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A type of multiple integral with log-gamma function

Duokui Yan, Rongchang Liu and Geng-zhe Chang

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In this paper, we give a general formula for the multiple integral

$$I=\int_0^1\int_0^1\dots\int_0^1 f(x_1+x_2+\dots+x_n)\,dx_1\,dx_2\dots\,dx_n.$$

As an application, the integral I with $f(x)=\log\Gamma(x)$ is evaluated for all $n\in\mathbb{N}$. The subsidiary computational challenges are interesting in their own right.

1. Introduction

A general idea, when faced with a multiple integral, is to lower its dimension. A well-known example, (see [Chang and Shi 2003], for instance) is the n-dimensional integral

$$\underset{\substack{x_1+x_2+\dots+x_n\le 1\\ x_1,x_2,\dots,x_n\ge 0}}{\int\cdots\int} f(x_1+x_2+\dots+x_n)\,dx_1\,dx_2\dots\,dx_n, \tag{1-1}$$

which can be simplified to a one-dimensional integral

$$\frac{1}{(n-1)!}\int_0^1 t^{n-1}f(t)\,dt.$$

However, to the best of our knowledge, a similar integral,

$$I=\int_0^1\int_0^1\dots\int_0^1 f(x_1+x_2+\dots+x_n)\,dx_1\,dx_2\dots\,dx_n, \tag{1-2}$$

has no such formula.

The aim of this paper is to find a formula for the above integral I and apply it to the special case when $f(x)=\log\Gamma(x)$. The main results are as follows. A general formula of I is obtained in Theorem 4.1.

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Theorem 4.1. *The integral I satisfies*

$$I = \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 + x_2 + \dots + x_n)\, dx_1\, dx_2 \dots dx_n = \frac{1}{(n-1)!} \sum_{m=1}^{n} \int_0^1 G_m(t) f(t+m-1)\, dt, \tag{1-3}$$

where

$$G_m(t) = \sum_{i=1}^{m} (-1)^{i-1} (t+m-i)^{n-1} \binom{n}{i-1}.$$

When $f(x) = \log\Gamma(x)$, the value of I is given in Theorem 5.1. The main challenge of the proof is to find appropriate combinatorial identities to simplify I.

Theorem 5.1.

$$I = I(n) = \int_0^1 \int_0^1 \dots \int_0^1 \log\Gamma(x_1 + x_2 + \dots + x_n)\, dx_1\, dx_2 \dots dx_n = \tfrac{1}{2}\log(2\pi) - \frac{n-1}{2} H_n + \sum_{k=2}^{n-1} \frac{(-1)^{n+k+1} k^n}{n!} \binom{n-1}{k} \log k,$$

where the last sum is missing when $n = 2$ and $H_n = \sum_{k=1}^{n} 1/k$.

The paper is organized as follows. In Sections 2 and 3, we explain the main ideas by using the cases $n = 2$ and 3. One can see from Figures 1 and 2 how we cut the square and the cube so that the integral I over each subset becomes a simple one-dimensional integral. In Section 4, a formula of I is derived in Theorem 4.1, and in Section 5, we evaluate I when $f(x) = \log\Gamma(x)$.

2. The case $n = 2$

When $n = 2$, the integral I becomes $\int_0^1 \int_0^1 f(x+y)\, dx\, dy$, where the integral domain is a unit square. Let $t = x + y$. The unit square can then be divided into two domains, D_1 and D_2 as in Figure 1, where

$$D_1 = \{(x,y) : 0 \le x+y \le 1,\ 0 \le x \le 1,\ 0 \le y \le 1\},$$
$$D_2 = \{(x,y) : 1 \le x+y \le 2,\ 0 \le x \le 1,\ 0 \le y \le 1\}.$$

The following lemma shows that $t_0^1 \int_0^1 f(x+y)\, dx\, dy$ is the sum of two one-dimensional integrals.

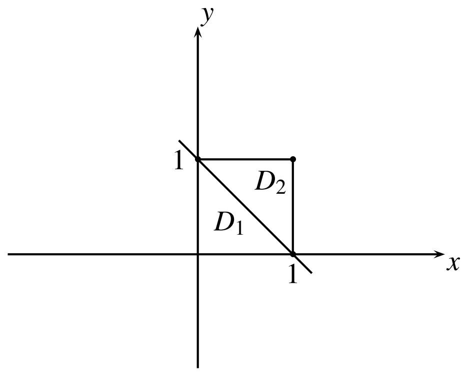

Figure 1. Domains D_1 and D_2.

Lemma 2.1.

$$\int_0^1 \int_0^1 f(x+y)\,dx\,dy = \iint_{D_1} f(x+y)\,dx\,dy + \iint_{D_2} f(x+y)\,dx\,dy$$
$$= \int_0^1 tf(t)\,dt + \int_0^1 (1-t)f(t+1)\,dt. \tag{2-1}$$

Proof. It is clear that

$$\int_0^1 \int_0^1 f(x+y)\,dx\,dy = \iint_{D_1} f(x+y)\,dx\,dy + \iint_{D_2} f(x+y)\,dx\,dy.$$

We first consider $\iint_{D_1} f(x+y)\,dx\,dy$. Note that $t = x + y$, and consider the transformation $(x, y) \mapsto (x, t)$. It is clear that the Jacobian is 1. Then

$$\iint_{D_1} f(x+y)\,dx\,dy = \int_0^1 \int_0^t f(t)\,dx\,dt = \int_0^1 tf(t)\,dt. \tag{2-2}$$

For the integral over domain D_2, we set $x_1 = 1 - x$ and $y_1 = 1 - y$. Then $(x_1, y_1) \in D_1$ and

$$\iint_{D_2} f(x+y)\,dx\,dy = \iint_{D_1} f(2-x_1-y_1)\,dx_1\,dy_1$$
$$= \int_0^1 tf(2-t)\,dt. \tag{2-3}$$

If one sets $u = 1-t$, it follows that $\int_0^1 tf(2-t)\,dt = \int_0^1 (1-u)f(u+1)\,du$. Then

$$\iint_{D_2} f(x+y)\,dx\,dy = \int_0^1 (1-u)f(u+1)\,du. \tag{2-4}$$

Then, identity (2-1) follows by identities (2-2) and (2-4). □

3. The case $n = 3$

When $n = 3$, the integral domain of I is a unit cube. The main idea is to cut the unit cube into several simplexes so that we can apply the integral formula (1-1) over each one.

Let $E = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ be the unit cube. Set

$$E_1 = \{(x, y, z) : 0 \le x + y + z \le 1,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\},$$
$$E_2 = \{(x, y, z) : 1 \le x + y + z \le 2,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\},$$
$$E_3 = \{(x, y, z) : 2 \le x + y + z \le 3,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}.$$

Then $E = E_1 \cup E_2 \cup E_3$ and the integral I satisfies

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_0^1 f(x+y+z)\,dx\,dy\,dz \\ &= \int_{E_1} f(x+y+z)\,dx\,dy\,dz + \int_{E_2} f(x+y+z)\,dx\,dy\,dz \\ &\qquad + \int_{E_3} f(x+y+z)\,dx\,dy\,dz. \end{aligned}$$

Using formula (1-1), it follows that $\int_{E_1} f(x+y+z)\,dx\,dy\,dz = \frac{1}{2}\int_0^1 t^2 f(t)\,dt$. The difficult parts are the integrals over E_2 and E_3. The following lemma explains how to simplify these two integrals to one-dimensional integrals.

Lemma 3.1.

$$\int_0^1 \int_0^1 \int_0^1 f(x+y+z)\,dx\,dy\,dz = \frac{1}{2}\int_0^1 t^2 f(t)\,dt + \frac{1}{2}\int_0^1 (-2t^2+2t+1) f(t+1)\,dt + \frac{1}{2}\int_0^1 (1-t)^2 f(t+2)\,dt. \tag{3-1}$$

Proof. We introduce the transformation $(x, y, z) \mapsto (x, y, t)$. By formula (1-1),

$$\int_{E_1} f(x+y+z)\,dx\,dy\,dz = \frac{1}{2}\int_0^1 t^2 f(t)\,dt. \tag{3-2}$$

Note that integral (3-2) can be applied to calculate the integral over E_3. Let $x_1 = 1 - x$, $y_1 = 1 - y$ and $z_1 = 1 - z$. The integral over E_3 becomes

$$\begin{aligned} \int_{E_3} f(x+y+z)\,dx\,dy\,dz &= \int_{E_1} f(3 - x_1 - y_1 - z_1)\,dx_1\,dy_1\,dz_1 \\ &= \frac{1}{2}\int_0^1 t^2 f(3-t)\,dt. \end{aligned} \tag{3-3}$$

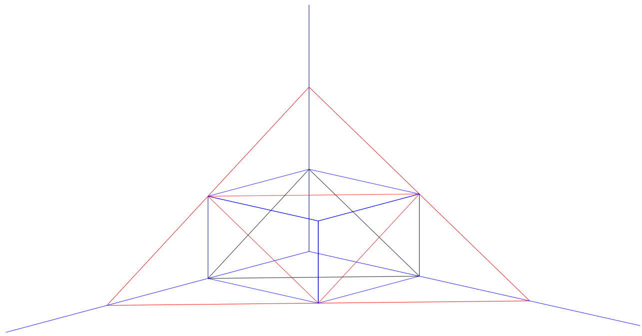

Figure 2. Region E_{20} and its partition: E_2, E_{21}, E_{22}, E_{23}.

If one sets $u = 1-t$, it implies that $\frac{1}{2}\int_0^1 t^2 f(3-t)\,dt = \frac{1}{2}\int_0^1 (1-u)^2 f(2+u)\,du$. Hence,

$$\int_{E_3} f(x+y+z)\,dx\,dy\,dz = \frac{1}{2}\int_0^1 (1-t)^2 f(t+2)\,dt. \tag{3-4}$$

By equalities (3-2) and (3-4), it is sufficient to show that

$$\int_{E_2} f(x+y+z)\,dx\,dy\,dz = \frac{1}{2}\int_0^1 (-2t^2+2t+1) f(t+1)\,dt. \tag{3-5}$$

Consider the domain

$$E_{20} = \{(x, y, z) : 1 \le x+y+z \le 2,\ 0 \le x \le 2,\ 0 \le y \le 2,\ 0 \le z \le 2\}.$$

Similar to Figure 1, we can cut E_{20} into 4 different domains, E_2, E_{21}, E_{22} and E_{23}, so that the integral over each domain can be handled easily. A picture of this partition is shown in Figure 2.

$$\begin{aligned}
E_2 &= \{(x, y, z) : 1 \le x+y+z \le 2,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\},\\
E_{21} &= \{(x, y, z) : 1 \le x+y+z \le 2,\ 1 \le x \le 2,\ 0 \le y \le 1,\ 0 \le z \le 1\},\\
E_{22} &= \{(x, y, z) : 1 \le x+y+z \le 2,\ 0 \le x \le 1,\ 1 \le y \le 2,\ 0 \le z \le 1\},\\
E_{23} &= \{(x, y, z) : 1 \le x+y+z \le 2,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 1 \le z \le 2\},
\end{aligned}$$

where $E_{20} = E_2 \cup E_{21} \cup E_{22} \cup E_{23}$.

Again by using formula (1-1), the integral over E_{20} is

$$\int_{E_{20}} f(x+y+z)\,dx\,dy\,dz = \int_1^2 \frac{1}{2}t^2 f(t)\,dt = \frac{1}{2}\int_0^1 (t+1)^2 f(t+1)\,dt. \tag{3-6}$$

On the other hand, the integral over E_{20} satisfies

$$\begin{aligned}\int_{E_{20}} & f(x+y+z)\,dx\,dy\,dz \\ &= \int_{E_{21}} f(x+y+z)\,dx\,dy\,dz + \int_{E_{22}} f(x+y+z)\,dx\,dy\,dz \\ &\qquad + \int_{E_{23}} f(x+y+z)\,dx\,dy\,dz + \int_{E_2} f(x+y+z)\,dx\,dy\,dz. \quad (3\text{-}7)\end{aligned}$$

By the definitions of E_{21}, E_{22} and E_{23}, it is clear that

$$\int_{E_{21}} f(x+y+z)\,dx\,dy\,dz = \int_{E_{22}} f(x+y+z)\,dx\,dy\,dz = \int_{E_{23}} f(x+y+z)\,dx\,dy\,dz.$$

So we only need to consider $\int_{E_{21}} f(x+y+z)\,dx\,dy\,dz$. Let $\tilde{x} = x-1$; then by equality (3-2),

$$\begin{aligned}\int_{E_{21}} f(x+y+z)\,dx\,dy\,dz &= \int_{E_1} f(\tilde{x}+y+z+1)\,d\tilde{x}\,dy\,dz \\ &= \frac{1}{2}\int_0^1 t^2 f(t+1)\,dt. \quad (3\text{-}8)\end{aligned}$$

Therefore, (3-6), (3-7) and (3-8) imply that

$$\begin{aligned}\int_{E_2} & f(x+y+z)\,dx\,dy\,dz \\ &= \int_{E_{20}} f(x+y+z)\,dx\,dy\,dz - 3\int_{E_{21}} f(x+y+z)\,dx\,dy\,dz \\ &= \frac{1}{2}\int_0^1 (t+1)^2 f(t+1)\,dt - \frac{3}{2}\int_0^1 t^2 f(t+1)\,dt \\ &= \frac{1}{2}\int_0^1 (-2t^2+2t+1) f(t+1)\,dt,\end{aligned}$$

which shows equality (3-5). □

4. The general case

In this section, we give a general formula for

$$I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n)\,dx_1\,dx_2 \ldots dx_n$$

in Theorem 4.1. In order to prove it, we first find a recursive formula for I in Theorem 4.3. The proof of Theorem 4.1 then follows by Theorem 4.4 and Theorem 4.3.

Theorem 4.1. *The integral I satisfies*

$$I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n)\, dx_1\, dx_2 \ldots dx_n$$
$$= \frac{1}{(n-1)!} \sum_{m=1}^{n} \int_0^1 G_m(t) f(t+m-1)\, dt, \tag{4-1}$$

where

$$G_m(t) = \sum_{i=1}^{m} (-1)^{i-1} (t+m-i)^{n-1} \binom{n}{i-1}.$$

The idea is to divide the n-dimensional unit box into n different polyhedrons and the integral I over each polyhedron can be simplified to a one-dimensional integral by applying the ideas in the 2D or 3D cases. The n different polyhedrons are defined as follows:

$$K_1 = \{(x_1, x_2, \ldots, x_n) : 0 \le x_1 + x_2 + \cdots + x_n \le 1,$$
$$0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ \ldots,\ 0 \le x_n \le 1\},$$
$$K_2 = \{(x_1, x_2, \ldots, x_n) : 1 \le x_1 + x_2 + \cdots + x_n \le 2,$$
$$0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ \ldots,\ 0 \le x_n \le 1\},$$
$$\vdots$$
$$K_n = \{(x_1, x_2, \ldots, x_n) : n-1 \le x_1 + x_2 + \cdots + x_n \le n,$$
$$0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ \ldots,\ 0 \le x_n \le 1\}.$$

By formula (1-1), the integral over K_1 satisfies the following proposition.

Proposition 4.2.

$$\int_{K_1} f(x_1 + x_2 + \cdots + x_n)\, dx_1\, dx_2 \ldots dx_n = \frac{1}{(n-1)!} \int_0^1 t^{n-1} f(t)\, dt.$$

Let

$$I_m = \int_{K_m} f(x_1 + x_2 + \cdots + x_n)\, dx_1\, dx_2 \ldots dx_n, \quad m = 1, 2, \ldots, n.$$

It is obvious that $I = \sum_{m=1}^{n} I_m$. Then the integral I reduces to the calculation of each I_m $(1 \le m \le n)$. Define

$$J_{s,m} = \int_{K_s} f(x_1 + \cdots + x_n + m - s)\, dx_1\, dx_2 \ldots dx_n, \tag{4-2}$$

where s is an integer and $1 \le s \le m$. Note that $J_{m,m} = I_m$. For any $1 \le s \le m-1$, $J_{s,m}$ can be calculated by I_s. The following theorem shows that I_m satisfies a recursive formula.

Theorem 4.3.

$$I_m = \frac{1}{(n-1)!}\int_0^1 (t+m-1)^{n-1} f(t+m-1)\,dt \\ - a_1 J_{1,m} - a_2 J_{2,m} - \cdots - a_{m-1} J_{m-1,m}, \quad (4\text{-}3)$$

where

$$a_i = \binom{m+n-i-1}{n-1}, \quad i = 1, 2, \ldots, m-1.$$

Proof. We consider the region

$$K_{m0} = \{(x_1, x_2, \ldots, x_n) : m-1 \le x_1 + x_2 + \cdots + x_n \le m, \\ 0 \le x_1 \le m,\ 0 \le x_2 \le m,\ \ldots,\ 0 \le x_n \le m\}.$$

By Proposition 4.2,

$$\int_{K_{m0}} f(x_1 + x_2 + \cdots + x_n)\,dx_1\,dx_2 \ldots dx_n \\ = \frac{1}{(n-1)!}\int_{m-1}^{m} t^{n-1} f(t)\,dt \\ = \frac{1}{(n-1)!}\int_0^1 (t+m-1)^{n-1} f(t+m-1)\,dt. \quad (4\text{-}4)$$

We define the subset $K_{i_1 i_2 \ldots i_n} \subset K_{m0}$ as follows:

$$K_{i_1 i_2 \ldots i_n} = \{(x_1, x_2, \ldots, x_n) : m-1 \le x_1 + x_2 + \cdots + x_n \le m, \\ i_1 - 1 \le x_1 \le i_1,\ i_2 - 1 \le x_2 \le i_2,\ \ldots,\ i_n - 1 \le x_n \le i_n\},$$

where $i_1, i_2, \ldots, i_n \in [1, m]$ are positive integers. It is easily seen that the intersection of any two subsets $K_{i_1 i_2 \ldots i_n}$ only happens on their boundaries. We then classify all possible $K_{i_1 i_2 \ldots i_n}$ so that the integral over each one can be evaluated easily. Note that by definition, $K_{1,1,\ldots,1} = K_m$. To find the integral over K_m, we need to subtract the integrals over all the other nonempty subsets $K_{i_1 i_2 \ldots i_n}$ ($i_1, i_2, \ldots, i_n \in [1, m]$) from $\int_{K_{m0}} f(x_1 + x_2 + \cdots + x_n)\,dx_1\,dx_2 \ldots dx_n$.

The first step is to determine when $K_{i_1 i_2 \ldots i_n}$ ($i_1, i_2, \ldots, i_n \in [1, m]$) is nonempty. For any set $K_{i_1 i_2 \ldots i_n}$, let

$$\tilde{x}_1 = x_1 - (i_1 - 1), \quad \tilde{x}_2 = x_2 - (i_2 - 1), \quad \ldots, \quad \tilde{x}_n = x_n - (i_n - 1). \quad (4\text{-}5)$$

Then $K_{i_1 i_2 \ldots i_n}$ becomes

$$\widetilde{K}_{i_1 i_2 \ldots i_n} = \{(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) : m+n-\alpha-1 \le \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n \le m+n-\alpha, \\ 0 \le \tilde{x}_1 \le 1,\ 0 \le \tilde{x}_2 \le 1,\ \ldots,\ 0 \le \tilde{x}_n \le 1\}.$$

where $\alpha = i_1 + i_2 + \cdots + i_n$. Let $s = m + n - \alpha$. It is clear that $K_{i_1 i_2 \dots i_n} \cong \widetilde{K}_{i_1 i_2 \dots i_n} = K_s$. Since $m + n - s = \sum_{j=1}^{n} i_j \geq n$, it follows that $s \leq m$. Note that if $s = m$, by equality (4-2), $J_{m,m} = I_m$. If $s = 0$, $K_{i_1 i_2 \dots i_n} \cong \widetilde{K}_{i_1 i_2 \dots i_n} = \{0\}$, and if $s < 0$, $K_{i_1 i_2 \dots i_n} \cong \widetilde{K}_{i_1 i_2 \dots i_n} = \varnothing$. So we only need to consider the case $1 \leq s \leq m-1$. For any given $s \in [1, m-1]$, it follows that

$$\begin{aligned}\int_{K_{i_1 i_2 \dots i_n}} & f(x_1 + x_2 + \cdots + x_n)\, dx_1\, dx_2 \dots dx_n \\ &= \int_{\widetilde{K}_{i_1 i_2 \dots i_n}} f(\tilde{x}_1 + \cdots + \tilde{x}_n + i_1 + \cdots + i_n - n)\, d\tilde{x}_1 \dots d\tilde{x}_n \\ &= \int_{K_s} f(x_1 + \cdots + x_n + m - s)\, dx_1 \dots dx_n \\ &= J_{s,m}. \end{aligned} \tag{4-6}$$

It implies that the subsets $K_{i_1 i_2 \dots i_n}$ $(i_1, i_2, \dots, i_n \in [1, m],\ i_1 + i_2 + \cdots + i_n \neq n)$ with nonzero measure can be classified into $m-1$ classes. In each class, every element is identical to some subset K_s after a shifting transformation in (4-5): $(x_1, x_2, \dots, x_n) \mapsto (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$.

Next step is to fix m and s $(1 \leq s \leq m-1)$, and find out how many subsets are identical to K_s. Since $s = m + n - (i_1 + i_2 + \cdots + i_n)$, we have

$$m + n - s = i_1 + i_2 + \cdots + i_n, \quad \text{where } i_1, i_2, \dots, i_n \text{ are positive integers.} \tag{4-7}$$

The number of positive integer solutions $(i_1, i_2, \dots, i_n)$ for (4-7) is $\binom{m+n-s-1}{n-1}$. It follows that the total number of subsets identical to K_s $(s \in [1, m-1])$ is

$$a_s = \binom{m+n-s-1}{n-1}. \tag{4-8}$$

Therefore, by equalities (4-4), (4-6) and (4-8), I_m satisfies

$$\begin{aligned} I_m &= \int_{K_m} f(x_1 + x_2 + \cdots + x_n)\, dx_1\, dx_2 \dots dx_n \\ &= \frac{1}{(n-1)!} \int_0^1 (t + m - 1)^{n-1} f(t + m - 1)\, dt \\ &\qquad - a_1 J_{1,m} - a_2 J_{2,m} - \cdots - a_{m-1} J_{m-1,m}, \end{aligned} \tag{4-9}$$

where a_s $(s = 1, \dots, m-1)$ is defined by (4-8). ☐

By using the cases $n = 2$ and 3, we can show by induction that

$$I_m = \frac{1}{(n-1)!} \int_0^1 G_m(t) f(t + m - 1)\, dt, \tag{4-10}$$

where $G_m(t)$ is a polynomial. It follows that

$$J_{s,m} = \int_{K_s} f(x_1 + \cdots + x_n + m - s)\, dx_1 \ldots dx_n = \frac{1}{(n-1)!} \int_0^1 G_s(t) f(t + m - 1)\, dt, \tag{4-11}$$

where s is an integer and $1 \le s \le m$. The integral I satisfies

$$I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n)\, dx_1\, dx_2 \ldots dx_n = \sum_{m=1}^{n} I_m. \tag{4-12}$$

In order to find a formula for I, we only need to compute the polynomial $G_m(t)$ in equality (4-10) for all $1 \le m \le n$. For $m = 1, 2$ and 3, a direct calculation shows that

$$\begin{aligned} G_1(t) &= t^{n-1}, \\ G_2(t) &= (t+1)^{n-1} - \binom{n}{1} t^{n-1}. \end{aligned} \tag{4-13}$$

By Theorem 4.3 and equality (4-11),

$$\begin{aligned} G_3(t) &= (t+2)^{n-1} - \binom{n+1}{n-1} G_1(t) - \binom{n}{n-1} G_2(t) \\ &= (t+2)^{n-1} - \binom{n}{1} (t+1)^{n-1} + \binom{n}{2} t^{n-1}. \end{aligned}$$

Similarly,

$$G_4(t) = (t+3)^{n-1} - \binom{n}{1} (t+2)^{n-1} + \binom{n}{2} (t+1)^{n-1} - \binom{n}{3} t^{n-1}.$$

It is reasonable to believe that $G_m(t)$ follows a pattern. The following theorem actually proves this fact.

Theorem 4.4.
$$G_m(t) = \sum_{i=1}^{m} (-1)^{i-1} (t + m - i)^{n-1} \binom{n}{i-1}. \tag{4-14}$$

Proof. The proof is based on the recursive formula (4-3) in Theorem 4.3 and the identity (4-11). By formula (4-3),

$$\begin{aligned} I_m &= \frac{1}{(n-1)!} \int_0^1 (t + m - 1)^{n-1} f(t + m - 1)\, dt - \sum_{i=1}^{m-1} a_i J_{i,m} \\ &= \frac{1}{(n-1)!} \int_0^1 G_m(t) f(t + m - 1)\, dt, \end{aligned}$$

where

$$G_m(t) = (t+m-1)^{n-1} - \sum_{i=1}^{m-1} a_i G_i(t), \quad \text{and} \quad a_i = \binom{m+n-i-1}{n-1}. \tag{4-15}$$

We show this theorem by induction. It is clear that formula (4-14) of $G_m(t)$ holds for $m = 1$. Assume that it holds for any $1 \le m \le k$. We need to show that formula (4-14) also holds for $m = k+1$.

By (4-15) and the induction assumption, the polynomial $G_{k+1}(t)$ satisfies

$$G_{k+1}(t) = (t+k)^{n-1} + \sum_{i=1}^{k} \binom{k+1+n-i-1}{n-1} \sum_{j=1}^{i} (-1)^j (t+i-j)^{n-1} \binom{n}{j-1}. \tag{4-16}$$

By formula (4-14), we can consider each $G_m(t)$ $(1 \le m \le k)$ as a polynomial of $(t+m-j)^{n-1}$ $(j = 1, 2, \ldots, m)$ with coefficient $(-1)^{j-1}\binom{n}{j-1}$. Then identity (4-16) implies that the coefficient of $(t+p)^{n-1}$ in $G_{k+1}(t)$ is

$$L_p(G_{k+1}(t)) = \sum_{i=p+1}^{k} \binom{k+1+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1}, \tag{4-17}$$

where $p \in [0, k-1]$ is an integer. Similarly, $G_k(t)$ satisfies

$$G_k(t) = (t+k-1)^{n-1} + \sum_{i=1}^{k-1} \binom{k+n-i-1}{n-1} \sum_{j=1}^{i} (-1)^j (t+i-j)^{n-1} \binom{n}{j-1},$$

and the coefficient of $(t+p)^{n-1}$ $(p \in [0, k-2])$ in $G_k(t)$ is

$$\sum_{i=p+1}^{k-1} \binom{k+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1}. \tag{4-18}$$

Note that $G_k(t) = \sum_{i=1}^{k} (-1)^{i-1} (t+k-i)^{n-1} \binom{n}{i-1}$. It follows that

$$\sum_{i=p+1}^{k-1} \binom{k+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1} = (-1)^{k-p-1} \binom{n}{k-p-1}. \tag{4-19}$$

If $p \neq 0$, let $q = p - 1$. By identity (4-19), the coefficient of $(t+p)^{n-1}$ in (4-17) satisfies

$$\begin{aligned} L_p(G_{k+1}(t)) &= \sum_{i=p+1}^{k} \binom{k+1+n-i-1}{n-1}(-1)^{i-p}\binom{n}{i-p-1} \\ &= \sum_{i=q+2}^{k} \binom{k+1+n-i-1}{n-1}(-1)^{i-q-1}\binom{n}{i-q-2} \\ &= \sum_{i=q+1}^{k-1} \binom{k+n-i-1}{n-1}(-1)^{i-q}\binom{n}{i-q-1} \\ &= (-1)^{k-q-1}\binom{n}{k-q-1} = (-1)^{k-p}\binom{n}{k-p}. \end{aligned} \tag{4-20}$$

Identity (4-20) holds for all integers $p \in [1, k-1]$. It remains to consider the case when $p = 0$.

If $p = 0$, by (4-17), the coefficient of t^{n-1} in $G_{k+1}(t)$ is

$$L_0(G_{k+1}(t)) = \sum_{i=1}^{k} \binom{k+1+n-i-1}{n-1}(-1)^i\binom{n}{i-1}. \tag{4-21}$$

Next, we show that $L_0(G_{k+1}(t)) = (-1)^k\binom{n}{k}$. Note that by the binomial theorem, the coefficient of the term x^{k+1} in $(1+x)^{-n}(1+x)^n$ is

$$\begin{aligned} &\sum_{i=0}^{k}(-1)^i\binom{n+i-1}{i}\binom{n}{k-i} \\ &\quad = \sum_{i=0}^{k}(-1)^i\binom{n+i-1}{n-1}\binom{n}{k-i} \\ &\quad = \sum_{j=1}^{k+1}\binom{k+1+n-j-1}{n-1}(-1)^{k+1-j}\binom{n}{j-1} \quad (j = k+1-i) \\ &\quad = (-1)^{k+1}\Big(L_0(G_{k+1}(t)) + (-1)^{k+1}\binom{n}{k}\Big). \end{aligned} \tag{4-22}$$

On the other hand, for a nonnegative integer k, the coefficient of the term x^{k+1} in $(1+x)^{-n}(1+x)^n = 1$ is always 0. Hence, (4-22) implies that

$$L_0(G_{k+1}(t)) = (-1)^k\binom{n}{k}. \tag{4-23}$$

Therefore, by identities (4-20) and (4-23), it follows that

$$\begin{aligned} G_{k+1}(t) &= (t+k)^{n-1} + \sum_{p=0}^{k-1} (-1)^{k-p} \binom{n}{k-p} (t+p)^{n-1} \\ &= \sum_{i=1}^{k+1} (-1)^{i-1} (t+k+1-i)^{n-1} \binom{n}{i-1}. \end{aligned} \tag{4-24}$$

This concludes the proof. □

5. Application to log-gamma function

In this section, we consider the integral of log-gamma function

$$I = \int_0^1 \int_0^1 \dots \int_0^1 \log \Gamma(x_1 + x_2 + \dots + x_n)\, dx_1\, dx_2 \dots dx_n. \tag{5-1}$$

The integral of log-gamma function has its own importance in many parts of mathematics [Amdeberhan et al. 2011; Choi and Srivastava 2005]. Actually, the case when $n = 2$ is a problem proposed by Ovidiu Furdui [2010] in the Problems and Solutions section of *The College Mathematics Journal*, and one of its solutions is proposed by Geng-zhe Chang [2011]. When it comes to general dimension n, it is quite a challenge to evaluate it.

After the preparation of Theorem 4.1 in Section 4, we can evaluate the integral (5-1). A nice formula is given in Theorem 5.1.

Theorem 5.1.

$$\begin{aligned} I = I(n) &= \int_0^1 \int_0^1 \dots \int_0^1 \log \Gamma(x_1 + x_2 + \dots + x_n)\, dx_1\, dx_2 \dots dx_n \\ &= \tfrac{1}{2} \log(2\pi) - \frac{n-1}{2} H_n + \sum_{k=2}^{n-1} \frac{(-1)^{n+k+1} k^n}{n!} \binom{n-1}{k} \log k, \end{aligned} \tag{5-2}$$

where the last sum is missing when $n = 2$ *and* $H_n = \sum_{k=1}^{n} 1/k$.

The proof of this theorem is based on Theorem 4.1 and several combinatorial identities in Jihuai Shi's book [2009].

Note that $\Gamma(t+1) = t\Gamma(t)$ and $G_m(t) = \sum_{i=1}^{m}(-1)^{i-1}(t+m-i)^{n-1}\binom{n}{i-1}$. By Theorem 4.1, the integral I becomes

$$\begin{aligned} I &= \frac{1}{(n-1)!}\sum_{m=1}^{n}\int_0^1 G_m(t)\log\Gamma(t+m-1)\,dt \\ &= \frac{1}{(n-1)!}\int_0^1 \sum_{m=1}^{n} G_m(t)\log\Gamma(t)\,dt \\ &\qquad + \frac{1}{(n-1)!}\int_0^1 \sum_{k=2}^{n}\sum_{m=k}^{n} G_m(t)\log(t+k-2)\,dt. \end{aligned} \tag{5-3}$$

Several combinatorial identities are introduced to simplify (5-3).

Lemma 5.2.

$$\sum_{m=k}^{n} G_m(t) = (n-1)! - \sum_{m=1}^{k-1}\binom{n-1}{k-m-1}(-1)^{k-m-1}(t+m-1)^{n-1},$$

and when $k=1$, $\sum_{m=1}^{n} G_m(t) = (n-1)!$.

Proof. Note that $G_m(t) = \sum_{i=1}^{m}(-1)^{i-1}(t+m-i)^{n-1}\binom{n}{i-1}$. It follows that

$$\begin{aligned} \sum_{m=1}^{k} G_m(t) &= \sum_{m=1}^{k}\sum_{i=1}^{m}(-1)^{i-1}(t+m-i)^{n-1}\binom{n}{i-1} \\ &= \sum_{m=1}^{k}\sum_{i=0}^{k-m}(-1)^{i}\binom{n}{i}(t+m-1)^{n-1}. \end{aligned}$$

By the combinatorial identity $\sum_{i=0}^{m}(-1)^{i}\binom{n}{i} = (-1)^m\binom{n-1}{m}$ $(m<n)$, we have

$$\sum_{m=1}^{k}\sum_{i=0}^{k-m}(-1)^{i}\binom{n}{i}(t+m-1)^{n-1} = \sum_{m=1}^{k}\binom{n-1}{k-m}(-1)^{k-m}(t+m-1)^{n-1}.$$

Hence,

$$\sum_{m=1}^{k} G_m(t) = \sum_{m=1}^{k}\binom{n-1}{k-m}(-1)^{k-m}(t+m-1)^{n-1}.$$

In the case when $k=n$, the combinatorial identity $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+n-k)^n=n!$ implies

$$\sum_{m=1}^{n}G_m(t)=\sum_{m=1}^{n}\binom{n-1}{n-m}(-1)^{n-m}(t+m-1)^{n-1}$$
$$=\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k(t+n-1-k)^{n-1}$$
$$=(n-1)!\,.$$

Therefore,

$$\sum_{m=k}^{n}G_m(t)=\sum_{m=1}^{n}G_m(t)-\sum_{m=1}^{k-1}G_m(t)$$
$$=(n-1)!-\sum_{m=1}^{k-1}\binom{n-1}{k-m-1}(-1)^{k-m-1}(t+m-1)^{n-1}. \qquad \square$$

Let

$$T_k=\sum_{m=1}^{k}\binom{n-1}{k-m}(-1)^{k-m}(t+m-1)^{n-1}=\sum_{m=0}^{k-1}\binom{n-1}{m}(-1)^m(t+k-m-1)^{n-1}.$$

Then

$$\sum_{m=k}^{n}G_m(t)=(n-1)!-T_{k-1}.$$

By applying Lemma 5.2, (5-3) becomes

$$I=\int_0^1\log\Gamma(t)\,dt+\int_0^1\sum_{k=0}^{n-2}\log(t+k)\,dt-\frac{1}{(n-1)!}\int_0^1\sum_{k=1}^{n-1}T_k\log(t+k-1)\,dt$$
$$=\tfrac{1}{2}\log(2\pi)+(n-1)\log(n-1)-n+1-\frac{1}{(n-1)!}\int_0^1\sum_{k=1}^{n-1}T_k\log(t+k-1)\,dt. \tag{5-4}$$

Then, the calculation of I reduces to the calculation of

$$\int_0^1\sum_{k=1}^{n-1}T_k\log(t+k-1)\,dt.$$

Note that $T_1 = t^{n-1}$ and

$$\int_0^1 T_k \log(t+k-1)\,dt = \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \int_0^1 (t+k-m-1)^{n-1} \log(t+k-1)\,dt.$$

When $k > 1$,

$$\begin{aligned}\int_0^1 &(t+k-m-1)^{n-1} \log(t+k-1)\,dt \\ &= \frac{(k-m)^n \log k - (k-m-1)^n \log(k-1)}{n} - \int_0^1 \frac{(t+k-m-1)^n}{n(t+k-1)}\,dt \\ &= \frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \\ &\qquad - \frac{1}{n} \sum_{r=1}^{n} \frac{k^r - (k-1)^r}{r} \binom{n}{r} (-m)^{n-r}.\end{aligned}$$

Let $S_1(1) = 0$,

$$S_1(k) = \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \left(\frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \log(k-1) \right),$$

and

$$S_2(k) = \frac{1}{n} \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \sum_{r=1}^{n} \frac{k^r - (k-1)^r}{r} \binom{n}{r} (-m)^{n-r}.$$

It follows that

$$\int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1)\,dt = \sum_{k=1}^{n-1} S_1(k) - \sum_{k=1}^{n-1} S_2(k). \tag{5-5}$$

The next lemma calculates $\sum_{k=1}^{n-1} S_1(k)$.

Lemma 5.3.

$$\sum_{k=1}^{n-1} S_1(k) = \frac{1}{n} \sum_{k=2}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \log k + \frac{\log(n-1)}{n} \big(n!\,(n-1) - (n-1)^n \big).$$

Proof. Note that $S_1(1) = 0$.

$$\sum_{k=1}^{n-1} S_1(k)$$
$$=\sum_{k=1}^{n-1}\sum_{m=0}^{k-1}\binom{n-1}{m}(-1)^m\left(\frac{(k-m)^n-(-m)^n}{n}\log k-\frac{(k-m-1)^n-(-m)^n}{n}\log(k-1)\right)$$
$$=\frac{1}{n}\sum_{k=2}^{n-2}\binom{n-1}{k}(-1)^k(-k)^n\log k$$
$$+\frac{1}{n}\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m\big((n-m-1)^n-(-m)^n\big)\log(n-1).$$

Using the combinatorial identity $\sum_{k=0}^{n}\binom{n}{k}(-1)^k(x-k)^{n+1} = (x-n/2)(n+1)!$, we have

$$\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m(n-1-m)^n = \sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m(n-1-m)^n = \frac{n-1}{2}n!,$$

and

$$\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m(-m)^n$$
$$= \sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m(-m)^n - (-1)^{n-1}(1-n)^n = (n-1)^n - \frac{n-1}{2}n!.$$

Hence,

$$\sum_{k=1}^{n-1} S_1(k) = \frac{1}{n}\sum_{k=2}^{n-2}\binom{n-1}{k}(-1)^k(-k)^n\log k + \frac{\log(n-1)}{n}\big(n!(n-1)-(n-1)^n\big).$$

□

The following lemma calculates $\sum_{k=1}^{n-1} S_2(k)$. Here we only give the result. For reader's convenience, the proof of it is given in the Appendix.

Lemma 5.4.

$$\sum_{k=1}^{n-1} S_2(k) = (n-1)!(n-1) - \frac{n-1}{2}H_n(n-1)!,$$

where $H_n = \sum_{k=1}^{n} 1/k$.

Using Lemma 5.3 and Lemma 5.4, we can prove Theorem 5.1 below.

Proof of Theorem 5.1. Let $H_n = \sum_{k=1}^{n} 1/k$. By identity (5-5), Lemma 5.3 and Lemma 5.4, we have that

$$\begin{aligned}
\int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1)\,dt
&= \sum_{k=1}^{n-1} S_1(k) - \sum_{k=1}^{n-1} S_2(k) \\
&= \frac{1}{n}\sum_{k=2}^{n-2}\binom{n-1}{k}(-1)^k(-k)^n \log k + \frac{\log(n-1)}{n}\big(n!\,(n-1)-(n-1)^n\big) \\
&\qquad - (n-1)!\,(n-1) + \frac{n-1}{2} H_n (n-1)!. \qquad (5\text{-}6)
\end{aligned}$$

By identities (5-4) and (5-6), it follows that

$$\begin{aligned}
I &= \tfrac{1}{2}\log(2\pi) + (n-1)\log(n-1) - n + 1 - \frac{1}{(n-1)!}\int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1)\,dt \\
&= \tfrac{1}{2}\log(2\pi) - \frac{n-1}{2} H_n + \frac{1}{n!}\sum_{k=2}^{n-1}\binom{n-1}{k}(-1)^{k+n+1} k^n \log k.
\end{aligned}$$

□

When $n = 2, 3$ and 4, the values of the integral I are

$$\begin{aligned}
I(2) &= -\tfrac{3}{4} + \tfrac{1}{2}\log(2\pi), \\
I(3) &= \tfrac{1}{2}\log(2\pi) + \tfrac{4}{3}\log 2 - \tfrac{11}{6}, \\
I(4) &= \tfrac{1}{2}\log(2\pi) - 2\log 2 + \tfrac{27}{8}\log 3 - \tfrac{25}{8}.
\end{aligned}$$

Appendix.

For reader's convenience, the proof of Lemma 5.4 is given here.

Lemma 5.4.

$$\sum_{k=1}^{n-1} S_2(k) = (n-1)!\,(n-1) - \frac{n-1}{2} H_n (n-1)!,$$

where $H_n = \sum_{k=1}^{n} 1/k$.

Proof. Note that

$$\sum_{k=1}^{n-1} S_2(k) = \frac{1}{n}\sum_{k=1}^{n-1}\bigg(\sum_{m=0}^{k-1}\binom{n-1}{m}(-1)^m \sum_{r=1}^{n}\frac{k^r-(k-1)^r}{r}\binom{n}{r}(-m)^{n-r}\bigg)$$

$$= -\frac{1}{n}\sum_{k=1}^{n-2}\binom{n-1}{k}(-1)^k(-k)^n\sum_{r=1}^{n}\binom{n}{r}\frac{(-1)^r}{r}$$

$$+ \frac{1}{n}\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m\sum_{r=1}^{n}\frac{(-m)^{n-r}(n-1)^r}{r}\binom{n}{r}.$$

Let

$$R_1 = -\frac{1}{n}\sum_{k=1}^{n-2}\binom{n-1}{k}(-1)^k(-k)^n\sum_{r=1}^{n}\binom{n}{r}\frac{(-1)^r}{r} \tag{A-1}$$

and

$$R_2 = \frac{1}{n}\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m\sum_{r=1}^{n}\frac{(-m)^{n-r}(n-1)^r}{r}\binom{n}{r}. \tag{A-2}$$

Then

$$\sum_{k=1}^{n-1} S_2(k) = R_1 + R_2. \tag{A-3}$$

By applying the combinatorial identities

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(x-k)^{n+1} = \Big(x-\frac{n}{2}\Big)(n+1)! \quad \text{and} \quad \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\binom{n}{k} = H_n,$$

the sum R_1 can be simplified to

$$R_1 = \frac{1}{n}\sum_{k=1}^{n-2}\binom{n-1}{k}(-1)^k(-k)^n\sum_{r=1}^{n}\binom{n}{r}\frac{(-1)^{r+1}}{r}$$

$$= \frac{H_n}{n}\bigg((n-1)^n - \frac{n-1}{2}n!\bigg). \tag{A-4}$$

To simplify R_2, we apply the combinatorial identity

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\binom{n}{k}\big(1-(1-x)^k\big) = \sum_{k=1}^{n}\frac{x^k}{k},$$

and it follows that

$$\begin{aligned}\sum_{r=1}^{n}\frac{(-m)^{n-r}(n-1)^r}{r}\binom{n}{r} &= -(-m)^n\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r}\binom{n}{r}\Big(1-\frac{m-n+1}{m}\Big)^r\\ &= (-m)^n\Big(\sum_{r=1}^{n}\frac{1}{r}\Big(\frac{m-n+1}{m}\Big)^r-\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r}\binom{n}{r}\Big)\\ &= \sum_{r=1}^{n}\frac{1}{r}(m-n+1)^r m^{n-r}(-1)^n-(-m)^n H_n.\end{aligned}$$

Recalling the formula of R_2 in (A-2), we have

$$nR_2=\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^{m+n}\sum_{k=1}^{n}\tfrac{1}{k}(m-n+1)^k m^{n-k}-H_n\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m(-m)^n. \tag{A-5}$$

By the combinatorial identity $\sum_{k=0}^{n}\binom{n}{k}(-1)^k(x-k)^{n+1}=(x-n/2)(n+1)!$, we see that

$$\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m(-m)^n=(n-1)^n-\frac{n-1}{2}n!. \tag{A-6}$$

We then simplify $\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^{m+n}\sum_{k=1}^{n}\frac{1}{k}(m-n+1)^k m^{n-k}$. Note that

$$\begin{aligned}&\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^{m+n}\sum_{k=1}^{n}\frac{1}{k}(m-n+1)^k m^{n-k}\\ &=\sum_{k=1}^{n}\frac{(-1)^n}{k}\Big(\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m\sum_{i=0}^{k}\binom{k}{i}m^{n-k+i}(n-1)^{k-i}(-1)^{k-i}\Big).\end{aligned} \tag{A-7}$$

Let

$$P(m)=\sum_{i=0}^{k}\binom{k}{i}m^{n-k+i}(n-1)^{k-i}(-1)^{k-i}.$$

We apply the combinatorial identity $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathbf{P}(k)=0$ for any polynomial $\mathbf{P}(k)$ with $\deg\mathbf{P}(k)<n$, and it follows that

$$\begin{aligned}\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m\sum_{i=0}^{k}\binom{k}{i}m^{n-k+i}(n-1)^{k-i}(-1)^{k-i}&\\ &=\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^m P(m)\\ &=\sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m P(m)-(-1)^{n-1}P(n-1)\\ &=\sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m(-k(n-1)m^{n-1}+m^n).\end{aligned}\tag{A-8}$$

By the combinatorial identity $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+n-k)^n=n!$, we have

$$-k(n-1)\sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m m^{n-1}=k(n-1)(-1)^n(n-1)!.$$

By the combinatorial identity $\sum_{k=0}^{n}\binom{n}{k}(-1)^k(x-k)^{n+1}=(x-n/2)(n+1)!$, we see that

$$\sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m m^n=(-1)^{n-1}\frac{n-1}{2}n!.$$

Then equality (A-7) becomes

$$\begin{aligned}\sum_{m=0}^{n-2}\binom{n-1}{m}(-1)^{m+n}\sum_{k=1}^{n}\frac{1}{k}(m-n+1)^k m^{n-k}&\\ &=\sum_{k=1}^{n}\frac{(-1)^n}{k}\left(k(n-1)(-1)^n(n-1)!+(-1)^{n-1}\frac{n-1}{2}n!\right)\\ &=n!(n-1)-\frac{n-1}{2}n!\,H_n,\end{aligned}\tag{A-9}$$

where $H_n=\sum_{k=1}^{n}1/k$.

Hence, by equalities (A-9) and (A-6), nR_2 in (A-5) can be simplified to

$$nR_2=n!(n-1)-(n-1)^n H_n.\tag{A-10}$$

That is,

$$R_2=(n-1)!(n-1)-\frac{H_n}{n}(n-1)^n.\tag{A-11}$$

Therefore, by equalities (A-3), (A-4) and (A-11), it follows that

$$\begin{aligned}\sum_{k=1}^{n-1} S_2(k) &= R_1 + R_2\\ &= \frac{H_n}{n}\left((n-1)^n - \frac{n-1}{2}n!\right) + (n-1)!(n-1) - \frac{H_n}{n}(n-1)^n\\ &= (n-1)!\,(n-1) - \frac{n-1}{2}H_n(n-1)!,\end{aligned}$$

where $H_n = \sum_{i=1}^{n} 1/i$. □

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