

# involve

a journal of mathematics

Adjacency matrices of zero-divisor graphs of integers modulo  
 $n$

Matthew Young





# Adjacency matrices of zero-divisor graphs of integers modulo $n$

Matthew Young

(Communicated by Kenneth S. Berenhaut)

We study adjacency matrices of zero-divisor graphs of  $\mathbb{Z}_n$  for various  $n$ . We find their determinant and rank for all  $n$ , develop a method for finding nonzero eigenvalues, and use it to find all eigenvalues for the case  $n = p^3$ , where  $p$  is a prime number. We also find upper and lower bounds for the largest eigenvalue for all  $n$ .

## 1. Introduction

Let  $R$  be a commutative ring with a unity. The notion of a *zero-divisor graph* of  $R$  was pioneered by Beck [1988]. It was later modified by Anderson and Livingston [1999] to be the following.

**Definition 1.1.** *The zero-divisor graph  $\Gamma(R)$  of the ring  $R$  is a graph with the set of vertices  $V(R)$  being the set of zero-divisors of  $R$  and edges connecting two vertices  $x, y \in R$  if and only if  $x \cdot y = 0$ .*

To each (finite) graph  $\Gamma$ , one can associate the *adjacency matrix*  $A(\Gamma)$  that is a square  $|V(\Gamma)| \times |V(\Gamma)|$  matrix with entries  $a_{ij} = 1$ , if  $v_i$  is connected with  $v_j$ , and zero otherwise. In this paper we study the adjacency matrices of zero-divisor graphs  $\Gamma_n = \Gamma(\mathbb{Z}_n)$  of rings  $\mathbb{Z}_n$  of integers modulo  $n$ , where  $n$  is not prime. We note that the adjacency matrices of zero-divisor graphs of  $\mathbb{Z}_p \times \mathbb{Z}_p$ ,  $\mathbb{Z}_p[i]$ , and  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ , where  $p$  is a prime number and  $i^2 = -1$ , were studied in [Sharma et al. 2011].

## 2. Properties of adjacency matrices of $\Gamma_n$

Let  $n = p_1^{t_1} \cdots p_s^{t_s}$ , where  $p_1, \dots, p_s$  are distinct primes. For any divisor  $d$  of  $n$ , we define  $S(d) = \{k \in \mathbb{Z}_n \mid \gcd(k, n) = d\}$ . If  $d = p_1^{a_1} \cdots p_s^{a_s}$ , we will also write  $S(d) = S(a_1, \dots, a_s)$ . We can easily compute the size of the sets  $S(d)$ .

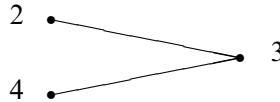
**Proposition 2.1.** *For a divisor  $d$  of  $n$ , the cardinality of the set  $S(d)$  is equal to  $|S(d)| = \phi(n/d)$ , where  $\phi$  denotes Euler's totient function.*

MSC2010: 05C50, 13M99.

Keywords: adjacency matrix, zero-divisor graph.

*Proof.* A positive integer  $m$  less than  $n$  is contained in the set  $S(d)$  if and only if  $\gcd(n, m) = d$ , which happens if and only if  $m = \hat{m}d$  and  $\gcd(n/d, \hat{m}) = 1$ . Thus, there is a one-to-one correspondence between the element  $m$  of  $S(d)$  and integers  $\hat{m}$ , where  $0 < \hat{m} < n/d$  and  $\gcd(n/d, \hat{m}) = 1$ .  $\square$

**Example 2.2.** We illustrate Proposition 2.1 with the zero-divisor graph  $\Gamma_6$  and its adjacency matrix:



$$A(\Gamma_6) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S(2) = \{2, 4\}, \quad |S(2)| = \phi(6/2) = 2.$$

**Theorem 2.3.** *If  $n > 4$ , then  $\det A(\Gamma_n) = 0$ .*

*Proof.* Write  $n = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$  as above. For  $i = 1, \dots, s$ , if  $p_i > 2$ , then  $|S(n/p_i)| = p_i - 1 > 1$ . If a vertex  $v$  of  $\Gamma_n$  corresponding to some divisor  $d$  of  $n$  is adjacent to one of the elements of  $S(n/p_i)$ , then  $d \cdot (n/p_i) = 0 \pmod{n}$ . Thus the product of  $d$  with any multiple of  $n/p_i$  is also zero, and  $v$  is adjacent to every element of  $S(n/p_i)$ . So  $A(\Gamma_n)$  will have repeated rows corresponding to each element of  $S(n/p_i)$ . Since  $|S(n/p_i)| > 1$ , we conclude that  $\det A(\Gamma_n) = 0$ .

If  $n=2^t$ , we must have  $t > 2$ . Then  $6 \in S(2)$ , and  $|S(2)| > 1$ . So  $A(\Gamma_n)$  will have repeated rows corresponding to 2 and 6, and  $\det A(\Gamma_n) = 0$ .  $\square$

The sets  $S(d)$  for all divisors  $d$  of a given integer  $n$  are an *equitable partition* of the set of vertices  $V(\Gamma_n)$ . That is, any two vertices in  $S(d_i)$  have the same number of neighbors in  $S(d_j)$  for all divisors  $d_i, d_j$  of  $n$ . This allows us to define a *projection graph*  $\pi\Gamma_n$  as a graph with vertices  $S(d)$  for all  $d|n$  and edges connecting  $S(d_i)$  with  $S(d_j)$  if every element in  $S(d_i)$  is connected with every element in  $S(d_j)$  in  $\Gamma_n$ .

**Example 2.4.** The projection graph  $\pi\Gamma_{15}$ :



**Proposition 2.5.** *The number of vertices in the graph  $\pi\Gamma_n$ , where  $n = \prod_{i=1}^s p_i^{t_i}$ , is*

$$|V(\pi\Gamma_n)| = \prod_{i=1}^s (t_i + 1) - 2.$$

*Proof.* The vertices of  $\pi\Gamma_n$  are the sets  $S(d)$  that are in one-to-one correspondence with the divisors  $d$  of  $n$ . If the prime decomposition for  $n$  is  $n = \prod_{i=1}^s p_i^{t_i}$  and the prime decomposition for a divisor  $d$  of  $n$  is  $d = \prod_{i=1}^s p_i^{a_i}$ , then we have  $t_i + 1$

choices for the exponent  $a_i$  of  $p_i$  in  $d$ . The choice of all  $a_i = 0$  leads to  $d = 1$ , and the choice of each  $a_i = t_i$  leads to  $d = n$ , neither of which are proper divisors of  $n$ . So the number of proper divisors  $d$  is  $\prod_{i=1}^s (t_i + 1) - 2$ .  $\square$

Let  $A(\pi(\Gamma_n))$  denote the adjacency matrix of  $\pi(\Gamma_n)$ . We will also consider the weighted adjacency matrix  $\mathcal{A}(\pi(\Gamma_n))$ , where  $a_{ij} = |S(d_j)|$  whenever  $S(d_i)$  is connected with  $S(d_j)$ . In the above example,

$$A(\pi(\Gamma_{15})) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{A}(\pi(\Gamma_{15})) = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}.$$

The following theorem relates the ranks of the various adjacency matrices.

**Theorem 2.6.** *Let  $n = \prod_{i=1}^s p_i^{t_i}$ . Then,*

$$\text{rank } A(\Gamma_n) = \text{rank } A(\pi\Gamma_n) = \text{rank } \mathcal{A}(\pi\Gamma_n) = \prod_{i=1}^s (t_i + 1) - 2.$$

*Proof.* Recall that vertices in  $V(\pi\Gamma_n)$  correspond to sets  $S(d)$ , where  $d|n$ . Since each element of  $S(d)$  contributes exactly the same row to the adjacency matrix  $A(\Gamma_n)$ , it follows that  $\text{rank } A(\Gamma_n) \leq |V(\pi\Gamma_n)|$ . On the other hand,  $\text{rank } A(\pi\Gamma_n) \leq \text{rank } A(\Gamma_n)$  since we just remove repeated rows and columns to get  $A(\pi\Gamma_n)$  from  $A(\Gamma_n)$ . Obviously,  $\text{rank } A(\pi\Gamma_n) = \text{rank } \mathcal{A}(\pi\Gamma_n)$ . So it is enough to show  $\text{rank } A(\pi\Gamma_n) = |V(\pi\Gamma_n)|$ .

Let  $n = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ . Since the rank of the matrix does not change with permutations of rows, assume that the rows of  $A(\pi\Gamma_n)$  correspond to the  $S(d)$ , with  $d|n$ , in the following order:  $S(p_1), S(n/p_1), S(p_2), S(n/p_2), \dots, S(p_s), S(n/p_s), S(p_i p_j)$  with  $i$  and  $j$  not necessarily distinct,  $S(n/(p_i p_j))$  for all possible pairs of  $i$  and  $j$ ,  $S(p_i p_j p_k), S(n/(p_i p_j p_k))$  for all possible triples  $i, j, k$  with  $i, j, k$  not necessarily distinct, etc.

We will compute the determinant of  $A(\pi\Gamma_n)$  and, by showing that it is not zero, will prove that  $|V(\pi\Gamma_n)|$  rows of  $A(\pi\Gamma_n)$  are linearly independent.

The first row corresponding to  $S(p_1)$  has 1 in the second column corresponding to  $S(n/p_1)$  and the rest of the entries are 0. Expand the determinant of  $A(\pi\Gamma_n)$  along the first row to get

$$\det A(\pi\Gamma_n) = -\det A_{1,2},$$

where  $-\det A_{1,2}$  is the cofactor of the  $(1, 2)$  entry of  $A(\pi\Gamma_n)$ . Note that the first column of  $A_{1,2}$  has 1 in the first row and the rest of the entries are 0. Expand  $\det A_{1,2}$  along the first column to get  $\det A_{1,2} = \det A^{(2)}$ , where  $A^{(2)}$  is a matrix obtained from  $A(\pi\Gamma_n)$  by deleting the first two rows and columns. So, we can conclude that  $\det A(\pi\Gamma_n) = -\det A^{(2)}$ .

We repeat this procedure for all  $S(p_i)$  and  $S(n/p_i)$  to get

$$\det A(\pi\Gamma_n) = (-1)^s \det A^{(2s)},$$

where  $A^{(2s)}$  is obtained from  $A(\pi\Gamma_n)$  by deleting the first  $2s$  rows and columns.

Now consider  $S(p_i p_j)$ . In  $\pi\Gamma_n$ , the vertex corresponding to  $S(p_i p_j)$  is adjacent to vertices of  $S(n/p_i)$ ,  $S(n/p_j)$  and  $S(n/(p_i p_j))$ . However, in the matrix  $A^{(2s)}$ , the row corresponding to  $S(p_i p_j)$  will have only one 1 in the column corresponding to  $S(n/(p_i p_j))$  since the columns corresponding to  $S(n/p_i)$  and  $S(n/p_j)$  were deleted. So we can repeat the procedure of expanding the determinant along the rows and columns corresponding to  $S(p_i)$  to expand the determinant along the rows and then the columns corresponding to  $S(n/(p_i p_j))$ .

Then continue to  $S(n/(p_i p_j p_k))$  in a similar fashion. In the end we will be left either with a  $2 \times 2$  matrix of determinant  $-1$ , or a  $1 \times 1$  matrix of determinant 1. So  $\det A(\pi\Gamma_n) = (-1)^m$ , where  $m$  is the number of distinct divisors  $d$  of  $n$  such that  $d < \sqrt{n}$ . It follows that  $\text{rank } A(\pi\Gamma_n) = |V(\pi\Gamma_n)|$ . The result follows from Proposition 2.5. □

**Corollary 2.7.** *We have  $\det A(\pi\Gamma_n) = (-1)^m$ , where  $m = \lfloor (\text{rank } A)/2 \rfloor$ .*

*Proof.* This follows from the proof of Theorem 2.6. □

**Corollary 2.8.** *We have  $\det \mathcal{A}(\pi\Gamma_n) = (-1)^m \prod_{d|n} |S(d)|$ , where  $m = \lfloor (\text{rank } \mathcal{A})/2 \rfloor$ .*

*Proof.* This result follows from the previous corollary and the fact that  $\mathcal{A}(\pi\Gamma_n)$  is obtained from  $A(\pi\Gamma_n)$  by multiplying the  $j$ -th column by  $|S(d_j)|$ . □

**Corollary 2.9.** *The multiplicity of the eigenvalue 0 of  $A(\Gamma_n)$  is*

$$n - \phi(n) - \prod_{i=1}^s (t_i + 1) + 2.$$

*Proof.* The multiplicity of the eigenvalue 0 of  $A(\Gamma_n)$  is  $|V(\Gamma_n)| - |V(\pi(\Gamma_n))|$ . The number of vertices of  $\Gamma_n$  is the number of positive integers less than  $n$  and not relatively prime to  $n$ , which is  $n - \phi(n)$ . The number of vertices of  $\pi\Gamma_n$  is

$$\prod_{i=1}^s (t_i + 1) - 2. \quad \square$$

Part of the following result is known, but we will include it for the convenience of the reader.

**Proposition 2.10.** *A nonzero  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A(\Gamma_n)$  if and only if it is an eigenvalue of  $\mathcal{A}(\pi(\Gamma_n))$ .*

Thus, to find all the nonzero eigenvalues of  $A(\Gamma_n)$ , it is enough to find all the eigenvalues of  $\mathcal{A}(\pi(\Gamma_n))$ . The proposition makes it especially easy when  $n$  has few factors. When  $n = pq$  is a product of distinct primes,  $\Gamma_n$  is a bipartite graph. It is known (and easy to see) that the nonzero eigenvalues of  $\mathcal{A}(\Gamma_{pq})$  are  $\pm\sqrt{(p-1)(q-1)}$  with multiplicity 1. When  $n = p^2$ , the matrix  $\mathcal{A}(\Gamma_n)$  is a  $1 \times 1$  matrix with  $p-1$  as a sole entry and hence the eigenvalue. So we consider the following two examples.

**Example 2.11.** Consider  $\Gamma_{p^3}$ . The matrix  $\mathcal{A}(\Gamma_{p^3})$  takes the form

$$A = \begin{bmatrix} 0 & p-1 \\ p(p-1) & p-1 \end{bmatrix}.$$

Its characteristic polynomial is  $p(\lambda) = \lambda^2 - (p-1)\lambda - p(p-1)^2$ , and the eigenvalues are  $\lambda_{1,2} = \frac{1}{2}(p-1)(1 \pm \sqrt{1+4p})$ .

**Example 2.12.** Consider  $\Gamma_{p^2q}$ , where  $p$  and  $q$  are distinct primes. In this case,

$$A(\Gamma_{p^2q}) = \begin{bmatrix} 0 & p-1 & 0 & 0 \\ (p-1)(q-1) & p-1 & q-1 & 0 \\ 0 & p-1 & 0 & p^2-p \\ 0 & 0 & q-1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \lambda^4 - (p-1)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p^2(p-1)(q-1)\lambda + p(p-1)^2(q-1)^3.$$

The roots can be found by the formulas for the roots of the fourth degree polynomial, but are too cumbersome to include here.

### 3. Estimates on eigenvalues

Since the increase in the number of factors of  $n$  leads to a rapid increase of the size of the adjacency matrix and the degree of the characteristic polynomial, one can use some known results to approximate the nonzero eigenvalues of  $A(\Gamma_n)$ . Since  $A(\Gamma_n)$  is symmetric, all its eigenvalues are real numbers. We will number them from largest to smallest  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ .

The *degree* of a vertex of a graph is the number of the edges incident to this vertex. Given a (finite) graph  $\Gamma$ , let  $\max\deg(\Gamma) = \max\{\deg v \mid v \in V(\Gamma)\}$  and

$$\text{avedeg}(\Gamma) = \frac{\sum_{v \in V(\Gamma)} \deg v}{|V(\Gamma)|}.$$

It is known (see [Brouwer and Haemers 2012, Proposition 3.1.2]) that

$$\text{avedeg}(\Gamma) \leq \lambda_1 \leq \max\deg(\Gamma).$$

We next compute  $\max\deg(\Gamma_n)$  and  $\text{avedeg}(\Gamma_n)$ .

**Proposition 3.1.** *Let  $n = p_1^{t_1} \cdots p_s^{t_s}$  with  $p_1 < p_2 < \cdots < p_s$ . Then*

$$\max\deg(\Gamma_n) = n/p_1 - 1.$$

*Proof.* For a divisor  $d$  of  $n$ , denote the corresponding vertex in  $\Gamma_n$  by  $v_d$ . The vertex  $v_d$  is connected to a vertex  $v_c$  corresponding to a divisor  $c$  of  $n$  by an edge in  $\Gamma_n$  if and only if  $dc \equiv 0 \pmod{n}$ . This happens if and only if  $c$  is a multiple of  $n/d$  (and is less than  $n$ ). There are  $d - 1$  such multiples, and so  $\deg(v_d) = d - 1$ . Then the vertex with the largest degree will correspond to the largest divisor of  $n$ , which is  $n/p_1$ . Since for any  $d|n$ , the degrees of all vertices in  $S(d)$  are the same and the sets  $S(d)$  partition  $V(\Gamma_n)$ ,  $\max\deg(\Gamma_n) = n/p_1 - 1$ .  $\square$

**Proposition 3.2.** *The average degree of the graph  $\Gamma_n$  is*

$$\text{avedeg}(\Gamma_n) = \frac{\sum_{d|n, d \neq n} \phi(n/d)(d - 1)}{n - \phi(n) - 1}.$$

*Proof.* To compute the average degree, we take the degree  $d - 1$  of a vertex in  $S(d)$ , multiply by the cardinality  $\phi(n/d)$  of the set  $S(d)$ , sum these products over all proper divisors  $d$  of  $n$ , and divide by the total number of vertices  $n - \phi(n) - 1$ .  $\square$

The estimate of the eigenvalue  $\lambda_1$  using the average degree of the graph is inconvenient to use. So we use the results on interlacing and on bipartite subgraphs of  $\Gamma_n$  for alternative estimates that are easier to use. Let  $\Gamma$  be any graph and  $\Delta$  an *induced subgraph*, that is, a subgraph obtained from  $\Gamma$  by deleting some vertices and all edges incident to the deleted vertices. The following result is known (see [Brouwer and Haemers 2012, Proposition 3.2.1] or [Godsil and Royle 2001, Theorem 9.1.1]). Let  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$  be eigenvalues of  $A(\Gamma)$  and  $\theta_1 \geq \cdots \geq \theta_l$  be the eigenvalues of  $A(\Delta)$ ; then  $\lambda_i \geq \theta_i \geq \lambda_{k-l+i}$  for  $i = 1, 2, \dots, l$ . With the interlacing result in mind, we prove the following proposition.

**Proposition 3.3.** *Let  $\lambda_1$  be the largest eigenvalue of  $A(\Gamma_n)$ :*

- (1) *If  $n$  is a product containing two or more distinct primes, then  $\lambda_1 \geq \sqrt{\phi(n)}$ .*
- (2) *If  $n = p^t$ , then  $\lambda_1 \geq p^{\lceil t/2 \rceil} - 1$ .*

*Proof.* Let  $n = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ . Consider the bipartite subgraph of  $\Gamma_n$  induced by the vertices in the sets  $S(p_1^{t_1})$  and  $S(n/p_1^{t_1})$ . The largest eigenvalue  $\mu_1$  of this subgraph is

$$\mu_1 = \sqrt{|S(p_1^{t_1})| \left| S\left(\frac{n}{p_1^{t_1}}\right) \right|} = \sqrt{\phi\left(\frac{n}{p_1^{t_1}}\right) \phi(p_1^{t_1})} = \sqrt{\phi(n)}.$$

The interlacing results give  $\mu_1 \leq \lambda_1$ .



To prove the second statement, notice that all vertices of  $S(p^i)$  are connected to all the vertices of  $S(p^j)$  whenever  $i + j \geq t$ ,  $1 \leq i, j \leq t - 1$ . In the case  $t$  is even,  $\Gamma_n$  contains a complete subgraph induced by vertices in the sets  $S(p^{t/2}), \dots, S(p^{t-1})$ . The largest eigenvalue  $\mu_1$  of this complete subgraph equals the number of vertices in the subgraph, which we compute next. By Proposition 2.1,  $|S(d)| = \phi(n/d)$ , so the number of vertices in the complete subgraph will be  $\phi(p) + \phi(p^2) + \dots + \phi(p^{t/2})$ . This can be expressed as  $(p - 1) + p(p - 1) + p^2(p - 1) + \dots + p^{t/2-1}(p - 1)$ , which after summing the arithmetic progression becomes  $p^{t/2} - 1 = \mu_1$ . In the case of  $t$  odd,  $\Gamma_n$  contains a complete subgraph induced by vertices in the sets  $S(p^{\lceil t/2 \rceil}), \dots, S(p^{t-1})$ . Using Proposition 2.1 and summing up the number of vertices as in case of  $t$  even gives  $p^{\lceil t/2 \rceil} = \mu_1$ .  $\square$

We can use the above estimates on the largest eigenvalue  $\lambda_1$  of  $A(\Gamma_n)$  to prove that there are only finitely many graphs with small eigenvalues.

**Theorem 3.4.** *For any positive integer  $k$ , there exists only a finite number of integers  $n$  such that all the eigenvalues of  $A(\Gamma_n)$  are less or equal than  $k$ .*

*Proof.* We will use the estimates  $\mu_1$  on  $\lambda_1$  obtained in Proposition 3.3. If, for a given  $k > 0$ , we have  $\lambda_1 \leq k$ , then  $\mu_1 \leq k$ . We will show  $\mu_1 \leq k$  is only possible for a finite number of integers  $n$ . Suppose  $n = p^t$ . Then  $p^{t/2} - 1$  must be less than or equal to  $k$ . Thus  $t \leq 2 \log_p(k + 1)$ , and there are only finitely many such positive integers.

Now suppose  $n$  is divisible by at least two distinct primes, say  $n = p_1^{t_1} \dots p_s^{t_s}$ . The Euler function on  $n$  can be computed as

$$\phi(n) = \phi(p_1^{t_1}) \dots \phi(p_s^{t_s}) = p_1^{t_1-1}(p_1 - 1) \dots p_s^{t_s-1}(p_s - 1).$$

Since there are only finitely many primes less than  $k + 1$ , and only a finite number of possible exponents  $t_1, \dots, t_s$  that satisfy  $0 < t_1, \dots, t_s \leq 2 \log_2 k$ , there are only finitely many positive integers  $n$  such that  $\phi(n) \leq k^2$ .  $\square$

**Example 3.5.** We will find all positive integers  $n$  such that all the eigenvalues of  $A(\Gamma_n)$  are less than or equal to  $k = 2$ . Suppose  $n = p^t$ . If  $t$  is even, then we must have  $p^{t/2} - 1 \leq 2$ . The only such possibilities are  $p = 2, t = 2$  and  $p = 3, t = 2$ . If  $t$  is odd, we must have  $p^{\lceil t/2 \rceil} \leq 2$ . This happens only if  $p = 2, t = 3$ . If  $n$  is divisible by at least two distinct primes, we must have  $\phi(n) \leq 4$ , and computations show that this is satisfied only for  $n = 6, 10, 12$ . For  $n = 4, 6, 8, 9, 10, 12$ , we compute the eigenvalues of  $A(\Gamma_n)$  and see that for all above  $n$  except  $n = 12$ , the eigenvalues are less or equal than 2. Thus, the adjacency matrices of  $\Gamma_4, \Gamma_6, \Gamma_8, \Gamma_9$  and  $\Gamma_{10}$  have all their eigenvalues less or equal than 2.

**Example 3.6.** For  $k = 3$ , the graphs all of whose eigenvalues are less than or equal to 3, in addition to the graphs of Example 3.5, are  $\Gamma_{12}, \Gamma_{14}$  and  $\Gamma_{15}$ . For  $k = 4$ ,

the new graphs (in addition to the ones with eigenvalues less or equal to 3) are  $\Gamma_{16}$ ,  $\Gamma_{21}$ ,  $\Gamma_{22}$ ,  $\Gamma_{25}$ ,  $\Gamma_{26}$  and  $\Gamma_{34}$ .

As we were considering the above examples, we noticed that for adjacency matrices of even rank, precisely half of the nonzero eigenvalues were positive and half negative. For adjacency matrices of odd rank, we always had one more positive eigenvalue than negative. We investigate this further.

The independence number  $\alpha(\Gamma)$  of a graph  $\Gamma$  is the size of the largest set of pairwise nonadjacent vertices. Let  $r$  denote the number of eigenvalues of a (weighted) adjacency matrix  $A(\Gamma)$  of a graph  $\Gamma$ ,  $r_+(A)$  the number of positive eigenvalues, and  $r_-(A)$  the number of negative eigenvalues. It is known (see [Brouwer and Haemers 2012, Theorem 3.5.4]) that  $\alpha(\Gamma) \leq r - r_+(A)$  and  $\alpha(\Gamma) \leq r - r_-(A)$ . We use this fact to show the following result.

**Theorem 3.7.** *Suppose the rank of  $A(\Gamma_n)$  is  $r$ . Then  $A(\Gamma_n)$  has  $\lfloor r/2 \rfloor$  positive eigenvalues and  $\lfloor r/2 \rfloor$  negative eigenvalues.*

*Proof.* We first note that it is enough to prove the theorem for  $\mathcal{A}(\pi\Gamma_n)$  since  $A(\Gamma_n)$  and  $\mathcal{A}(\pi\Gamma_n)$  have the same nonzero eigenvalues. We start by computing the independence number of  $\pi\Gamma_n$ . Recall that the vertices of  $\pi\Gamma_n$  are the sets  $S(d)$ , for all divisors  $d$  of  $n$ , and  $S(d_1)$  is connected to  $S(d_2)$  by an edge if  $n$  divides the product  $d_1d_2$ . So for all  $d \leq \sqrt{n}$ , the vertices  $S(d)$  are pairwise not connected. It is easy to see that all the vertices of  $\pi\Gamma_n$  can be split into pairs  $S(d)$  and  $S(n/d)$ , with  $S(\sqrt{n})$  without a pair if  $\sqrt{n}$  is a divisor of  $n$ . So the set of  $S(d)$  with  $d|n$  and  $d < \sqrt{n}$  is the maximal nonadjacent set of cardinality  $\lfloor r/2 \rfloor$ , where  $r$  denotes the number of vertices of  $\pi\Gamma_n$ . The number of vertices of  $\pi\Gamma_n$  is equal to the rank of  $\mathcal{A}(\pi\Gamma_n)$  by Theorem 2.6. So the independence number  $\alpha(\pi\Gamma_n)$  is equal to  $\lfloor r/2 \rfloor$ .

For  $r$  even, we have  $r/2 \leq r - r_+(A)$  and  $r/2 \leq r - r_-(A)$ , which implies the statement of the theorem. For  $r$  odd, it remains to show that we have one more positive eigenvalue than negative. By Corollary 2.8, we know that the sign of  $\det \mathcal{A}(\pi\Gamma_n)$  is given by  $(-1)^{\lfloor r/2 \rfloor}$ . On the other hand,  $\det \mathcal{A}(\pi\Gamma_n)$  is equal to the product of eigenvalues. So the parity of the number of negative eigenvalues must determine the sign of  $(-1)^{\lfloor r/2 \rfloor}$ . Since  $\lfloor r/2 \rfloor \leq r - r_-(A)$  and  $\lfloor r/2 \rfloor \leq r - r_+(A)$ , we must have  $r_-(A) = \lfloor r/2 \rfloor$ .  $\square$

### Acknowledgement

I offer my utmost gratitude to Dr. Lucy Lifschitz who introduced this topic to me, taught me everything I know about it that I didn't discover for myself, asked all of the right questions that led me to my results, provided all of the sources and previously known results, and converted my disorganized original draft into a cohesive, legible paper. None of this would have been possible without her guidance.

## References

- [Anderson and Livingston 1999] D. F. Anderson and P. S. Livingston, “The zero-divisor graph of a commutative ring”, *J. Algebra* **217**:2 (1999), 434–447. MR 2000e:13007 Zbl 0941.05062
- [Beck 1988] I. Beck, “Coloring of commutative rings”, *J. Algebra* **116**:1 (1988), 208–226. MR 89i:13006 Zbl 0654.13001
- [Brouwer and Haemers 2012] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Springer, New York, 2012. MR 2882891 Zbl 1231.05001
- [Godsil and Royle 2001] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics **207**, Springer, New York, 2001. MR 2002f:05002 Zbl 0968.05002
- [Sharma et al. 2011] P. Sharma, A. Sharma, and R. K. Vats, “Analysis of adjacency matrix and neighborhood associated with zero divisor graph of finite commutative rings”, *Int. J. Comput. Appl.* **14**:3 (2011), Article #7.

Received: 2014-02-07

Revised: 2014-08-27

Accepted: 2014-09-19

mjy5068@psu.edu

*Mathematics Department, Penn State University,  
University Park, 008 McAllister Building,  
State College, PA 16802, United States*



## MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

## BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

## PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or [msp.org/involve](http://msp.org/involve) for submission instructions. The subscription price for 2015 is US \$140/year for the electronic version, and \$190/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

# involve

2015

vol. 8

no. 5

A simplification of grid equivalence NANCY SCHERICH	721
A permutation test for three-dimensional rotation data DANIEL BERO AND MELISSA BINGHAM	735
Power values of the product of the Euler function and the sum of divisors function LUIS ELESBAN SANTOS CRUZ AND FLORIAN LUCA	745
On the cardinality of infinite symmetric groups MATT GETZEN	749
Adjacency matrices of zero-divisor graphs of integers modulo $n$ MATTHEW YOUNG	753
Expected maximum vertex valence in pairs of polygonal triangulations TIMOTHY CHU AND SEAN CLEARY	763
Generalizations of Pappus' centroid theorem via Stokes' theorem COLE ADAMS, STEPHEN LOVETT AND MATTHEW McMILLAN	771
A numerical investigation of level sets of extremal Sobolev functions STEFAN JUHNKE AND JESSE RATZKIN	787
Coalitions and cliques in the school choice problem SINAN AKSOY, ADAM AZZAM, CHAYA COPPERSMITH, JULIE GLASS, GIZEM KARAALI, XUEYING ZHAO AND XINJING ZHU	801
The chromatic polynomials of signed Petersen graphs MATTHIAS BECK, ERIKA MEZA, BRYAN NEVAREZ, ALANA SHINE AND MICHAEL YOUNG	825
Domino tilings of Aztec diamonds, Baxter permutations, and snow leopard permutations BENJAMIN CAFFREY, ERIC S. EGGE, GREGORY MICHEL, KAILEE RUBIN AND JONATHAN VER STEEGH	833
The Weibull distribution and Benford's law VICTORIA CUFF, ALLISON LEWIS AND STEVEN J. MILLER	859
Differentiation properties of the perimeter-to-area ratio for finitely many overlapped unit squares PAUL D. HUMKE, CAMERON MARCOTT, BJORN MELLEM AND COLE STIEGLER	875
On the Levi graph of point-line configurations JESSICA HAUSCHILD, JAZMIN ORTIZ AND OSCAR VEGA	893



1944-4176(2015)8:5;1-2