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Knight's tours on boards with odd dimensions

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A closed knight's tour of a board consists of a sequence of knight moves, where each square is visited exactly once and the sequence begins and ends with the same square. For boards of size $m \times n$ where m and n are odd, a tour is impossible because there are unequal numbers of white and black squares. By deleting a square, we can fix this disparity, and we determine which square to remove to allow for a closed knight's tour.

1. Introduction

One popular form of recreational mathematics deals with chess problems [Elkies and Stanley 2003]. While these problems can take many different forms (e.g., placing nonattacking queens or solving endgames), one of the most well-known variations is the knight's tour. In chess, a knight can move in a very restricted way. Namely, it must move one unit in one direction and two units in the perpendicular direction (see Figure 1).

A *knight's tour* is a sequence of legal knight moves where each square on the board is visited once; further, a *closed knight's tour* has the additional condition that it begins and ends with the same square. The problem of determining when a board has a closed knight's tour dates back several hundred years (see for example the work of Euler [1759]), and a full solution using a simple inductive argument was given by Schwenk.

Theorem 1 [Schwenk 1991]. *For $m \leq n$, an $m \times n$ rectangular board has a closed knight's tour unless one of the three following conditions hold*:

(1) *mn is odd.*

(2) $m \in \{1, 2, 4\}$.

(3) $m = 3$ *and* $n \in \{4, 6, 8\}$.

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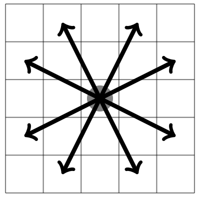


Figure 1. Legal knight moves.

Variations of this result have been studied including looking at closed knight's tours on a torus [Watkins and Hoenigman 1997], cylinders [Watkins 2000], spheres [Cairns 2002], and other boards [Lam et al. 1999].

When a knight moves on an $m \times n$ board, it will alternate between squares which are white and black. When we add in the requirement that we must start and stop at the same square this means that we must take an even number of steps in a closed tour (i.e., to return to our original colored square). However there are mn steps needed to cover the $m \times n$ board, and this establishes the first condition of Theorem 1. However, by deleting one square it is possible to leave an equal number of white and black squares on the board opening up the possibility of having a closed knight's tour. This leads to the following pair of questions:

Question. *Let m, n be odd with $m, n \geq 3$. Given an $m \times n$ board, when is it possible to delete one square so that the remaining board has a closed knight's tour? When it is possible to delete a square, which square(s) can we delete?*

An answer to the first question was given by DeMaio and Hippchen [2009] who showed that it is always possible except for the 3×5 board. The purpose of this paper is to give an answer to the second question, namely which squares can be deleted when it is possible, which we summarize in the theorem below.

For convention, we will label the squares of the board (i, j) as we would a matrix, i.e., $1 \leq i \leq m$ indicates the row going from top to bottom while $1 \leq j \leq n$ indicates the column going from left to right. With this labeling we note that a knight move will go from (i, j) to (k, ℓ), where $i + j$ and $k + \ell$ have different parity. Since there is one more square with $i + j$ even than there is with $i + j$ odd, in order for a knight's tour to exist, a necessary condition is that we must delete a square with $i + j$ even.

Theorem 2. *Let m, n be odd with $3 \leq m \leq n$. Then we can delete the square (i, j) from the $m \times n$ board and have a closed knight's tour in the remaining board for the following situations:*

(1) *For the 3×3 board, $(i, j) = (2, 2)$.*

(2) *For the 3×5 board, there is no single square which can be deleted.*

(3) *For the 3×7 board, $(i, j) \in \{(2, 2), (2, 6)\}$.*

(4) *For the* 3×9 *board*, $(i, j) \in \{(1,1), (1,5), (1,9), (3,1), (3,5), (3,9)\}$.

(5) *For the* $3 \times n$ *board with* $n \geq 11$, $i + j$ *is even and* $j \notin \{3, 4, n-3, n-2\}$.

(6) *For the* 5×5 *board*, $(i, j) \in \{(1,1), (1,5), (5,1), (5,5)\}$.

(7) *For* $m \geq 5$ *and* $n \geq 7$, $i + j$ *is even.*

The problem of which squares can be deleted from a $3 \times n$ board and having a knight's tour on the remaining board was independently done by Miller and Farnsworth [2013]. We include those results here for completeness and also because the proof of Miller and Farnsworth overlooked the case of removing the (2, 8) square from the 3×15 board.

The rest of this paper is organized as follows. In Section 2 we introduce a method that allows us to expand a closed knight's tour from a smaller board to a larger board. In Sections 3, 4, and 5 we handle the cases of $3\times$(odd), $5\times$(odd), and finally, the remaining cases. Lastly, in Section 6, we give some concluding remarks.

In the remainder of the paper we will make extensive use of symmetry, i.e., if we rotate a board by 90° or take a mirror image, we will still have a closed knight's tour.

2. Gluing on expanders

Our general approach mirrors that which was given in [Schwenk 1991]. Namely, we will form a large collection of base cases and show how to expand these base cases to get the remaining results. Our base cases have been relegated to the appendices, while in this section, we will show how we can expand a board.

Our tool of choice will be $m \times p$ *expanders* which correspond to open knight's tours of the $m \times p$ board that start at (2, 1) and end at (3, 1). This type of board can be easily connected to corners (since the moves at corners are forced). The following shows how to take a closed knight's tour that uses all or part of a board (i.e., a sub-board) and extend the board in one direction.

Lemma 3. *Given a closed knight's tour on a sub-board of the* $m \times n$ *board which visits the square* $(1, n)$ *and an* $m \times p$ *expander, we can find a closed knight's tour on the* $m \times (n + p)$ *board which, when restricted to the first* n *columns, covers the same sub-board as the original* $m \times n$ *board.*

Proof. By assumption, our tour visits the $(1, n)$ square. Therefore, we know that one move on the knight's tour is from $(1, n)$ to $(3, n-1)$. Deleting this move will result in an open knight's tour that starts at $(1, n)$ and ends at $(3, n-1)$. Now sequentially place the two boards, first placing the $m \times n$ board and then the $m \times p$ expander. Note that the expander is now an open tour that starts at $(2, n+1)$ and ends at $(3, n+1)$. Finally, we combine these two open tours to form one single closed tour that visits every square by adding the moves $(1, n)$ to $(3, n+1)$ and $(3, n-1)$ to $(2, n+1)$. By construction this will cover the same sub-board as the original $m \times n$ board. □

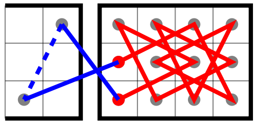


Figure 2. An illustration of Lemma 3 for a 3×4 expander.

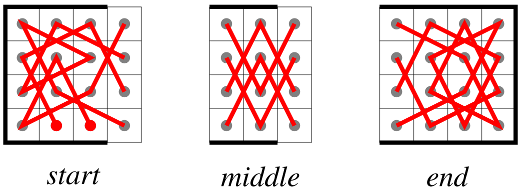


Figure 3. The three building blocks to form expanders.

An illustration of Lemma 3 which has a 3×4 expander is shown in Figure 2. Note by symmetry that we can also use other corners to glue. Since we will only be deleting one square from the board, we will always have at least one corner on a side available to use. We note that DeMaio and Hippchen [2009] used a similar gluing in their approach.

Following Schwenk, we want to be able to add four rows or columns to boards, which means we want to show that $n \times 4$ expanders exist. Unfortunately, they do not exist for all n. However, we will show that they exist when $n \geq 7$ and is odd. This will be done by appropriately combining the three pieces shown in Figure 3 (where for convenience we have rotated by 90°).

Proposition 4. *A $n \times 4$ expander exists for odd values $n \geq 7$.*

Proof. We will use the pieces given above along with induction to show how to do this. First note that these pieces are designed to overlap in a column, so if we take the *start* and *end* together we get the 7×4 expander shown in Figure 4.

To finish the proof it suffices to show how we can take an expander and increase its width by 2; i.e., given that we have $n \times 4$, we can construct $(n+2) \times 4$. To do this we move the *end* piece over by two spots and in the gap insert a middle. For example, for $n = 9$ and $n = 11$, we now get the expanders shown in Figure 5.

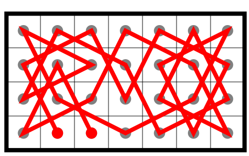


Figure 4. A 4×7 expander.

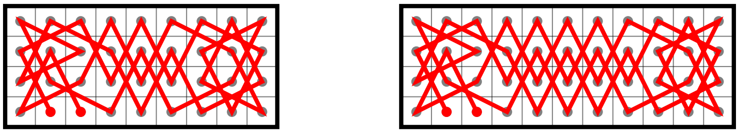


Figure 5. 4×9 and 4×11 expanders.

Because of the format of the pieces, as we glue these pieces together, we will have degree two at each vertex except for the two special vertices coming from the *start* piece. To show that this is a valid expander, we only need to make sure that we have an open knight's tour (i.e., we visit every square once and we begin and end in different squares). The key to see why this holds is to note that for the *middle* piece we have the relationship shown in Figure 6

This indicates that the relative ordering of the four "tracks" is the same. In particular, the addition of the *middle* piece will not effect whether or not we have an open knight's tour outside of that piece. But by induction, since we started with an open knight's tour, we still have an open knight's tour, and hence this construction gives a valid expander. □

3. Closed tours on $3 \times (\text{odd})$ boards

In this section we will work through the cases of $3 \times n$ for n odd. We will first look at what happens when $n \leq 9$ where there are extra constraints on what can be deleted, and then we will establish the general case for $n \geq 11$.

When $n = 3$, we note that there is no legal knight move from $(2, 2)$ to another square. Thus, it cannot be involved in a tour, so it is the only square which can be deleted. Further, there is a closed knight's tour with this square deleted (in Appendix A), establishing the result.

When $n = 5$, each corner would have a move to $(2, 3)$ and since we only delete one square, we would have to visit the center square multiple times, which is impossible for a closed knight's tour.

When $n = 7$, if we keep *both* $(2, 2)$ and $(2, 6)$, then the moves shown in Figure 7 (among others) would be forced to occur. This is impossible to extend to a closed knight's tour of the 3×7 board as we already have a cycle just among these four

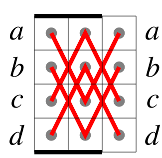


Figure 6. The relationship of the central pieces.

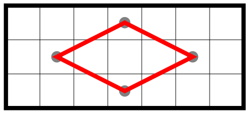


Figure 7. Forced moves for 3×7.

vertices. Therefore, we must delete either $(2, 2)$ or $(2, 6)$ (which up to symmetry are equivalent). Starting with the 3×3 closed knight's tour in Appendix A and gluing on the 3×4 expander as in Lemma 3 to the left (or right) will give a 3×7 closed knight's tour with $(2, 6)$ (or $(2, 2)$) deleted.

Before moving on to analyze the 3×9 case, we will establish a general restriction about which square can be deleted.

Lemma 5. *It is not possible to construct a closed knight's tour on a* $3 \times n$ *board*, n *odd*, *with a deleted square in column* 3, 4, $n-3$, *or* $n-2$.

Proof. By symmetry it suffices to show that we cannot delete a square in columns 3 or 4. Further note that by parity, we only need to show that $(1, 3)$, $(3, 3)$ and $(2, 4)$ cannot be deleted.

Note that to make a complete tour, each square must have an ingoing and outgoing move. This restriction forces the moves of several squares including $(1, 1)$, $(2, 1)$ and $(3, 1)$, as shown in Figure 8 (assuming they have not been deleted).

In particular, since $(2, 1)$ cannot be deleted, both $(1, 3)$ and $(3, 3)$ need to be present to be able to connect to $(2, 1)$. Thus, we cannot delete a square in column 3.

If we delete $(2, 4)$, then the squares $(1, 2)$ and $(3, 2)$ *must* connect to $(3, 3)$ and $(1, 3)$ respectively. This then forces a small cycle (as shown in Figure 8) which we cannot then extend to a closed knight's tour. Therefore, we cannot delete $(2, 4)$. $\square$

Applying Lemma 5, we see that for the 3×9 board, we cannot delete a square in columns 3, 4, 6, or 7. In Appendix A, we give closed knight's tours for the cases when we delete $(1, 9)$ and $(1, 5)$ (which, by symmetry, give tours for when $(1, 1)$, $(3, 1)$, $(3, 9)$ or $(3, 5)$ are deleted). It remains to show that we cannot delete $(2, 2)$. This is done by examining forced moves. The process is illustrated in Figure 9. First we add in all moves which are forced (near the ends). After this is done, we note that each of the squares $(1, 5)$ and $(3, 5)$ only have two possible moves available to them, so their moves are also forced. Finally, this leaves $(2, 4)$ with

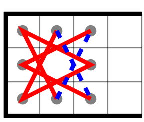


Figure 8. The forced moves from the left-hand column of a $3 \times n$ board.

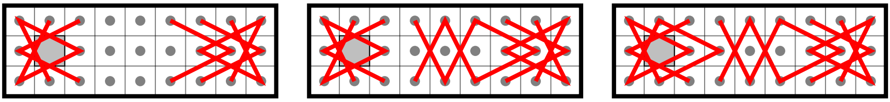


Figure 9. Forced moves for the 3×9 board.

only two available moves and so those moves are also forced. But we are now left with a closed cycle that does not cover the entire board and so we cannot extend this to a closed knight's tour.

We are now ready to establish a general result for larger $3 \times n$ boards.

Theorem 6. *Suppose we have a $3 \times n$ board with $n \geq 11$ and odd. Then a closed knight's tour is possible on the board after removing the square (i, j) if and only if $i + j$ is even and $j \notin \{3, 4, n-3, n-2\}$.*

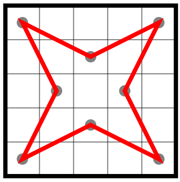
Proof. By Lemma 5, we cannot delete a square in column 3, 4, $n-3$ or $n-2$.

It remains to show that the deletion of every other square results in a board containing a closed knight's tour. For $n = 11$, we show in Appendix A closed knight's tours with squares $(1, 1)$ and $(1, 5)$ deleted (which by symmetry also gives $(1, 11)$, $(3, 1)$, $(3, 11)$, $(1, 7)$, $(3, 5)$ and $(3, 7)$). In addition, we can take the 3×3 board and using Lemma 3, glue on a 3×4 expander either twice to the left, twice to the right, or once on each side, giving a closed knight's tours with squares $(2, 10)$, $(2, 2)$, or $(2, 6)$, respectively, deleted.

For $n = 13$, we can use the known solutions for the 3×9 board and use Lemma 3 with the 3×4 expander to get solutions for the 3×13 board with a deleted square. Doing this we get everything except (up to symmetry) boards with squares $(1, 7)$, $(2, 6)$ or $(2, 2)$ deleted. These boards are given in Appendix A, establishing this case.

Now assume the result holds true for $3 \times n$. Then by taking the collection of closed knight's tours and applying Lemma 3 with a 3×4 expander on the left, we will get every closed knight's tour for the $3 \times (n+4)$ board which does not have a deleted square in column $1, 2, 3, 4, 7, 8, n+1$, or $n+2$. Similarly, if we apply Lemma 3 with a 3×4 expander on the right, we will get every closed knight's tour for the $3 \times (n+4)$ board which does not have a deleted square in column $3, 4, n-3, n-2, n+1, n+2, n+3$, or $n+4$. The intersection of these sets of columns will contain the mutually common columns $3, 4, n+1$, and $n+2$. It might also contain additional term(s) if $\{7, 8\} \cap \{n-3, n-2\}$ is nonempty. Because $n \geq 11$ by assumption, this can only occur when $n = 11$ and the common column is 8, giving that for $n \geq 13$, the intersection is $\{3, 4, n+1, n+2\}$ and $\{3, 4, 8, 12, 13\}$ if $n = 11$.

Therefore, we can get all solutions by building off of the base cases, except for the case when we have a 3×15 board and we delete the square $(2, 8)$. In Appendix A we show a closed knight's tour for such a board, and therefore we can construct all such boards. $\square$



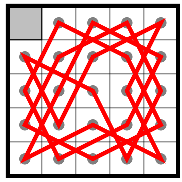


Figure 10. Knight's tour on the 5×5 board.

4. Closed tours on 5 x (odd) boards

In this section we will work through the cases of $5 \times n$. We first handle the exceptional case of 5×5 by noting that if we do not delete one of the corner squares and then we draw in the forced moves, we get the board shown on the left in Figure 10. This board has a closed cycle, so we will not be able to form a closed knight's tour. Therefore, we must delete a corner, and by symmetry, we can delete any corner. On the right in Figure 10 we have given a closed knight's tour with $(1, 1)$ deleted.

The remaining cases are handled in the following theorem which makes use of the 5×6 expander given in Figure 11.

Theorem 7. *Given any $5 \times n$ board where $n \geq 7$ is odd, a closed knight's tour exists after deleting (i, j) if and only if $i + j$ is even.*

Proof. In Appendix B we have given a knight's tour for any appropriate deleted square (up to symmetry) for the 5×7, 5×9 and 5×11 boards.

Now suppose we have a $5 \times n$ board with $n \geq 13$ and a square (i, j) with $i + j$ even. Then we show how to form a closed knight's tour for this board. First we note that on either the left or the right of the deleted square, there are six full columns in the board. So we repeatedly pull off sets of six columns from one side or the other of the deleted square *until* we have a 5×7, 5×9 or 5×11 board with a deleted square (which by construction will be at (i', j') with $i' + j'$ even). We now take the closed knight's tour for this board (which we have already found) and we repeatedly add back in the sets of six columns that we deleted by use of Lemma 3 and the expander shown in Figure 11. The end result will be our desired closed knight's tour. $\square$

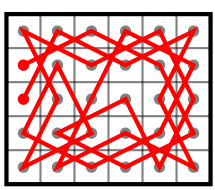


Figure 11. A 5×6 expander.

The proof we have just given works by showing how to start with a large board and then showing how to reduce down to a base case which we know is true. An alternative proof approach would be to start with the base cases and then use the expanders in all possible ways to construct a collection of boards and then show that all of the desired boards are in our collection. The latter approach can work but we have opted for the first approach as it gives a simple constructive approach to building the boards. Namely take the board, reduce down to a base case which we know and then reverse the steps to build the desired board. Using the second approach, it is not obvious a priori which board to build off of or how to build up to a larger board; this is especially true for the final result in the next section.

5. Closed tours on larger boards

In this section we finish establishing the main result.

Theorem 8. *Given an $m \times n$ board with $m \leq n$, $m \geq 5$ and $n \geq 7$ and any square (i, j) with $i + j$ even, there is a closed knight's tour of the $m \times n$ board with (i, j) deleted.*

Proof. We will make use of the $n \times 4$ expanders from Proposition 4, for odd $n \geq 7$, to mimic the proof of the last theorem. By the previous theorem, we know the result holds if $m = 5$, so we can assume that $m \geq 7$. Further, in Appendix C we give (up to symmetry) closed knight's tours for the 7×7 board. So we know the result also holds for $m = n = 7$.

Now, for any (i, j), there are either four columns to the left or four columns to the right. We can pull off those four columns and consider the resulting smaller board. By Lemma 3, it follows that if we have a closed knight's tour for this smaller board, we can use the expander to recover a closed knight's tour of our original board. (Note that we might possibly interchange the dimensions by rotating after pulling off these extra columns to maintain that $m \leq n$.)

In particular, after finitely many iterations (at most $(m + n)/4$ since we can only repeat this at most $m/4$ times for rows and at most $n/4$ for columns) we will have shrunk the board down to either a $5 \times n$ or a 7×7, in which case we have a solution. We now take this solution and work backwards to recover the desired original knight's tour. □

6. Conclusion

In this paper we have determined which squares can be deleted in a board with odd dimensions to allow the existence of a closed knight's tour. Reexamining Schwenk's result [1991], we note that there are no closed knight's tours of the $4 \times n$ board for any n. DeMaio and Hippchen [2009] were able to show that there are closed tours that exist after deleting two squares (as long as $n \geq 3$). In light of our discussion this raises the following natural question:

Question. *For the* $4 \times n$ *board with* $n \geq 3$, *which pairs of squares can be deleted that result in the existence of a closed knight's tour on the remaining board*?

We note that there is the obvious restriction that there must be one square of each parity. There is also a more subtle constraint.

Proposition 9. *If two squares in the* $4 \times n$ *board are deleted and a closed knight's tour exists for the remaining board*, *then neither square could come from the middle two rows.*

Proof. In the $4 \times n$ board, if we have a closed knight's tour, then any move from the first or fourth row must go into the middle two rows. By orienting the tour, we can then create a one-to-one pairing between squares in the first and fourth rows with a subset of the squares in the middle two rows (i.e., by what square follows after in the order given by the tour). Therefore, we can not have deleted both squares from the middle two rows.

Similarly, if we have one square deleted from the middle two rows, then we deleted one square from the first or fourth rows. Therefore, in the closed knight's tour, squares alternate between being in the middle or not. But we also know that squares alternate between different parities, which would imply that the squares in the middle two rows are all the same parity. But this is impossible. □

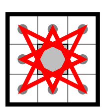
This shows that we must delete our two squares from the first and fourth row. Yet, when n is small, this is not sufficient. However, computational evidence suggests the following.

Conjecture. Consider the $4 \times n$ board with $n \geq 7$. For any pair of squares, with one of each parity and neither coming from the middle two rows, there is a closed knight's tour on the board that avoids only these two squares.

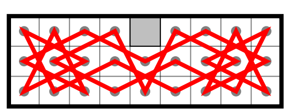
We look forward to seeing the next move in this area.

Appendix A: Base cases for 3 × (odd)

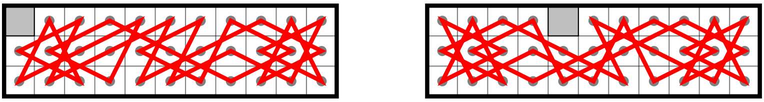
The following is the closed knight's tour of the 3×3 board:



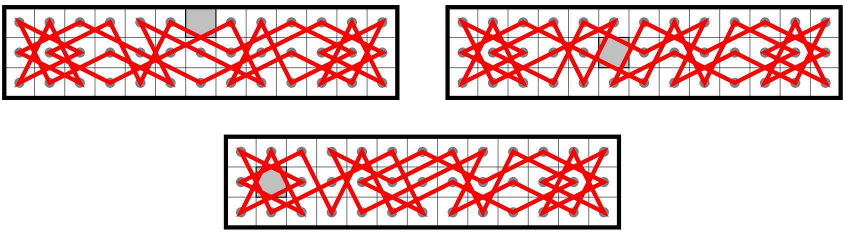
The following are closed knight's tours of the 3×9 boards with $(1, 9)$ and $(1, 5)$, respectively, deleted:



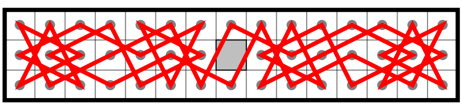
The following are closed knight's tours of the 3×11 boards with $(1, 1)$ and $(1, 5)$, respectively, deleted:



The following are closed knight's tours of the 3×13 boards with $(1, 7)$, $(2, 6)$ and $(2, 2)$, respectively, deleted:

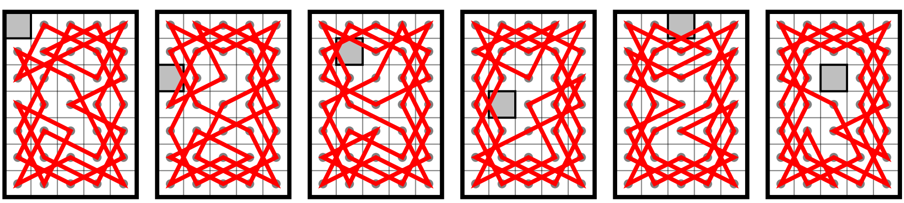


The following is a closed knight's tour of the 3×15 board with $(2, 8)$ deleted:

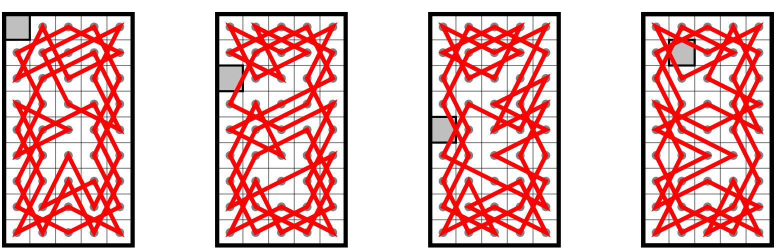


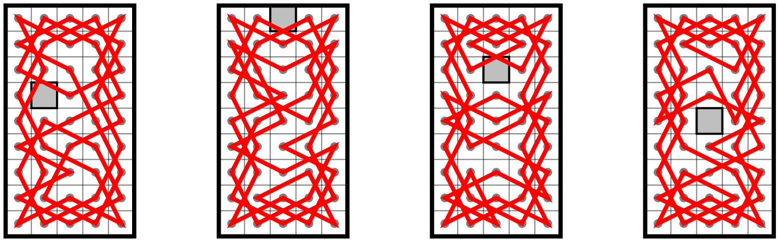
Appendix B: Base cases for $5 \times (\text{odd})$

The following cover the cases (up to symmetry) for the 5×7 board:

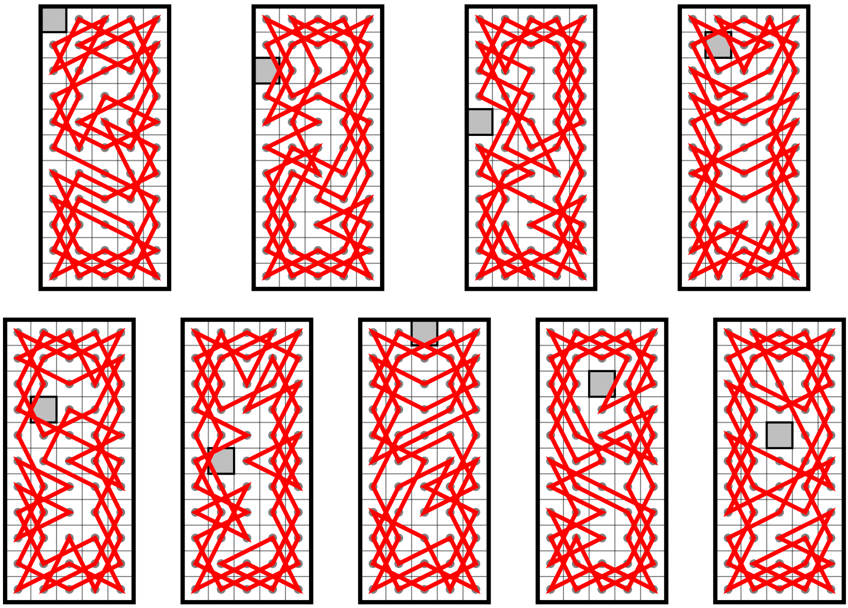


The following cover the cases (up to symmetry) for the 5×9 board:



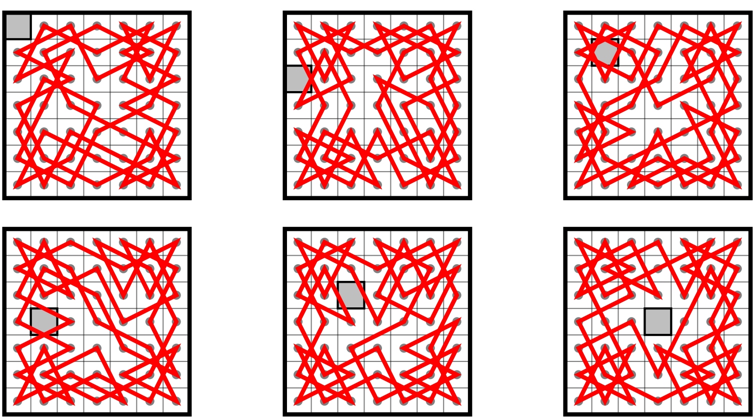


The following cover the cases (up to symmetry) for the 5×11 board:



Appendix C: Cases for 7 x 7

The following cover the cases (up to symmetry) for the 7×7 board:



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