a nonzero integer k, we get

$$\int_0^{2\pi} |\phi_n(e^{i\theta})|^2\, d\theta = \int_0^{2\pi} \sum_{k=0}^{n} a_k^2\, d\theta \ge \int_0^{2\pi} a_n^2\, d\theta = 2\pi.$$

Hence, $\|\phi_n(z)\|_{d\mu}^2 \ge \delta$.

Combining this with the upper bound on $\|\phi_n(z)\|_{d\mu}^2$ from (1-5) gives

$$\delta \le \|\phi_n(z)\|_{d\mu}^2 \le 2,$$

and as a result

$$\frac{|\phi_n(z)|}{\sqrt{2}} \le |\varphi_n(z)| \le \frac{|\phi_n(z)|}{\sqrt{\delta}}. \qquad \square$$

2. Review of Rahmanov's result

We begin by reviewing Rahmanov's argument [1979] to show that the growth of the monic polynomials under Rahmanov's scheme is bounded below by $c \log n$, where c is a constant. Before doing that, though, we need to prove two lemmas that simplify our future calculations.

We now characterize the roots of the Christoffel–Darboux kernel (which we defined on page 2) for a certain measure:

Lemma 2.1. *For $d\mu = d\theta/2\pi$, the roots of the Christoffel–Darboux kernel are the n-th roots of unity times a constant of modulus one.*

Proof. Recall the definition of the Christoffel–Darboux kernel and its roots from (1-2) and (1-3). For our $d\mu$, $\varphi_j = z^j$, so assuming ξ_j is of modulus one for all j,

$$\begin{aligned} K_{n-1}(\xi_i, \xi_j) &= \sum_{j=0}^{n-1} \xi_i^j / \xi_j^j \\ &= \frac{\xi_i^n/\xi_j^n - 1}{\xi_i/\xi_j - 1} \qquad \text{(since this is a geometric series)} \\ &= \frac{\xi_i^n - \xi_j^n}{(\xi_i - \xi_j)\xi_j^{n-1}}. \end{aligned} \tag{2-1}$$

Therefore, by (2-1), $\xi_j = e^{2i\pi j/n}\xi_0$, where $1 \le j \le n$ and ξ_0 is an arbitrary point on the unit circle, and so ξ_j is an n-th root of unity times a constant. □

Lemma 2.2. *We need only assess $\phi_n(z; d\mu)$ at $z = 1$, since*

$$\sup_{\mu\in S_\delta} \max_{z\in\mathbb{T}} |\phi_n(z; d\mu)| = \sup_{\mu\in S_\delta} |\phi_n(1; d\mu)|.$$

Proof. Let

$$d\mu_1 = p(\theta)\, d\theta + \sum_{j=1}^{m} m_j\, \delta(\theta - \theta_j),$$

where $d\mu_1 \in S_\delta$. Then $d\mu_2 \in S_\delta$, where

$$d\mu_2 = p(\theta - \theta^*)\, d\theta + \sum_{j=1}^{m} m_j\, \delta(\theta - \theta^* - \theta_j), \quad \theta^* \in [0, 2\pi).$$

In particular, $\phi_n(e^{i\theta}; d\mu_1) = \phi_n(e^{i(\theta+\theta^*)}; d\mu_2)$.

Hence,

$$\max_{z\in\mathbb{T}} |\phi_n(z; d\mu_1)| = \max_{z\in\mathbb{T}} |\phi_n(z; d\mu_2)|,$$

$$\sup_{\mu\in S_\delta} \max_{z\in\mathbb{T}} |\phi_n(z; d\mu)| = \sup_{\mu\in S_\delta} |\phi_n(1; d\mu)|.$$

Henceforth, we will only look at $\phi_n(z; d\mu)$ evaluated at $z = 1$. □

Theorem 2.3. *Under a finite measure $d\eta = d\mu + \sum_{j=1}^{\lfloor n/4\rfloor} m_j\, \delta(\theta - \theta_j)$, the monic polynomials are not uniformly bounded from above; specifically, there exists a $d\eta$ such that the maximums are greater than or equal to $8/(5\pi^2)\log(\lfloor n/4\rfloor - 1)$.*

Remark 2.4. This is Rahmanov's result [1979], whose proof we have included for the reader's convenience.

Proof. First, we will deal generally with some $d\eta$ without specifying the added masses.

In light of Lemma 2.1, let $\theta_j = (2\pi j - \pi)/n$ for $1 \le j \le \lfloor n/4 \rfloor$. Then, using Rahmanov's formula in (1-4), we have

$$\phi_n(z; d\eta) = \phi_n(z; d\mu) - \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j \phi_n(\xi_j; d\mu)}{1 + m_j K_{n-1}(\xi_j, \xi_j)} K_{n-1}(z, \xi_j), \tag{2-2}$$

which, by noting that $K_{n-1}(\xi_j, \xi_j) = \sum_{j=1}^{n-1} 1 = n$ and substituting z and ξ_j into (2-2), becomes

$$\begin{aligned} \phi_n(z; d\eta) &= z^n - \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j \xi_j^n}{1 + m_j n} \frac{-z^n - 1}{z e^{-i\theta_j} - 1} \\ &= z^n + \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j}{1 + m_j n} \frac{z^n + 1}{1 - z e^{-i\theta_j}}. \end{aligned}$$

Now we want to find a lower bound for $|\phi_n|$:

$$\max_{z \in \mathbb{T}} |\phi_n(z; d\eta)| \ge \max_{z \in \mathbb{T}} \left| \operatorname{Im}\left(z^n + \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j}{1 + m_j n} \frac{z^n + 1}{1 - z e^{-i\theta_j}} \right) \right|.$$

We take $z = 1$, in line with Lemma 2.2, to get

$$\begin{aligned} \max_{z \in \mathbb{T}} |\phi_n(z; d\eta)| &\ge \left| \operatorname{Im}\left(1 + 2 \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j}{1 + m_j n} \frac{1}{1 - e^{-i\theta_j}} \right) \right| \\ &= \left| \operatorname{Im}\left(1 - 2 \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j}{1 + m_j n} \frac{e^{i\theta_j} - 1}{|1 - e^{-i\theta_j}|^2} \right) \right|. \end{aligned}$$

Note that $0 < \theta_j < \pi/2$, $|1 - e^{-i\theta_j}| \le \theta_j$, and

$$\operatorname{Im}(e^{i\theta_j} - 1) = \sin\theta_j \ge \frac{2\theta_j}{\pi} \quad \text{for } \theta \in \left(0, \frac{\pi}{2}\right),$$

which gives

$$\left| \operatorname{Im}\left(1 - 2 \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j}{1 + m_j n} \frac{e^{i\theta_j} - 1}{|1 - e^{-i\theta_j}|^2} \right) \right| \ge 2 \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{m_j}{1 + m_j n} \frac{2}{\pi \theta_j}. \tag{2-3}$$

Now, we specify the masses of $d\eta$ to get a precise bound. Let $m_j = 4/n$ for all j. This simplifies (2-3) to

$$\max_{z \in \mathbb{T}} |\phi_n(z; d\eta)| \ge \frac{16}{5\pi n} \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{1}{(2j-1)\frac{\pi}{n}} \ge \frac{8}{5\pi^2} \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{1}{j}.$$

Note that $\log a = \int_1^a 1/x\, dx \geq \sum_{j=1}^{a-1} 1/j$ since $1/x$ is decreasing.

Therefore,

$$\max_{z\in\mathbb{T}} |\phi_n(z; d\eta)| \geq \frac{8}{5\pi^2} \log\Big(\Big\lfloor \frac{n}{4} \Big\rfloor - 1\Big). \tag{2-4}$$

Since $4/(5\pi^2)\log(\lfloor n/4\rfloor - 1)$ is strictly increasing in n, $\max_{z\in\mathbb{T}} |\phi_n(z, d\eta)|$ is not uniformly bounded from above. □

3. Finding a general upper bound

In this section, we find a general upper bound for the growth of the monic orthogonal polynomials under a $d\eta$ which differs from $d\theta/2\pi$ only in the discrete portion. We prove the following theorem by making a sequence of overestimates of $|\phi_n(1, d\eta)|$.

Theorem 3.1. *Let $d\eta$ be a measure such that*

$$d\eta = \frac{1}{2\pi}\, d\theta + \sum_{j=1}^{n} m_j\, \delta(\theta - \theta_j),$$

where $m_j \geq 0$, $\theta_j = 2\pi j/n + \theta_0$ for $1 \leq j \leq n$, and $\theta_0 \in [0, 2\pi)$.

Then

$$|\phi_n(1, d\eta)| \leq \tfrac{1}{\pi} \log n + C,$$

where C is a constant uniformly bounded in n.

Remark 3.2. Note the generalized offset θ_0 in the theorem. In Section 2, we used the specific offset of $\theta_0 = -\pi/n$, but here, we find a general upper bound under any offset.

We prove the theorem using two lemmas. The first, Lemma 3.3, finds an overestimate for $|\phi_n(1, d\eta)|$ using Rahmanov's formula [1979] . The second, Lemma 3.4, makes another overestimate using Taylor series.

Lemma 3.3. *Let $d\eta$ be a measure such that*

$$d\eta = \frac{1}{2\pi}\, d\theta + \sum_{j=1}^{n} m_j\, \delta(\theta - \theta_j),$$

where $m_j \geq 0$ and $\theta_j = 2\pi j/n + \theta_0$, $1 \leq j \leq n$.

Then

$$|\phi_n(1)| = \begin{cases} \frac{|1-e^{in\theta_0}|}{2} \left| \sum_{j=1}^{n} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j} \right| + c_n & \text{if } \theta_0 \neq 0, \\ c_n & \text{if } \theta_0 = 0, \end{cases}$$

where $|c_n| < 2$ for all n.

Proof. We first consider the case where $\theta_0 = 0$. If $\theta_0 = 0$, then $K_{n-1}(1, e^{i\theta_j}) = 0$ for $1 \le j < n$ and $K_{n-1}(1, e^{i\theta_n}) = n$. From Rahmanov's formula (1-4),

$$|\phi_n(1)| = \left|1 - \frac{m_n}{1 + m_n n} n\right| < 1.$$

Therefore, $\phi_n(1)$ is not increasing in n for $\theta_0 = 0$. Henceforth, we restrict ourselves to working with $\theta_0 \neq 0$.

From Rahmanov's formula in (1-4) and applying Lemma 2.1, we derive

$$\phi_n(1) = 1 - \sum_{j=1}^{n} \frac{m_j}{1 + m_j n} e^{in\theta_j} \frac{\overline{e^{in\theta_j}} - 1}{\overline{e^{i\theta_j}} - 1}.$$

Then, using algebra, we find that

$$\begin{aligned} \phi_n(1) &= 1 + \sum_{j=1}^{n} \frac{m_j}{1 + m_j n} \frac{1 - e^{in\theta_j}}{1 - \overline{e^{i\theta_j}}} \\ &= 1 + \frac{1 - e^{in\theta_0}}{2} \sum_{j=1}^{n} \frac{m_j}{1 + m_j n} \left(1 - i \frac{\sin\theta_j}{1 - \cos\theta_j}\right), \end{aligned} \tag{3-1}$$

which implies by the triangle inequality that

$$\left|\phi_n(1) - \frac{1 - e^{in\theta_0}}{2} i \sum_{j=1}^{n} \frac{m_j}{1 + m_j n} \frac{\sin\theta_j}{1 - \cos\theta_j}\right| \le 1 + \frac{|1 - e^{in\theta_0}|}{2} \left|\sum_{j=1}^{n} \frac{m_j}{1 + m_j n}\right|.$$

Note that

$$|1 - e^{in\theta_0}| \le 2 \quad \text{and} \quad 0 \le \frac{m_j}{1 + m_j n} \le \frac{1}{n}, \quad \text{so} \quad \frac{|1 - e^{in\theta_0}|}{2} \left|\sum_{j=1}^{n} \frac{m_j}{1 + m_j n}\right| \le 1.$$

Hence,

$$\left|\phi_n(1) - \frac{1 - e^{in\theta_0}}{2} i \sum_{j=1}^{n} \frac{m_j}{1 + m_j n} \frac{\sin\theta_j}{1 - \cos\theta_j}\right| \le 2. \qquad \square$$

Thus, it is sufficient to consider the growth of

$$\frac{|1 - e^{in\theta_0}|}{2} \left|\sum_{j=1}^{n} \frac{m_j}{1 + m_j n} \frac{\sin\theta_j}{1 - \cos\theta_j}\right|. \tag{3-2}$$

We want to eliminate the magnitude around the sum in (3-2).

Since $m_j \ge 0$, and $\sin\theta_j/(1 - \cos\theta_j)$ is positive on $(0, \pi)$ and negative on $(\pi, 2\pi)$, we have

$$\left|\sum_{j=1}^{n} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j}\right| \le \max\left\{\left|\sum_{\theta_j\in(0,\pi)} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j}\right|, \left|\sum_{\theta_j\in(\pi,2\pi)} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j}\right|\right\}.$$

Now, if we alter $d\eta$ so that the masses are instead located at $\theta_j* = -2\pi j/n - \theta_0$, essentially reflecting the discrete portion of the measure over the real axis, we see that (3-2) does not change.

Hence, since we are looking to find an upper bound of (3-1) that is independent of m_j and θ_0, we can assume without loss of generality that we are only looking at $\theta_j \in (0, \pi)$, and thus take

$$\frac{|1-e^{in\theta_0}|}{2}\left|\sum_{j=1}^{n} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j}\right| \le \frac{|1-e^{in\theta_0}|}{2}\sum_{\theta_j\in(0,\pi)} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j}.$$

Since replacing θ_0 with $\theta_0 + 2\pi/n$ and then shifting the index of the m_j does not affect the value of (3-2), assume $\theta_0 \in (-2\pi/n, 0)$. Having made these simplifications, we can now move on to the main lemma, which finds an upper bound as described in the theorem.

Lemma 3.4.

$$\frac{|1-e^{in\theta_0}|}{2}\sum_{\theta_j\in(0,\pi)} \frac{\sin\theta_j}{1-\cos\theta_j} \le n\left(1+\frac{1}{\pi}+\frac{1}{\pi}\log\left\lfloor\frac{n}{2}\right\rfloor\right),$$

where $\theta_j = 2\pi j/n + \theta_0, \quad \theta_0 \in (-2\pi/n, 0)$.

Proof. We separate the first term from the sum, since that term contributes the most to the magnitude. Recall that $\theta_1 = 2\pi/n + \theta_0$. Thus,

$$\frac{|1-e^{in\theta_0}|}{2}\sum_{\theta_j\in(0,\pi)} \frac{\sin\theta_j}{1-\cos\theta_j} \le \frac{|1-e^{in\theta_0}|}{2}\frac{\sin\theta_1}{1-\cos\theta_1} + \sum_{\theta_j\in(2\pi/n,\pi)} \frac{\sin\theta_j}{1-\cos\theta_j}$$

since $|1-e^{in\theta_0}| \le 2$. We now bound these two terms of the sum separately.

We claim that

$$\frac{|1-e^{in\theta_0}|}{2}\frac{\sin\theta_1}{1-\cos\theta_1} \le n$$

for $\theta_0 \in (-2\pi/n, 0)$. Recall $\theta_1 = \theta_0 + 2\pi/n$, so hence $|1-e^{in\theta_0}| = |1-e^{in\theta_1}|$. Denote θ_1 by t, where $t \in (0, 2\pi/n)$. We do the calculation

$$\frac{|1-e^{int}|}{2}\frac{\sin t}{1-\cos t}$$
$$=\frac{\sqrt{(1-\cos nt)^2+(\sin nt)^2}}{2}\frac{\sin t}{2\left(\sin\frac{t}{2}\right)^2}=\frac{\sqrt{2-2\cos nt}}{2}\frac{\sin t}{2\left(\sin\frac{t}{2}\right)^2}.$$

Because $\sin(nt/2)$ is nonnegative for $t\in(0,2\pi/n)$, we have

$$\sin\frac{nt}{2}\frac{\sin t}{2\left(\sin\frac{t}{2}\right)^2}=\frac{\sin\frac{nt}{2}\left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)}{2\left(\sin\frac{t}{2}\right)^2}=\frac{\sin\frac{nt}{2}\cos\frac{t}{2}}{\sin\frac{t}{2}},$$

$\sin(nt/2)/\sin(t/2)$ is nonnegative for $t\in(0,2\pi/n)$ and $\cos t/2$ is bounded above by 1. Hence, the expression is bounded above by $\sin(nt/2)/\sin(t/2)$. It remains to show this is bounded above by n.

This is clearly true for $n=1$. Let $n>1$. Recall that $nt\in(0,2\pi)$. Consider an $(n+1)$-gon inscribed in a unit circle in which n of the sides of the polygon form a central angle of t. The last side of the polygon forms a central angle of nt (this angle may be reflexive.) Recall that the length of a chord of a unit circle which forms a central angle of t is $2\sin(t/2)$. Similarly the length of the chord which forms a central angle of nt is $2\sin(nt/2)$. As the polygon is not degenerate, the sum of the lengths of the n equal side lengths is greater than the length of the remaining side length. Namely $2n\sin(t/2)\geq 2\sin(nt/2)$ as desired.

We now handle the second term. To bound

$$\sum_{\theta_j\in(2\pi/n,\pi)}\frac{\sin\theta_j}{1-\cos\theta_j},$$

note that $\sin\theta_j/(1-\cos\theta_j)$ is decreasing on $(0,2\pi)$, so

$$\sum_{\theta_j\in(2\pi/n,\pi)}\frac{\sin\theta_j}{1-\cos\theta_j}\leq\sum_{j=1}^{\lfloor n/2\rfloor}\frac{\sin\left(\frac{2\pi}{n}j\right)}{1-\cos\left(\frac{2\pi}{n}j\right)}.$$

Recall from the Taylor expansion that we can approximate $\sin x/(1-\cos x)$ near 0 by $2/x$. In fact, since

$$\lim_{x\to 0}\frac{\sin x}{1-\cos x}-\frac{2}{x}=0$$

and for $x\in(0,\pi]$, we have

$$\frac{d}{dx}\left(\frac{\sin x}{1-\cos x}-\frac{2}{x}\right)<0,$$

we arrive at the inequality

$$\frac{\sin x}{1-\cos x}\leq\frac{2}{x}\quad\text{for } x\in(0,\pi].$$

Therefore,

$$\sum_{j=1}^{\lfloor n/2\rfloor} \frac{\sin\left(\frac{2\pi}{n} j\right)}{1-\cos\left(\frac{2\pi}{n} j\right)} \le \sum_{j=1}^{\lfloor n/2\rfloor} \frac{2}{\frac{2\pi}{n} j} = \frac{n}{\pi} \sum_{j=1}^{\lfloor n/2\rfloor} \frac{1}{j}.$$

Recall that $\log a = \int_1^a 1/x\, dx \ge \sum_{j=2}^{a} 1/j$ since $1/x$ is decreasing, so that

$$\frac{n}{\pi} \sum_{j=1}^{\lfloor n/2\rfloor} \frac{1}{j} = \frac{n}{\pi} + \frac{n}{\pi} \sum_{j=2}^{\lfloor n/2\rfloor} \frac{1}{j} \le \frac{n}{\pi} + \frac{n}{\pi} \log\left\lfloor \frac{n}{2} \right\rfloor,$$

and thus

$$\frac{|1-e^{in\theta_0}|}{2} \sum_{\theta_j \in (0,\pi)} \frac{\sin\theta_j}{1-\cos\theta_j} \le n\left(1+\frac{1}{\pi}+\frac{1}{\pi}\log\left\lfloor\frac{n}{2}\right\rfloor\right). \qquad \square$$

Returning to the statement of the theorem,

$$\begin{aligned} |\phi_n(1)| &\le 2 + \frac{|1-e^{in\theta_0}|}{2} \left| \sum_{j=1}^{n} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j} \right| \\ &\le 2 + \frac{|1-e^{in\theta_0}|}{2} \left| \sum_{j=1}^{n} \frac{1}{n} \frac{\sin\theta_j}{1-\cos\theta_j} \right| \\ &\le 3 + \frac{1}{\pi} + \frac{1}{\pi} \log\left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Since $\log\lfloor n/2 \rfloor$ is equal to $\log n$ plus some uniformly bounded term, we can conclude that

$$|\phi_n(1)| \le \tfrac{1}{\pi} \log n + C,$$

where C is constant in n, which completes the proof of the theorem.

Remark 3.5. Note that here we used that

$$\frac{m_j}{1+m_j n} \le \frac{1}{n}.$$

If we were to use the Rahmanov scheme of distributing masses and set all $m_j = 1/n$ then

$$\frac{m_j}{1+m_j n} = \frac{1}{2n},$$

and the monic orthogonal polynomials given by the Rahmanov type of measure would have growth bounded from above by $1/2\pi \log n + b$, where b is a bounded constant.

4. Proving the lower bound

In this section, we construct a measure that achieves the upper bound of $1/\pi \log n$ plus a bounded term, as described in Theorem 3.1. We accomplish this primarily by applying the technique of Lagrange multipliers to find an optimal measure.

Theorem 4.1. *For all $n \in \mathbb{N}$, there exists a measure*

$$d\eta = \frac{1}{2\pi}\, d\theta + \sum_{j=1}^{n} m_j\, \delta(\theta - \theta_j),$$

where $m_j \geq 0$ and $\sum_{j=1}^{n} m_j = 1$ such that

$$|\phi_n(1, d\eta)| \geq \tfrac{1}{\pi} \log n + c,$$

where c is a bounded constant.

We will prove this theorem as a sequence of lemmas.

The first lemma, Lemma 4.2, finds a lower bound for the expression from Lemma 3.3 which is simpler to manipulate. In the second lemma, Lemma 4.4, we apply the technique of Lagrange multipliers to that lower bound to find a critical "point", in our case a scheme of m_js. Finally, in the third lemma, Lemma 4.6, we insert those derived m_j into the approximation and find that we achieve the growth stated in the theorem.

Set $\theta_j = (2\pi j - \pi)/n$. Inserting those θ_j into (3-1), we have that

$$|\phi_n(1)| = \frac{|1-e^{-\pi i}|}{2} \left| \sum_{j=1}^{n} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j} \right| + c_n = \left| \sum_{j=1}^{n} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j} \right| + c_n$$

for some constant $|c_n| < 2$. We know that $\sin\theta_j/(1-\cos\theta_j)$ is positive for $\theta_j \in (0, \pi)$ and negative for $\theta_j \in (\pi, 2\pi)$. Thus, in order to maximize $|\phi_n(1)|$, we set $m_j = 0$ for all j such that $\theta_j \in (\pi, 2\pi)$, which prevents destructive interference from the other side of the circle.

Under this setting, we can say that

$$\left| \sum_{j=1}^{n} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j} \right| = \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j}.$$

We next bound this equation from below with a simpler expression.

Lemma 4.2. *For $\theta_j = (2\pi j - \pi)/n$ and $m_j \geq 0$,*

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \frac{m_j}{1+m_j n} \frac{\sin\theta_j}{1-\cos\theta_j} \geq \frac{1}{\pi} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{n m_j}{1+m_j n} \frac{1}{j} + d,$$

where d is some constant.

Remark 4.3. It may appear contradictory that we first find a lower bound when we want the n-th degree monic polynomial to be as large as possible. However, this lower bound is easier to manipulate, and we show in the subsequent lemmas that it actually achieves the growth stated in the theorem.

Proof. We prove this lemma using two approximations. We first approximate $\sin\theta_j/(1-\cos\theta_j)$ by $2/\theta_j$, from the Taylor series; we then approximate $2/\theta_j$ by $1/(\pi j)$.

First, we show that $2/\theta_j$ is a good approximation of $\sin\theta_j/(1-\cos\theta_j)$. Let

$$M = \max_{\theta_j\in[0,\pi]}\left|\frac{\sin\theta_j}{1-\cos\theta_j} - \frac{2}{\theta_j}\right|.$$

This maximum, M, is achieved because

$$\left|\frac{\sin\theta_j}{1-\cos\theta_j} - \frac{2}{\theta_j}\right|$$

is continuous in an open neighborhood containing $[0,\pi]$. Thus, $2/\theta_j$ is a good approximation and we can bound the following difference by a constant:

$$\begin{aligned}&\left|\sum_{j=1}^{\lfloor n/2\rfloor}\frac{m_j}{1+m_jn}\frac{\sin\theta_j}{1-\cos\theta_j} - \sum_{j=1}^{\lfloor n/2\rfloor}\frac{m_j}{1+m_jn}\frac{2}{\theta_j}\right| \\ &\qquad\le \sum_{j=1}^{\lfloor n/2\rfloor}\frac{m_j}{1+m_jn}\left|\frac{\sin\theta_j}{1-\cos\theta_j} - \frac{2}{\theta_j}\right| \le \sum_{j=1}^{\lfloor n/2\rfloor}\frac{1}{n}M \le M.\end{aligned}$$

Having established this, we can now replace $2/\theta_j$ with $2n/((2j-1)\pi)$ and attain the inequality

$$\sum_{j=1}^{\lfloor n/2\rfloor}\frac{m_j}{1+m_jn}\frac{2}{\theta_j} \ge \frac{n}{\pi}\sum_{j=1}^{\lfloor n/2\rfloor}\frac{m_j}{1+m_jn}\frac{1}{j}.$$

Combining this and the previous inequality proves the lemma. □

Now that we have the simplified lower bound

$$\frac{1}{\pi}\sum_{j=1}^{\lfloor n/2\rfloor}\frac{nm_j}{1+m_jn}\frac{1}{j},$$

we can apply the method of Lagrange multipliers to it in order to construct the m_j that prove the theorem.

Lemma 4.4. *Let $n\in\mathbb{N}$. Consider any $l\in\mathbb{N}$ with $l\le n$. Under the constraints $m_j\ge 0$ and $\sum_{j=1}^{l}m_j = 1$, we achieve the maximum of*

$$\sum_{j=1}^{l} \frac{m_j n}{(1+nm_j)j}$$

by setting

$$m_j = \frac{m_1}{\sqrt{j}} + \frac{1}{n}\left(\frac{1}{\sqrt{j}} - 1\right)$$

for all $1 \le j \le l$, *where*

$$m_1 = \left(1 + \frac{l}{n}\right) \frac{1}{\sum_{j=1}^{\lfloor n/2 \rfloor} 1/\sqrt{j}} - \frac{1}{n}.$$

Proof. Set up f, the function to be maximized, and the constraint g, where $\boldsymbol{m}$ is the vector listing all m_j:

$$\begin{aligned} f(\boldsymbol{m}) &= \sum_{j=1}^{l} \frac{m_j n}{(1+nm_j)j}, \\ g(\boldsymbol{m}) &= \sum_{j=1}^{l} m_j - 1 = 0. \end{aligned} \tag{4-1}$$

If f under the constraint g has a local extremum at $\boldsymbol{m}'$ and $\boldsymbol{m}'$ is not on the boundary, for example $m'_j > 0$ for all $1 \le j \le l$, then there is a $\lambda \in \mathbb{R}$ such that

$$\nabla f(\boldsymbol{m}') = \lambda \nabla g(\boldsymbol{m}'). \tag{4-2}$$

To simplify the following expressions, denote

$$\sum_{j=1}^{l} \frac{1}{\sqrt{j}} = \alpha(l).$$

Calculations yield that, for all j,

$$\frac{n}{(1+nm'_1)^2} = \frac{n}{j(1+nm'_j)^2},$$

which, substituting m'_1 and m'_j, gives m'_j in terms of m'_1, that is,

$$m'_j = \frac{m'_1}{\sqrt{j}} + \frac{1}{n}\left(\frac{1}{\sqrt{j}} - 1\right). \tag{4-3}$$

Inserting that expression for m'_j into $g(\boldsymbol{m}') = \sum_{j=1}^{l} m'_j - 1 = 0$ yields

$$m'_1 = \left(1 + \frac{l}{n}\right) \frac{1}{\alpha(l)} - \frac{1}{n}.$$

Remark 4.5. For all $1 \le j \le l$, we have $m'_j > 0$ since $\alpha(l) < \int_0^l 1/\sqrt{l}\,dl = 2\sqrt{l}$. Thus, we satisfy the condition that $m'_j \ge 0$.

To insure that, in the computation for $\boldsymbol{m}'$, the Lagrange multipliers method did in fact give us the $\boldsymbol{m}$ that maximized $f(\boldsymbol{m})$ under the constraint $m_j \ge 0$ and $\sum_{j=1}^{l} m_j = 1$, we must check the boundary. We next provide a quick proof that the maximum is not achieved at the boundary.

Consider the Lagrangian $L(\boldsymbol{m}) = f(\boldsymbol{m}) - \lambda g(\boldsymbol{m})$ defined on $(-1/n, \infty)^l$, where λ is the constant in (4-2). Note that $\boldsymbol{m}'$ is a critical point of L since $\boldsymbol{m}'$ satisfies $\nabla L = \nabla f - \lambda \nabla g = 0$. It suffices to show that L is concave on $(-1/n, \infty)^l$.

We first calculate the entries of the Lagrangian L:

$$\frac{\partial^2 L}{\partial m_j^2} = -\frac{2n^2}{j}\frac{1}{(1+nm_j)^3} < 0,$$
$$\frac{\partial^2 L}{\partial m_j \partial m_k} = 0 \quad \text{for } j \ne k.$$

The Hessian of L is then negative definite and hence L is concave on $(-1/n, \infty)^l$. Therefore, $\boldsymbol{m}'$ as computed in (4-3) is a point where L achieves a global maximum on the open neighborhood $(-1/n, \infty)^l$. In particular, $L(\boldsymbol{m}')$ is the maximum of L on the region defined by $m_j \ge 0$ and $\sum_{j=1}^{l} m_j = 1$, a subset of $(-1/n, \infty)^l$. On this region, $g = 0$, so $L = f$. Hence f, constrained to the aforementioned region, achieves a global maximum at $\boldsymbol{m}'$. □

We conclude the proof by calculating the value of

$$\sum_{j=1}^{l} \frac{m_j n}{(1+nm_j)j}$$

for m_j as described in Lemma 4.4. Since this function evaluated at $l = \lfloor n/2 \rfloor$ is a lower bound of $|\phi_n(1, d\eta)|$, as proved in Lemma 4.2, this final lemma concludes the proof of the theorem.

Lemma 4.6. *For the m_j described in Lemma 4.4 in (4-3),*

$$\sum_{j=1}^{l} \frac{m_j n}{(1+nm_j)j} = \tfrac{1}{\pi}\log l + c,$$

where c is uniformly bounded.

Proof. We simply evaluate f from (4-1) at the $\boldsymbol{m}'$ given by (4-3):

$$\begin{aligned} f(\boldsymbol{m}') &= \sum_{j=1}^{l} \frac{m'_j}{1+nm'_j}\,\frac{n}{j} = \sum_{j=1}^{l} \frac{\frac{1}{n}\big(\frac{1}{\sqrt{j}}(1+nm'_1)-1\big)}{1+n\frac{1}{n}\big(\frac{1}{\sqrt{j}}(1+nm'_1)-1\big)}\,\frac{n}{j} \\ &= \sum_{j=1}^{l} \frac{\frac{1}{\sqrt{j}}(1+nm'_1)-1}{\sqrt{j}(1+nm'_1)} = \sum_{j=1}^{l} \frac{1}{j} - \frac{1}{1+nm'_1}\,\alpha(l) \\ &= \sum_{j=1}^{l} \frac{1}{j} - \frac{\alpha(l)^2}{\big(1+\frac{l}{n}\big)n} \\ &= \sum_{j=1}^{l} \frac{1}{j} - \frac{\alpha(l)^2}{n+l}. \end{aligned}$$

Now $\sum_{j=1}^{l} 1/j$ differs from $\log l$ by at most 1, and $\alpha(l)^2/(n+l)$ is bounded in n and l since

$$0 \le \frac{\alpha(l)^2}{n+l} < \frac{(2\sqrt{l})^2}{n+l} = \frac{4l}{n+l} \le \frac{4n}{n} = 4.$$

Therefore, for the $\boldsymbol{m}'$ given by (4-3), $f(\boldsymbol{m}') = \log l + d_l$, where d_l is a constant bounded uniformly in l. In light of Lemma 4.2, we have constructed a $d\eta$ such that $|\phi_n(1, d\eta)| \ge 1/\pi \log n + c$, where c is a bounded constant, completing the proof of Theorem 4.1. □

5. Investigating higher degree polynomials

In the previous sections, we described the magnitude of monic polynomials of degree less than or equal to n, where n is the number of discrete masses in the measure, using Rahmanov's formula in (1-4). However, we also want to describe the higher degree monic polynomials, i.e., $\phi_{n'}(z; d\eta)$, where $n' > n$. Unfortunately, we are not able to do this for all $n' > n$, but we can partially describe $\phi_{n'}(z; d\eta)$, where $n' = kn$, $k \in \mathbb{N}$.

Recall the definition of Verblunsky coefficients [Simon 2005]:

$$\phi_{n+1}(z) = z\phi_n(z) - \bar{\alpha}_n \phi_n^*(z), \tag{5-1}$$

where

$$\begin{aligned} \phi_n(z) &= \beta_n z^n + \cdots + \beta_0, \quad 0 \le j \le n,\ \beta_j \in \mathbb{C}, \\ \phi_n^*(z) &= \bar{\beta}_0 z^n + \cdots + \bar{\beta}_n. \end{aligned}$$

In the $n' = kn$ case, we are able to derive the corresponding Verblunsky coefficients, and do so explicitly for a $d\rho$ similar to that of Rahmanov's in Section 2.

Theorem 5.1. *For a measure $d\eta = d\theta/2\pi + \sum_{j=1}^{n} m_j \delta(\theta-\theta_j)$, with masses located at $\xi_j = e^{i\theta_j}$ and $\theta_j = 2\pi j/n + \theta_0$ (cf. Lemma 2.1),*

$$\phi_{nk}(z, d\eta) = z^{nk} - \xi_0^{nk} \sum_{j=1}^{n} \frac{m_j}{1+m_j nk} K_{nk-1}(z, \xi_j),$$

and

$$\alpha_{nk-1} = \overline{\xi_0^{nk}} \sum_{j=1}^{n} \frac{m_j}{1+m_j nk},$$

where α_{nk-1} is a Verblunsky coefficient. Furthermore, under Rahmanov's scheme, where $\theta_j = 2\pi j/n$ and

$$d\rho = \frac{d\theta}{2\pi} + \sum_{j=1}^{n} \frac{\delta(\theta-\theta_j)}{n},$$

the Verblunsky coefficients are

$$\alpha_{nk-1} = \frac{1}{1+k}.$$

Proof. Note that, since $\phi_n(z; d\eta)$ is a monic polynomial, β_n from the above definition of the Verblunsky coefficients is 1, so

$$\phi_n^*(0; d\eta) = 1,$$

which by (5-1) implies

$$\phi_{n+1}(0; d\eta) = -\bar{\alpha}_n. \tag{5-2}$$

Having set out these preliminaries, we can simply apply Rahmanov's formula [1979] from (1-4) to find a formula for $\phi_{nk}(z; d\eta)$ under a measure $d\eta$ as described in the statement of Theorem 5.1:

$$\phi_{nk}(z; d\eta) = z^{nk} - \sum_{j=1}^{n} \frac{m_j \phi_{nk}(\xi_j; d\mu)}{1+m_j K_{nk-1}(\xi_j, \xi_j)} K_{nk-1}(z, \xi_j) \tag{5-3}$$

$$= z^{nk} - \xi_0^{nk} \sum_{j=1}^{n} \frac{m_j}{1+m_j nk} K_{nk-1}(z, \xi_j). \tag{5-4}$$

Remark 5.2. The simplification of the numerator from (5-3) to (5-4) depends upon the ξ_j being roots of unity times a constant (as in Lemma 2.1). Such a simplification is only possible in the ϕ_{nk} case, which is why the description of other higher-degree monic polynomials is considerably more complicated.

Now consider $z = 0$ to find the Verblunsky coefficients:

$$\phi_{nk}(0, d\eta) = -\xi_0^{nk} \sum_{j=1}^{n} \frac{m_j}{1 + m_j nk} K_{nk-1}(0, \xi_j) = -\xi_0^{nk} \sum_{j=1}^{n} \frac{m_j}{1 + m_j nk},$$

and, applying (5-2), we obtain

$$\begin{aligned} -\bar{\alpha}_{nk-1} &= \phi_{nk}(0, d\eta) = -\xi_0^{nk} \sum_{j=1}^{n} \frac{m_j}{1 + m_j nk}, \\ \alpha_{nk-1} &= \overline{\xi_0^{nk}} \sum_{j=1}^{n} \frac{m_j}{1 + m_j nk}. \end{aligned} \tag{5-5}$$

If we now take $\theta_0 = 0$, as Rahmanov does, and

$$d\rho = \frac{d\theta}{2\pi} + \sum_{j=1}^{n} \frac{\delta(\theta - \theta_j)}{n},$$

then (5-5) simplifies to

$$\alpha_{nk-1} = \frac{1}{1 + k}. \tag{5-6}$$

□

Remark 5.3. It is noteworthy that, as k grows, the α_{nk-1} decay at the rate of $1/(1 + k)$. In light of the fact that $\sum_{j=1}^{\infty} \alpha_j^2 < \infty$ [Simon 2005], this suggests that the α_j are small for $j \in (n(k-1), nk)$, where $k \in \mathbb{N}$, and increase rapidly near $j = kn$. However, as mentioned above, describing $\phi_j(z; d\eta)$ for $j \neq kn$ is much more complicated.

Appendix: Numerical appendix

In order to help visualize the results of this paper, the graphs of the magnitudes of four orthogonal monic polynomials induced by four respective measures have been included at the end of this section. Each measure has a continuous portion of $d\theta/2\pi$ as well as masses placed at $\theta_j = \pi/n(2j - 1)$, where $1 \leq j \leq n/2$ (cf. Lemma 2.1). For simplicity, throughout this section, we will consider only even n. For the first two polynomials (displayed in Figure 1), masses of uniform size $2/n$ are used as suggested by Rahmanov (see Section 2). For the second two (Figure 2), the masses are given their weights according to (4-3).

These graphs have several key features in common, including the presence of two peaks that grow in n: one at $\theta = 0$ and another at $\theta = \pi$. Also, both have much lower minimums in the range $0 \leq \theta \leq \pi$ than in $-\pi \leq \theta \leq 0$. Upon closer inspection, it can be seen that the two peaks in Figure 1 are equal; in contrast, in

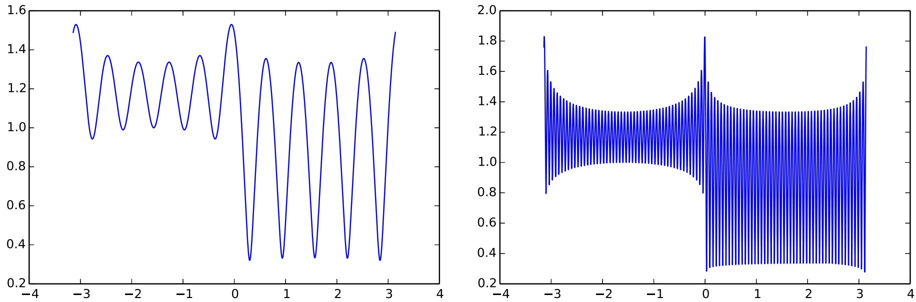

Figure 1. Left: $|\phi_{10}(\theta)|$ for $\theta_j = \frac{\pi}{10}(2j-1)$ and $m_j = \frac{1}{5}$, where $1 \le j \le 5$. Right: $|\phi_{100}(\theta)|$ for $\theta_j = \frac{\pi}{100}(2j-1)$ and $m_j = \frac{1}{50}$, where $1 \le j \le 50$.

Figure 2, the peak at $\theta = 0$ is larger than the peak at $\theta = \pi$. Additionally, the peak at $\theta = 0$ in the latter case is higher than in the former, as predicted by Theorem 4.1.

To explain some of these features, first note that with the above choice of placement of the masses, Rahmanov's formula (1-4) [1979] reduces to

$$\operatorname{Re}(\phi_n(e^{i\theta})) = (1+\cos(n\theta))\left(1+\frac{1}{2}\sum_{j=1}^{\frac{n}{2}} \frac{m_j}{1+nm_j}\right)-1-\tfrac{1}{2}\sin(n\theta)\sum_{j=1}^{n/2}\frac{m_j}{1+nm_j}\,\frac{\sin(\theta-\theta_j)}{1-\cos(\theta-\theta_j)},$$

$$\operatorname{Im}(\phi_n(e^{i\theta})) = \sin(n\theta)\left(1+\frac{1}{2}\sum_{j=1}^{n/2}\frac{m_j}{1+nm_j}\right)+\tfrac{1}{2}(1+\cos(n\theta))\sum_{j=1}^{n/2}\frac{m_j}{1+nm_j}\,\frac{\sin(\theta-\theta_j)}{1-\cos(\theta-\theta_j)}.$$

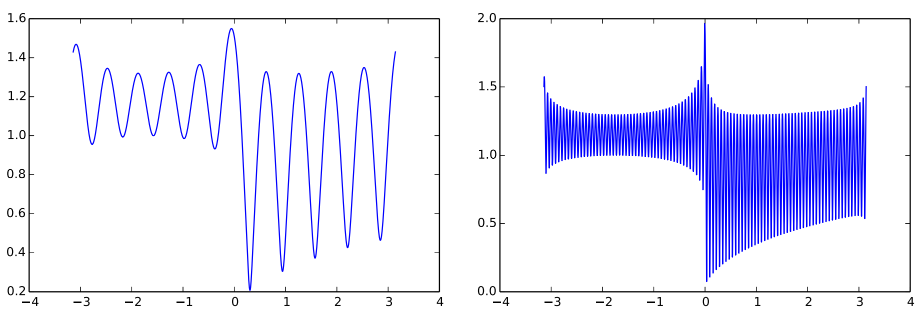

Figure 2. Left: $|\phi_{10}(\theta)|$ for $\theta_j = \frac{\pi}{10}(2j-1)$ and m_j chosen optimally, where $1 \le j \le 5$. Right: $|\phi_{100}(\theta)|$ for $\theta_j = \frac{\pi}{100}(2j-1)$ and m_j chosen optimally, where $1 \le j \le 50$.

Analysis of the minima. Due to its prominent role in each term, let us evaluate both the real and imaginary parts at the extrema of $1+\cos(n\theta)$, that is, $\theta=\theta_k=(\pi/n)(2k-1)$ and $\theta=\theta_k^*=2\pi k/n$. For $\theta=\theta_k$, $\sin(n\theta_k)$ and $1+\cos(n\theta_k)$ are each zero. However, we must be careful, because for $1\le k\le n/2$, one of the terms in the sum will have a denominator of zero. Thus, using L'Hôpital's rule, we take the limits

$$\lim_{\theta\to\theta_k}\frac{\sin(n\theta)\sin(\theta-\theta_k)}{1-\cos(\theta-\theta_k)}=-2n,$$
$$\lim_{\theta\to\theta_k}\frac{(1+\cos(n\theta))\sin(\theta-\theta_k)}{1-\cos(\theta-\theta_k)}=0.$$

Substituting these values into our formulae, we then have that

$$|\phi_n(e^{i\theta_k})|=\begin{cases}1-nm_k/(1+nm_k) & \text{if } 1\le k\le n/2,\\ 1 & \text{otherwise.}\end{cases}$$

Thus, the minima will be lower in the region where the masses are placed than outside that region. Also, we can now see the reason for the minima increasing as θ increases in the cases where the choice of $\boldsymbol{m}_j$ is optimal, as in Figure 2.

Analysis of peaks at $\boldsymbol{\theta=0,\pi}$. Now, let us examine the values of the polynomials at $\theta=\theta_k^*$. In this case, $\sin(n\theta_k^*)$ is still zero, but $1+\cos(n\theta_k^*)$ is instead 2, so we need not worry about zero denominators. Immediately, we have that our previous formulae reduce to

$$\begin{aligned}\operatorname{Re}\big(\phi_n(e^{i\theta_k^*})\big)&=1+\sum_{j=1}^{n/2}\frac{m_j}{1+nm_j},\\ \operatorname{Im}\big(\phi_n(e^{i\theta_k^*})\big)&=\sum_{j=1}^{n/2}\frac{m_j}{1+nm_j}\,\frac{\sin(\theta_k^*-\theta_j)}{1-\cos(\theta_k^*-\theta_j)}.\end{aligned}\tag{A-1}$$

The real part is constant in θ_k^* and can be ignored. For $k=0$, we have precisely the sum that was analyzed in Section 4. For $k=n/2$, we obtain the sum

$$\begin{aligned}\operatorname{Im}\big(\phi_n(e^{i\theta_{n/2}^*})\big)&=\sum_{j=1}^{n/2}\frac{m_j}{1+nm_j}\,\frac{\sin(\pi-\theta_j)}{1-\cos(\pi-\theta_j)}\\ &=\sum_{j=1}^{n/2}\frac{m_j}{1+nm_j}\,\frac{\sin\theta_j}{1+\cos\theta_j}.\end{aligned}$$

It can easily be seen that, if m_j is constant, this sum will be identical to the sum for $k=0$, and so the result will be two peaks of equal amplitude as we observed before in Figure 1. If m_j decreases proportionally to $1/\sqrt{j}$, however, this sum will be very different from the sum for $k = 0$, since the largest terms of the sum will now be those θ_j close to π rather than zero. The m_j with corresponding θ_j close to π will all be of the order $1/n$, and so we would expect that the value of the polynomial here will behave something more similarly to the peaks of the uniform mass case than to those of the optimal $\boldsymbol{m}$ case.

Analysis of peaks away from $\theta = 0, \pi$. However, we have not yet explained why the peaks away from $\theta = 0$ and $\theta = \pi$ are all smaller, so now we consider the case where $\theta = \theta_k^*$ for $0 < k < n/2$. First, note that

$$\theta_k^* - \theta_j = \frac{\pi}{n}(2(k-j)+1),$$

and consider the terms in the sum (A-1), where $j = k$ and $j = k+1$. These terms will be

$$\frac{m_k}{1+nm_k}\,\frac{\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}$$

and

$$-\frac{m_{k+1}}{1+nm_{k+1}}\,\frac{\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}.$$

In the case that all the masses have equal weight, these terms will cancel out completely, and, even in the case of the optimal choice of $\boldsymbol{m}_j$, they still mostly cancel out since the difference of m_{k+1} and m_k will be small. In general, for the $j = k-l$ and $j = k+l+1$ terms, as long as $k-l \geq 1$ and $k+l+1 \leq n/2$ are satisfied, similar cancellations will occur. Thus, the values at these peaks will be less than those at $\theta = 0$ and $\theta = \pi$.

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