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Coalitions and cliques in the school choice problem

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The school choice problem (SCP) looks at assignment mechanisms matching students in a public school district to seats in district schools. The Gale–Shapley deferred acceptance mechanism applied to the SCP, known as the student optimal stable matching (SOSM), is the most efficient among stable mechanisms yielding a solution to the SCP. A more recent mechanism, the efficiency adjusted deferred acceptance mechanism (EADAM), aims to address the well-documented tension between efficiency and stability illustrated by SOSM. We introduce two alternative efficiency adjustments to SOSM, both of which necessarily sacrifice stability. Our discussion focuses on the mathematical novelty of new efficiency modifications rather than any practical superiority of implementation or outcome. That is, our contribution lies in process rather than outcome. Yet we argue that the demonstration of multiple processes yielding common outcomes is, in itself, a measure of the quality of that outcome. More specifically the consistency of outcome from different processes strengthens the argument that Pareto dominations of SOSM can be supported as "fair" despite the resulting priority violations.

1. Introduction

Since the mid-eighties, in cities across the United States, public school assignment policies have shifted towards providing students the opportunity to influence their school assignment. The main objective of these *school choice* policies is to allow all students to attend more desirable schools. A standard theoretical framework for studying such policies is two-sided matching (see [Gale 2001; Roth and Sotomayor 1990]). Presented in this context, the practical goal of the school choice problem (SCP) is to devise a matching mechanism (designed by or for the school district) that

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allocates available resources (seats in schools) among players (students or parents) subject to district priorities and legal requirements. Mathematically, it is interesting to consider ways in which the mechanisms might be modified that, while arguably consistent with the societal objectives of the SCP, present novel approaches to the underlying process.

In the economics literature, the SCP is viewed as a standard prototype for priority-based allocation problems (see [Kesten 2006]) and many of the school choice mechanisms in use or under investigation tolerate a large number of students receiving low preference schools (inefficiency) in order to respect school priority structures (stability). The ultimate purpose of these priorities is to benefit the students, but in many practical situations they are also the direct cause of efficiency losses, thus resulting in students receiving less desirable assignments than might have been possible. This suggests that taking a stable solution as a starting point and then making improvements for efficiency may be a reasonable compromise resulting in more desirable matchings.[1] Our discussion here will focus on the mathematical nuances of different efficiency modifications rather than any practical superiority of implementation or outcome. We will also suggest that stability loss may be justified to key stakeholders by arguing that the mathematical modifications are unbiased and incorporated as part of the overall process, and thus they do not constitute a "breach of contract".

Throughout, we employ the language and methods of mechanism design as applied to the SCP following in the footsteps of, for example, [Abdulkadiroğlu and Sönmez 2003]. In this context the designer/principal is the school district (or whoever is choosing the mechanism to be used), students are the players, and schools are merely items to be consumed.

The Gale–Shapley deferred acceptance mechanism applied to the SCP, known as the student optimal stable matching (SOSM), is the most efficient among stable mechanisms yielding a solution to the SCP [loc. cit.]. In this article we examine two concrete processes that modify the outcome of SOSM and improve efficiency at the cost of stability. Our goal is to situate in a common framework a range of ideas introduced recently by several different authors, so that the mathematical connections between different outcomes and processes are more visible. More specifically, we focus here on using multiple cooperation/collaboration methods to obtain Pareto improvements of SOSM. We are interested in the process as well as the outcome and, in particular, we argue that examining multiple pathways strengthens the case for those outcomes both in theory and in practice.

[1]A relevant quote from [Abdulkadiroğlu et al. 2009]: "Pareto efficiency for the students is the primary welfare goal, but [. . .] stability of the matching, and strategyproofness in the elicitation of student preferences, are incentive constraints that likely have to be met for the system to produce substantial welfare gains over the [current] system."

We begin in Section 1B by introducing two standard mechanisms used in this area of investigation: SOSM and its close neighbor, the efficiency adjusted deferred acceptance mechanism (EADAM), a more recently introduced mechanism which aims to address the well-documented tension between efficiency and stability illustrated by SOSM. In Section 2 we introduce the first of our approaches by studying the use of "coalitions" in order to modify SOSM school assignments. This section closely follows [Huang 2006], where it is shown that while the Gale–Shapley deferred acceptance algorithm (DA) disincentivizes strategic action by individuals, it is still feasible for groups to beat the system by coming together and strategizing. We adapt Huang's methods to the SCP and describe a process which we call the *coalition improvement procedure* in Section 2A. Using coalitions in the SCP allows us to approach efficiency modifications to SOSM in a new way and offers an alternative argument in support of previously known matching mechanisms. For example, this approach can result in the EADAM outcome along with other Pareto improvements of SOSM. We focus on properties of coalition improvements and comparisons to EADAM in Section 2B.

Following up on the coalition/cooperation theme, in Section 3 we introduce a second and related approach which focuses on groups of students who form trading cycles ("cliques") to improve their own assignments.[2] We examine the impact of these cliques as applied to the SOSM outcome. Once again, our approach deploys mathematical tools in a new context to produce several Pareto improvements on SOSM. We take the opportunity to show that the coalition improvements of Section 2 can also be integrated into this new framework, which proves to be a powerful construct to study cycle improvements of various kinds from a common point of view.

1A. ***Notation and basic terms used.*** Let I denote a nonempty set of students, and S a nonempty set of schools. A *matching* $M : I \to S \cup \{\text{null}\}$ is a function that associates every student $i \in I$ with exactly one school $M(i)$, or potentially no school at all, in which case $M(i) = \text{null}$. Write $\mathfrak{M}$ for the set of matchings. We will also occasionally want to talk about school quotas, which we will encode in a function $q : S \to \mathbb{N}$; in other words, for $s \in S$, $q(s)$ is the number of seats to be filled at school s.

A *preference profile* P_i for student $i \in I$ is a tuple $(S_1, \dots, S_n)$ where the S_j form a partition of S and every element of S_j is preferred to every element of S_k if and only if $j < k$.[3] Define the *ranking function* $\varphi_i : S \to \mathbb{N}$ of a student $i \in I$

[2]The term *clique* has a specific meaning in graph theory, unrelated to our work here.

[3]We will assume that student preference lists are complete, so it makes sense to define a preference list as a partition of the set of all schools. This is not always realistic however. Some students may wish to submit truncated lists, and this may or may not be allowed by school district policies. In fact, complete preference profiles in this context are rare. Often families are only permitted to list 3 to 7

by letting $\varphi_i(s)$ denote i's ranking of $s \in S$. In other words, $\varphi_i(s) = j$ if $s \in S_j$. If i prefers s_k to s_l, we write $s_k \succ_i s_l$, or simply $s_k \succ s_l$ if i is unambiguous. Note that the notation $\succ$ denotes a strict preference order; if we want to describe a weak order, we will write $\succeq$. We denote a set consisting of preference profiles for each student in I by $\boldsymbol{P} = \{P_i : i \in I\}$, and the space of all such sets is denoted by $\mathfrak{P}$.

A *priority structure* Π_s for school $s \in S$ is a tuple $(I_1, \dots, I_n)$ where the I_j form a partition of I and every element of I_j is preferred to every element of I_k if and only if $j < k$. If s prefers i_k to i_l, we write $i_k \succ_s i_l$, or simply $i_k \succ i_l$ if s is unambiguous. Once again, the notation $\succ$ denotes a strict preference order; if we want to describe a weak order, we will write $\succeq$. We denote a set consisting of priority structures for each school in S by $\boldsymbol{\Pi} = \{\Pi_s : s \in S\}$, and the space of all such complete sets is denoted by $\mathfrak{I}$.

A matching M' (*Pareto*) *dominates* M if $M'(i) \succeq_i M(i)$ for all i and $M'(j) \succ_j M(j)$ is strict for some j. A (*Pareto*) *efficient matching* is a matching that is not (Pareto) dominated.

A *matching mechanism* $\mathcal{M} : \mathfrak{P} \times \mathfrak{I} \to \mathfrak{M}$ is a function that takes an ordered pair $(\boldsymbol{P}, \boldsymbol{\Pi})$ of preferences and priorities and produces a matching.

Let Π_s be a priority structure for school s. A matching M *violates the priority* of $i \in I$ for s if there exist some $j \in I$ and $s' \in S$ such that

(1) $M(j) = s$, $M(i) = s'$: j gets assigned s under M and i gets assigned s' under M,

(2) $s \succ_i s'$: i prefers attending s over s', and

(3) $i \succ_s j$: s prioritizes i over j.

We say that a matching M is *stable* if

(1) M does not violate any priorities,

(2) no student is matched to a lower-ranked school when a more preferred school is unfilled, or more precisely, if $M(i) = s$, then for any school $s' \in S$ with $s' \succ_i s$, $\#\{j \in I \mid M(j) = s'\} = q(s')$,

(3) no student remains unmatched when a school is unfilled; that is, if $M(i) = \text{null}$, then for any school $s \in S$, $\#\{j \in I \mid M(j) = s\} = q(s)$.[4]

A *stable mechanism* is one that always produces stable matchings.

schools, choosing among many more. Then the district "completes" the student's profile, by first adding any school in her walk zone (if not already listed), and then "padding" the list with the schools that remain unlisted added to the end of the preference list, strictly below any listed by the student herself. Here we shall assume that when incomplete lists are allowed or unavoidable, the student preference lists are padded in this manner; our results will then work without modification.

[4]We assume that all students prefer being placed anywhere to being unassigned.

1B. ***Background.*** In this section we describe two well-studied mechanisms in the SCP: the student optimal stable matching (SOSM) and the efficiency adjusted deferred acceptance mechanism (EADAM). All mechanisms presented in this section use strict preference lists for students.

The first of these, SOSM, is based on the Gale–Shapley deferred acceptance algorithm (DA) [1962]. See [Roth and Sotomayor 1990] for an extensive review of the various applications of the DA algorithm and [Roth 2008] for a more recent historical overview. Gale and Shapley first described their method in the context of the *stable-marriage problem* (see [Knuth 1997]) and proposed applying it to the *college admissions problem*, a problem that in some ways resembles the SCP. Abdulkadiroğlu and Sönmez [2003] adapted the DA algorithm to the SCP and called it the student optimal stable mechanism (SOSM). Below is a brief description of this procedure.

Student optimal stable mechanism:

<u>Round 1</u>: Each student applies first to his or her first choice school. Each school then tentatively accepts the student(s) highest on its preference list among those who applied that round (such students are now waitlisted) and rejects the rest beyond its quota. We remove each waitlisted student from the market. All unwaitlisted students move on to the next round.

And in general:

<u>Round k</u>, $k \geq 1$: Each unassigned student applies to his or her next choice school. Each school considers the new applicants together with the current waitlist and repopulates the waitlist with those applicants who are highest on its priority list and rejects the rest beyond its quota. We remove each waitlisted student from the market. All unwaitlisted students move on to the next round. The algorithm runs until all students have been assigned.

SOSM performs well when evaluated for Pareto efficiency[5], stability, and strategyproofness and is viewed as a practical mechanism for implementation. In fact,

[5]We should qualify this assertion about the efficiency performance of SOSM. In the school choice problem as in many other matching markets, the preference and priority classes often are not singleton sets [Irving 1994; Manlove 2002]. In other words, there are many students in the same priority level for a given school, and it is conceivable that a student may wish to classify two or more schools in the same level of preference. The way SOSM and similar mechanisms deal with ties in such scenarios (often randomly and only on the school side, assuming students will submit strict preferences) creates arbitrary rankings, introduces artificial conditions, and results in a sizable efficiency loss (see [Erdil and Ergin 2008] for a study of tie-breaking in the school choice context and its efficiency cost). More generally it is known that many desirable properties of stable matching mechanisms are automatic only in the strict-preferences and strict-priorities scenario; once we allow indifferences, the problem often gets much more complicated [Manlove et al. 2002] and one might need to devise new goals and new extensions of the notion of stability (see [Chen 2012; Irving 1994]). We will say a bit more about indifferences in the final section of this paper.

several large districts such as New York City and Boston [Abdulkadiroğlu et al. 2009; 2006; 2005a; 2005b] have adopted SOSM as their mechanism of choice. As we have mentioned, SOSM offers a stable strategyproof mechanism whose outcomes Pareto dominate all other stable matchings.[6]

As motivation for investigating efficiency adjustments to the SOSM outcome and for introducing a powerful and well-respected model mechanism (EADAM), we give an example, due to Roth, that illustrates the problem of efficiency versus stability in SOSM (see [Abdulkadiroğlu et al. 2009; Kesten 2010]). This example also suggests that one could consider alternative processes that maintain appropriate respect for the (players') input while allowing for viable algorithmic alternatives.

Assume there are three schools, s_1, s_2, s_3 and three students i_1, i_2, i_3. The priorities of the schools and the preferences of the students are given by

$$\mathrm{SCP}_1 : \begin{array}{ll} i_1 : s_2 \succ s_1 \succ s_3, & s_1 \colon i_1 \succ i_3 \succ i_2, \\ i_2 : s_1 \succ s_2 \succ s_3, & s_2 \colon i_2 \succ i_1 \succ i_3, \\ i_3 : s_1 \succ s_2 \succ s_3, & s_3 \colon i_2 \succ i_1 \succ i_3, \end{array}$$

where $a \succ b$ stands for "*a is preferable to b*". Here, the only stable matching is

$$M_S^{\mathrm{SCP}_1} = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix},$$

but this matching is (Pareto) dominated by

$$M_E^{\mathrm{SCP}_1} = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}.$$

We see that $M_E^{\mathrm{SCP}_1}$ (Pareto) dominates $M_S^{\mathrm{SCP}_1}$ because it assigns i_1 and i_2 schools they prefer over their $M_S^{\mathrm{SCP}_1}$ assignment. Furthermore, $M_E^{\mathrm{SCP}_1}$ is (Pareto) efficient. However, the matching is no longer stable because i_2 is in the position of violating i_3's priority for s_1.

In part to address the weakness illustrated by the example above, Kesten [2010] proposed a new mechanism, and called it the efficiency adjusted deferred acceptance mechanism (EADAM). In order to understand EADAM, we must first define an *interrupter*. Let student i be one who is tentatively placed in a school s at some step t while running the SOSM, and rejected from it at some later step t'. If there

[6]Note that a stable mechanism can never really be strategyproof in the complete sense. More specifically, *no stable matching mechanism exists for which stating the true preferences is always a best response for every agent where all other agents state their true preferences* (see for instance [Roth and Sotomayor 1990, Corollary 4.5]). However the DA/SOSM is practically strategyproof as we only view the students as strategic players and the student optimality implies that there is no incentive for the students to misrepresent their preferences (see [Roth 1982]). This perspective does not take into account manipulation by schools in capacity (see [Sönmez 1997]) or preferences, see [Ehlers 2010] for recent work addressing these issues.

exists at least one other student who is rejected from school s after step $t-1$ and before step t', then we call student i an *interrupter* for school s and the pair (i, s) is an *interrupting pair* of step t'. An interrupter is *consenting* if she allows the mechanism to violate her priorities at no expense to her, that is, if she allows the mechanism to drop her from the running for schools she was an interruptor for, thus ignoring her priority standing with such schools. Note that the student's actual assignment would remain the same if not improve, and the consent would cost her nothing; by definition, she would not have been assigned to any school for which she was an interrupter in the first place.

EADAM then runs as follows:

Efficiency adjusted deferred acceptance mechanism:

Round 0: Run SOSM.

Round 1: Find the last step (of SOSM run in Round 0) at which a consenting interrupter is rejected from the school for which he/she is an interrupter. Identify all interrupting pairs in that step which contain a consenting interrupter. If there are no such pairs, then stop. Otherwise for each identified interrupting pair (i, s), remove school s from the preference list of student i without changing the relative order of the remaining schools. Rerun SOSM with the new preference profile for all such i until all students have been assigned.

And in general:

Round k, $k \geq 1$: Find the last step (of SOSM run in the previous round) at which a consenting interrupter is rejected from the school for which he/she is an interrupter. Identify all interrupting pairs in that step which contain a consenting interrupter. If there is no such pair, stop. Otherwise for each identified interrupting pair (i, s), remove school s from the preference list of student i without changing the relative order of the remaining schools. Rerun SOSM with the new preference profile until all students have been assigned.

In SCP_1, (i_3, s_1) is an interrupting pair and EADAM with the consent of i_3 outputs the Pareto efficient matching $M_E^{\text{SCP}_1}$. Note that this result improves the assignments for i_1 and i_2 while leaving i_3 with the same assignment. This mechanism deploys a balanced approach to priorities and preferences and points towards the possibility of introducing alternative pathways to these outcomes, which leads us to our next section where we do just that.

2. Coalitions in the school choice problem

Huang [2006] discusses a weakness of the Gale–Shapley algorithm in the context of the stable marriage problem and introduces the idea of *coalition cheating in the marriage problem*. More specifically he shows that a coalition can be formed where some men, without forgoing their own Gale–Shapley stable matching assignment, can cheat (misrepresent their preferences) so that some other men marry women who are higher on their preference list.

In this section we apply these ideas to the school choice problem. In this way we develop an alternative process for improving on the SOSM outcome. In Section 2A we give some background and an example that will motivate Huang's construction. We then introduce the elements of what Huang calls cheating coalitions in the context of the SCP, and discuss some implementation issues. From here onward, we resist the use of the term "cheating" in this context because we believe that these coalitions could be systematically incorporated into the design of a mechanism since they improve outcomes for some with no adverse effects on others. If the goal is for a "benevolent" district mechanism, then improving efficiency beyond that of a stable matching (e.g., SOSM), might simply be a part of the process. In Section 2B we compare the possible outcomes of coalitions to that of EADAM. Our presentation and general approach here are consistent with our focus on process as the primary area of interest while maintaining loyalty to the practical needs of the SCP framework.

2A. ***Huang's construction, coalitions and school choice.*** The following theorem establishes that in the stable marriage problem, there exists no coalition of men that may falsify their preferences such that every member of the coalition receives a *strictly better* assignment:

Theorem 2.1 [Dubins and Freedman 1981]. *In the Gale–Shapley men-optimal algorithm, no subset of men can improve their assignment by falsifying their preference lists.*

Translating the stable marriage problem to the context of the SCP, as done in [Abdulkadiroğlu and Sönmez 2003] by replacing men with s students, we get as an immediate corollary:

Corollary 2.2. *In the SOSM algorithm, no subset of students can improve their assignment by falsifying their preference lists.*[7]

In light of results of this nature, Huang [2006] introduces a nuanced notion of coalitions that falsify preferences to improve assignments. In the following, we carry over to the SCP setting this coalition model, which distinguishes between two main groups of players: those who falsify their preferences, and those who benefit from the falsifications.

Let I and S be the set of students and schools respectively in a given SCP. Let M be the SOSM stable matching assignment for the case where all students submit their true preferences. A *coalition* C is defined in terms of a pair (K, A) of subsets of the

[7] One can nonetheless prove that SOSM (DA as applied to school choice) is not group-strategyproof. We choose not to go further into strategy discussions here.

set I of students. The first subset, the *cabal* $K = (i_1, i_2, \ldots, i_{|K|})$ of a coalition C, is a list of students such that each student i_k, $1 \le k \le |K|$, prefers $M(i_{k+1})$ to $M(i_k)$, indices taken modulo $|K|$. In other words, we have $M(i_{k+1}) \succ_{i_k} M(i_k)$ for $1 \le k \le |K|$, and a *cabal loop*, written $(i_1 \to i_2 \to \cdots \to i_{|K|} \to i_1)$, a closed chain of students each of whom would prefer the stable assignment of the person following him to his own stable assignment. The second subset, the *accomplice set* $A = A(K)$ of cabal $K = (i_1, i_2, \ldots, i_{|K|})$, is a set of students $A(K) \subset I$ such that $i \in A(K)$ if for some $i_k \in K$, we have $M(i_{k+1}) \succ_i M(i)$ and $i \succ_{M(i_{k+1})} i_k$. In other words, an *accomplice* is a student who in his truthful preference list ranks the stable assignment of someone in the cabal (i_{k+1}) higher than his own stable assignment, while he himself is ranked higher by that school than another member of the cabal (the one pointing toward i_{k+1}) who would prefer it to his own school. Note that K and $A(K)$ may or may not be disjoint.

For any student $i \in I$, we can write the preference profile of i as a disjoint union of three sets: $(P_L[i], M(i), P_R[i])$. Here the set $P_L[i]$ (respectively $P_R[i]$) is simply the list of schools on i's preference profile to the left (respectively to the right) of his stable assignment $M(i)$. Let π_r denote a random permutation of S. We can now prove the following (as an easy adaptation from the analogous result of Huang):

Theorem 2.3 (cf. [Huang 2006]). *Let M be the SOSM matching for a given SCP when students submit their true preferences. Consider a coalition $C = (K, A(K))$, and suppose that each accomplice $i \in A(K)$ submits a falsified list of the form $(\pi_r(P_L[i] - X), M(i), \pi_r(P_R[i] \cup X))$, where*

- *if $i \notin K$, then $X = \{s \in M(K) \mid s = M(i_k), s \succ_i M(i), i \succ_s i_{k-1}\}$, and*
- *if $i = i_k \in A(K) \cap K$, then*

$$X = \{s \in M(K) \mid s = M(i_j), j \neq k, s \succ_{i_k} M(i_k), i_k \succ_s i_{j-1}\}.$$

Then in the resulting matching M', $M'(i_k) = M(i_{k+1})$ for $i_k \in K$ and $M'(i) = M(i)$ for $i \notin K$.

We observe that accomplices modify their preference profiles by moving schools on the left of their stable assignment to the right of their stable assignment if they are desirable to other students in the cabal. In particular, if i is an accomplice, then the set X of schools i moves to the right of his stable assignment will consist of all the stable assignments of the members of the cabal that rank i higher than the student following their stable assignment in the cabal loop. Note that the falsified preference lists incorporate a random permutation π_r of the preferences to the left and the right of the stable partner. The coalition procedure is quite robust, in that such a random permutation will not affect the outcome. In other words, the resulting matching creates a cyclical reassignment of those within the cabal loop while leaving all other assignments as they were.

We call each outcome of the improvement process described in Theorem 2.3 a *coalition improvement* and formalize the concept in the coalition improvement procedure (CIP).

Coalition improvement procedure for a given sequence of coalitions $(C_1,\dots,C_k)$[8]:

Round 0: Given a preference and priority profile, run the SOSM algorithm and obtain a temporary matching M_0.

Round t, $1 \leq t \leq k$: Given M_{t-1}, apply Theorem 2.3 with the coalition $C_t = (K_t, A(K_t))$. Return the resulting matching M'_{t-1} as the outcome M_t.

Let us now consider an example, which we will label SCP_2. This example demonstrates what CIP might look like in practice and also points out which set of consenting interruptors would result in EADAM having the same outcome. Let $I = \{i_1, i_2, i_3, i_4, i_5\}$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$ be the sets of students and schools, respectively, and let their respective preference and priority profiles be given as follows:

$$\mathrm{SCP}_2: \quad \begin{array}{ll} i_1: s_2 \succ s_5 \succ s_4 \succ s_3 \succ s_1, & s_1: i_3 \succ i_2 \succ i_4 \succ i_1 \succ i_5, \\ i_2: s_2 \succ s_5 \succ s_4 \succ s_1 \succ s_3, & s_2: i_4 \succ i_5 \succ i_1 \succ i_2 \succ i_3, \\ i_3: s_5 \succ s_2 \succ s_1 \succ s_3 \succ s_4, & s_3: i_2 \succ i_3 \succ i_4 \succ i_5 \succ i_1, \\ i_4: s_4 \succ s_1 \succ s_2 \succ s_3 \succ s_5, & s_4: i_1 \succ i_2 \succ i_3 \succ i_5 \succ i_4, \\ i_5: s_5 \succ s_4 \succ s_2 \succ s_3 \succ s_1, & s_5: i_1 \succ i_2 \succ i_5 \succ i_3 \succ i_4. \end{array}$$

Note that the matching output by SOSM for SCP_2 is

$$M_S^{\mathrm{SCP}_2} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_5 & s_4 & s_1 & s_2 & s_3 \end{pmatrix}.$$

We now consider the following coalition $C = (K, A(K))$: Let $K = \{i_1, i_2, i_4\}$ with the cabal loop $(i_1 \to i_4 \to i_2 \to i_1)$. The accomplice set $A(K)$ is $\{i_5\}$ and the set X for i_5 is $\{s_2, s_4\}$. In other words, the only student who modifies his preference profile is i_5. We display his old and new profiles:

$$i_5\text{'s old profile}: s_5 \succ s_4 \succ s_2 \succ \underline{s_3} \succ s_1,$$
$$i_5\text{'s new profile}: s_5 \succ \underline{s_3} \succ s_1 \succ s_2 \succ s_4.$$

(We underlined i_5's stable assignment s_3.) The outcome matching when we rerun SOSM is

$$M_C^{\mathrm{SCP}_2} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_2 & s_5 & s_1 & s_4 & s_3 \end{pmatrix},$$

[8]It should be apparent that there may be multiple outcomes of CIP for a given SCP depending on the particular sequence of coalitions we input. For simplicity we will assume that the cabals in each of the C_i are disjoint.

which improves the outcome for all members of the cabal and does not affect the remaining students. We note that this is also the EADAM outcome if i_5 consents. We will discuss this example further in Section 3A.

2B. ***Coalitions and EADAM.*** SOSM's strict adherence to stability and the resulting inefficiency has already been mentioned here (and documented in [Abdulkadiroğlu et al. 2009; Kesten 2010] and elsewhere). In this section we compare CIP (described in Section 2A) and EADAM from [Kesten 2010] (described in Section 1B), both of which model efficiency adjustments to SOSM. Specifically, we show that the common outcome of CIP and EADAM demonstrated by SCP_2 holds more generally by proving that for any SCP, there exists a coalition so that CIP yields the EADAM outcome with full consent. This fact may justify "fairness" arguments despite the sacrifice of stability.

Here is our general statement:

Theorem 2.4. *For any possible combination of consenters, the associated EADAM outcome may be obtained by forming an appropriately designed coalition and running CIP.*

The intuition behind this is that accomplices can be viewed as interrupters who consent to waive their priority so that they do not start a rejection chain. But coalitions reframe the argument so that the players are given the power to improve the outcome of the mechanism rather than being asked to waive their priority as with consenters.

Proof of Theorem 2.4. Let I and S be the sets of students and schools, respectively. Let $(\boldsymbol{P}, \boldsymbol{\Pi})$ be a given school choice problem for the pair (I, S), and let W be the set of students who consent to waiving their priorities under EADAM. Denote by M_S and M_E the SOSM and the EADAM outcome matchings of this problem, respectively. We will now construct a coalition C which will result in the same outcome M_E. First define the cabal set K to be the set of all students whose assignments are different under M_S and M_E:

$$K = \{i \in I \mid M_S(i) \neq M_E(i)\}.$$

These are the students who benefit from EADAM; they will also be the students who will benefit from the coalition C. Since every student whose assignment changes under EADAM is in K, we can partition K into cabal loops. This is equivalent to the basic algebraic fact that any finite permutation can be written as the product of disjoint cycles. Hence an elementary algorithm to decompose K into its individual cabal loops can be described as follows:

Step 0: Define a permutation π_K of K by setting $\pi_K(i') = i$ (i' *points to* i) if $M_S(i) = M_E(i')$. In words, *i' points to i if EADAM matches i' to the school to which SOSM matches i.*

Step 1: Pick a student $i \in K$ and label her $i_{1,1}$. Then let $i_{1,2}$ be the student $\pi_K(i_{1,1})$ and more generally label $i_{1,j+1} = \pi_K(i_{1,j})$. This process will stop at some j_1 with $\pi_K(i_{1,j_1}) = i_{1,1}$, as π_K is a finite permutation. Then

$$K_1 = (i_{1,1} \to i_{1,2} \to \cdots \to i_{1,j_1} \to i_{1,1})$$

is a cabal loop.

And in general:

Step k, $k \geq 1$: Pick a student $i \in K$ who has not yet been assigned to a cabal loop and label her $i_{k,1}$. If none exists then the algorithm stops. Otherwise, label $\pi_K(i_{k,1})$ as $i_{k,2}$ and more generally label $i_{k,j+1} = \pi_K(i_{k,j})$. This process stops at some j_k with $\pi_K(i_{k,j_k}) = i_{k,1}$ as π_K is finite. Then $K_k = (i_{k,1} \to i_{k,2} \to \cdots \to i_{k,j_k} \to i_{k,1})$ is a cabal loop.

Note that the algorithm has to stop because K is finite. Furthermore each student in K shows up in exactly one round and hence in exactly one cabal loop, because π_K is invertible.

Next we describe how to form the accomplice set $A(K)$. A student i will be in $A(K)$ if and only if the following two conditions are both satisfied:

- $i \in W$, or equivalently, i consents to waive her priorities in EADAM.
- There is a school s such that (i, s) is a last interrupter pair at some round of EADAM.

The new preference profile for an accomplice $i \in A(K)$ will be of the form

$$(P_L[i] - X, M_S(i), P_R[i] \cup X),$$

where

- if $i \notin K$, then $X = \{s \in M_S(K) \mid s = M_S(i_k), s \succ_i M_S(i), i \succ_s i_{k+1}\}$, and
- if $i = i_k \in A(K) \cap K$, then

$$X = \{s \in M_S(K) \mid s = M_S(i_j), j \neq k, s \succ_{i_k} M_S(i_k), i_k \succ_s i_{j+1}\}.$$

Here we are using the notation of Section 2A where $P_L[i]$ (respectively $P_R[i]$) is the list of schools on i's preference profile to the left (respectively to the right) of his stable assignment $M_S(i)$.

Finally Theorem 2.3 allows us to conclude that the outcome matching M_C of $C = (K, A(K))$ will be as follows: $M_C(i) = M_S(i)$ for all $i \notin K$, and $M_C(i_k) = M_S(i_{k+1})$ for i_k, i_{k+1} in some cabal loop K_j in K. But then $M_C = M_E$ and we are done. $\square$

It is interesting to observe that CIP can produce outcomes that cannot be obtained via EADAM no matter which students consent. That is, the converse of Theorem 2.4 is not true. To see this we analyze a minor modification of SCP_2 which we label SCP_3. Let $I = \{i_1, i_2, i_3, i_4, i_5\}$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$ be given with the following preference and priority structures, respectively:

$$\mathrm{SCP}_3: \quad \begin{array}{ll} i_1 : s_1 \succ s_2 \succ s_5 \succ s_4 \succ s_3, & s_1 : i_3 \succ i_2 \succ i_4 \succ i_1 \succ i_5, \\ i_2 : s_2 \succ s_5 \succ s_4 \succ s_1 \succ s_3, & s_2 : i_4 \succ i_5 \succ i_1 \succ i_2 \succ i_3, \\ i_3 : s_5 \succ s_2 \succ s_1 \succ s_3 \succ s_4, & s_3 : i_2 \succ i_3 \succ i_4 \succ i_5 \succ i_1, \\ i_4 : s_4 \succ s_1 \succ s_2 \succ s_3 \succ s_5, & s_4 : i_1 \succ i_2 \succ i_3 \succ i_5 \succ i_4, \\ i_5 : s_5 \succ s_4 \succ s_2 \succ s_3 \succ s_1, & s_5 : i_1 \succ i_2 \succ i_5 \succ i_3 \succ i_4. \end{array}$$

The SOSM outcome is

$$M_S^{\mathrm{SCP}_3} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_5 & s_4 & s_1 & s_2 & s_3 \end{pmatrix}.$$

EADAM with full consent (in fact we only need i_5's consent) returns the matching

$$M_E^{\mathrm{SCP}_3} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_1 & s_2 & s_5 & s_4 & s_3 \end{pmatrix}.$$

This corresponds to a coalition with the cabal set $\{i_1, i_2, i_3, i_4\}$ and the singleton accomplice set $\{i_5\}$. The set X for i_5 will be $X = \{s_2, s_4, s_5\}$. Note that there are two cabal loops: $(i_1 \to i_3 \to i_1)$ and $(i_2 \to i_4 \to i_2)$. There are indeed other coalitions that could be used for the same SCP. Take, for instance, the cabal to be $\{i_2, i_4\}$ and let $\{i_5\}$ be the singleton accomplice set. Then $X = \{s_2, s_4\}$ and we get

$$M_C^{\mathrm{SCP}_3} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_5 & s_2 & s_1 & s_4 & s_3 \end{pmatrix}.$$

This outcome cannot be obtained via EADAM because once i_5 consents to waive his priorities, he has to consent fully, and all Pareto improvements involving the interrupter pairs he was a part of will also be made.

3. Cliques for school choice

EADAM and CIP provide us with ways to systematically improve upon SOSM matching. Both involve complicated procedures requiring the identification of problematic preference profiles (of interruptors or possible coalition members) and subsequent modification of preference profiles and/or priority violations. The ultimate goal in either case is the same: to Pareto improve upon SOSM in a way that justifies the resulting priority violation(s). In this section we propose another way to improve efficiency starting from the SOSM outcome. There are, again necessarily, priority violations in the final matching. The main idea is as follows: We begin by applying SOSM to the given SCP. Next, with no further consideration of priorities, we enter students into a trading market designed purely to improve school assignments from the point of view of student preferences.

In Section 3A, we describe in more detail our new theoretical approach, the *trading adjusted deferred acceptance procedure* (TADAP). While doing so, we explicitly associate a directed graph to a given matching to provide a visual tool to describe possible efficiency improvements. We investigate basic properties of TADAP and compare outcomes of TADAP with those of other methods in Section 3B. In particular, in keeping with our focus on process and the relationship between outcomes, we discuss how coalitions and cliques relate to one another and to other mechanisms involving cycle improvements. We also comment on implications for the school choice context.

3A. ***The trading adjusted deferred acceptance procedure.*** We now develop a systematic way to find all Pareto improvements upon a predetermined matching M in a given SCP. We will of course be particularly interested in the case where M is the outcome of SOSM.

We start by associating a directed weighted graph (V, E, w) to M as follows: Each student i is assigned a unique vertex v_i in V. There is an edge from vertex v_i to vertex v_j if student i desires student j's assignment under the given matching at least as much as, if not more than, the school to which he himself was assigned. An edge e from vertex v_i to vertex v_j has weight $w(e) = 0$ if student i desires student j's assignment under the given matching as much as, but not more than, the school to which he himself was assigned, and $w(e) = 1$ if the preference is strict.

In the above we can identify V with the set of students. With this in mind we now introduce the following:

Definition 3.1. Let I and S be a set of n students and a set of m schools, respectively, with respective preference and priority structures $(\boldsymbol{P}, \boldsymbol{\Pi})$. Let M be a matching for the associated SCP. We say that the directed weighted graph $G_M = (V, E, w)$ is the *(directed weighted) graph of the matching M* if $V = I$; for any pair of students (i, j), there is an edge e_{ij} from i to j if and only if $M(j) \succeq_i M(i)$; and for each edge $e_{ij} \in E$, $w(e_{ij}) = 0$ if $M(i) \succeq_i M(j)$, and $w(e_{ij}) = 1$ otherwise.

Using this terminology, we can make the following definition:

Definition 3.2 (cf. [Ergin 2002, Definition 1]). Let I, S, $(\boldsymbol{P}, \boldsymbol{\Pi})$, M and G_M be given as in Definition 3.1 and let $k \in \mathbb{N}$. A *clique of length k* consists of a sequence $(i_1, i_2, \ldots, i_k)$ of k distinct students such that for each $s < k$, there is an edge in E from v_{i_s} to $v_{i_{s+1}}$, there is an edge in E connecting v_{i_k} back to v_{i_1}, and for some $s < k$, we have $w(e_{i_s,i_{s+1}}) = 1$ or $w(e_{i_k,i_1}) = 1$.[9] A similar cycle where $w = 0$ on

[9]What we call a *clique* is occasionally called a *trading cycle* in some of the literature. We use the former for brevity and also as a hint to the social context.

all edges is called a *null clique*. A matching whose graph contains no cliques (null or otherwise) is *acyclical*.

A straightforward result then follows:

Theorem 3.3. *If there exists a matching M (Pareto) dominating a matching M', then the directed graph $G_{M'}$ of M' admits a clique. Equivalently, if the directed graph of M' is acyclical, then M' is Pareto efficient. Conversely, if M' admits a clique, we can always find a matching M which Pareto dominates M' (equivalently, the directed graph of a Pareto efficient matching is acyclical).*[10]

Consider now the following procedure:

Trading adjusted deferred acceptance procedure:

Round 0: Given a preference and priority profile, run the SOSM algorithm and obtain a temporary matching M_0.

Round t, $t \geq 1$: Given M_{t-1}, consider the graph (V_t, E_t, w_t) of M_{t-1}. If there exists a student with no path through him, remove that student from the graph; his assignment under M_t will remain his assignment at the beginning of this round. If there are any cliques in the graph (V_t, E_t), pick one (note that different choices here may yield different results). For each edge from i to j in this clique, let M_t be the matching that assigns student i the school to which j was matched under M_{t-1}. If there is no clique, return M_{t-1} as the outcome M_t and stop.

It is apparent from the description above that there may be multiple outcomes of TADAP for a given SCP. In particular, in cases with multiple cliques, the procedure may output different matchings depending on which cycles are selected at rounds $t \geq 1$. Because following a clique yields a Pareto improvement, all outcomes of TADAP in which a nonempty clique exists will Pareto dominate the SOSM matching. In fact, any final outcome of TADAP will be Pareto efficient Pareto dominations of the initial SOSM matching. A district might choose to select cliques in an arbitrary manner and/or select cliques that include certain student populations over others (reinforcing their original priority structure) in order to define a trading adjusted deferred mechanism.

We begin with an example where the preference and priority structures are strict. (In such a situation, the weight function on the graph is uniformly 1 and can be ignored.) Consider once again SCP_2 (Section 2A) with five students and five schools

[10] In this theorem and in the rest of this section, we do not consider the case when there are some unassigned students and/or some unfilled places at a given school. If, on the other hand, this happens, some students can improve their assignment by taking a more preferred free place at a school without harming others. This means that a matching M may be Pareto dominated even in the case when the directed graph of M is acyclic; see [Abraham et al. 2005], where a necessary and sufficient condition for a matching to be Pareto optimal is proved.

each with one seat:

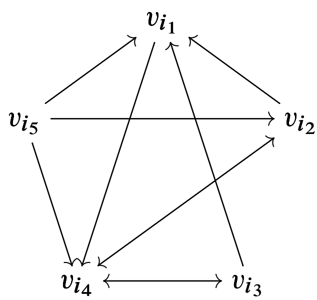
$$\mathrm{SCP}_2: \quad \begin{array}{ll} i_1 : s_2 \succ s_5 \succ s_4 \succ s_3 \succ s_1, & s_1 : i_3 \succ i_2 \succ i_4 \succ i_1 \succ i_5, \\ i_2 : s_2 \succ s_5 \succ s_4 \succ s_1 \succ s_3, & s_2 : i_4 \succ i_5 \succ i_1 \succ i_2 \succ i_3, \\ i_3 : s_5 \succ s_2 \succ s_1 \succ s_3 \succ s_4, & s_3 : i_2 \succ i_3 \succ i_4 \succ i_5 \succ i_1, \\ i_4 : s_4 \succ s_1 \succ s_2 \succ s_3 \succ s_5, & s_4 : i_1 \succ i_2 \succ i_3 \succ i_5 \succ i_4, \\ i_5 : s_5 \succ s_4 \succ s_2 \succ s_3 \succ s_1, & s_5 : i_1 \succ i_2 \succ i_5 \succ i_3 \succ i_4. \end{array}$$

The matching under SOSM is

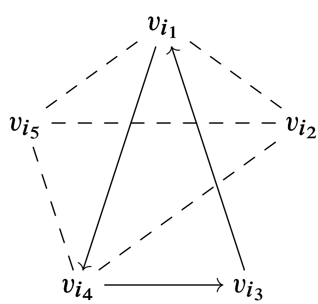
$$M_S^{\mathrm{SCP}_2} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_5 & s_4 & s_1 & s_2 & s_3 \end{pmatrix}.$$

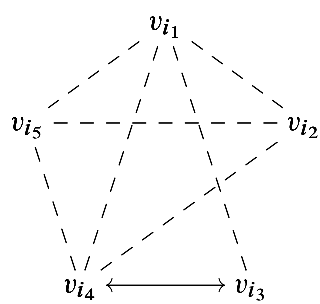
SOSM does a poor job with student preferences here. One student gets his fourth choice, three get their third choice and one gets his second choice.

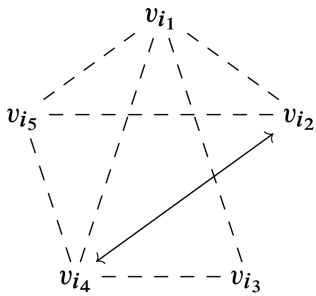
For SCP_2, the associated SOSM matching can thus be translated into the following graph:

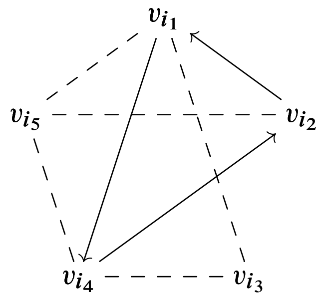


We see that if there is an arrow from i_l to i_j then i_l would (weakly) prefer to be assigned to $M(i_j)$. Such a swap can only be allowed if another student, i_k, prefers $M(i_l)$ to his own assignment, that is, only if there is a directed edge from some v_{i_k} to v_{i_l}. In this manner, a group of students can form a "swap market" and they can trade their SOSM assignments among themselves consistent with the directed graph. Such a swap market would correspond to a cycle in the graph. Here are four different cliques within the directed graph above (cliques denoted by unbroken arrows):

Cycle 1: 

Cycle 2: 

Cycle 3: 

Cycle 4: 

We list the assignments corresponding to each of the four cliques (note that the students' assignments are underlined in each matching):

$$M_1 = \begin{cases} i_1 : \underline{s_2} \succ s_5 \succ s_4 \succ s_3 \succ s_1, \\ i_2 : s_2 \succ s_5 \succ \underline{s_4} \succ s_1 \succ s_3, \\ i_3 : \underline{s_5} \succ s_2 \succ s_1 \succ s_3 \succ s_4, \\ i_4 : s_4 \succ \underline{s_1} \succ s_2 \succ s_3 \succ s_5, \\ i_5 : s_5 \succ s_4 \succ s_2 \succ \underline{s_3} \succ s_1, \end{cases} \qquad M_2 = \begin{cases} i_1 : s_2 \succ \underline{s_5} \succ s_4 \succ s_3 \succ s_1, \\ i_2 : s_2 \succ s_5 \succ \underline{s_4} \succ s_1 \succ s_3, \\ i_3 : s_5 \succ \underline{s_2} \succ s_1 \succ s_3 \succ s_4, \\ i_4 : s_4 \succ \underline{s_1} \succ s_2 \succ s_3 \succ s_5, \\ i_5 : s_5 \succ s_4 \succ s_2 \succ \underline{s_3} \succ s_1, \end{cases}$$

$$M_3 = \begin{cases} i_1 : s_2 \succ \underline{s_5} \succ s_4 \succ s_3 \succ s_1, \\ i_2 : \underline{s_2} \succ s_5 \succ s_4 \succ s_1 \succ s_3, \\ i_3 : s_5 \succ s_2 \succ \underline{s_1} \succ s_3 \succ s_4, \\ i_4 : \underline{s_4} \succ s_1 \succ s_2 \succ s_3 \succ s_5, \\ i_5 : s_5 \succ s_4 \succ s_2 \succ \underline{s_3} \succ s_1, \end{cases} \qquad M_4 = \begin{cases} i_1 : \underline{s_2} \succ s_5 \succ s_4 \succ s_3 \succ s_1, \\ i_2 : s_2 \succ \underline{s_5} \succ s_4 \succ s_1 \succ s_3, \\ i_3 : s_5 \succ s_2 \succ \underline{s_1} \succ s_3 \succ s_4, \\ i_4 : \underline{s_4} \succ s_1 \succ s_2 \succ s_3 \succ s_5, \\ i_5 : s_5 \succ s_4 \succ s_2 \succ \underline{s_3} \succ s_1. \end{cases}$$

Observe that M_1, M_3, and M_4 are Pareto efficient but M_2 is not. In fact, if we draw the directed graph of M_2, we see that there is another cycle between i_3 and i_1. Thus we could continue with another clique, which would result in M_1. This raises the question of what efficient matching should be chosen in case of multiple efficient matchings. In this specific example, all three matchings give two students their top choice, one student her second choice, one student her third choice, and one student her fourth choice. Note that M_4 is the one obtained earlier via EADAM with the consent of i_5 and, equivalently, via a coalition with the cabal $K = \{i_1, i_2, i_4\}$ (the cabal loop is $(i_1 \to i_4 \to i_2 \to i_1)$), the accomplice set $A(K) = \{i_5\}$, and the set $X = \{s_2, s_4\}$ for i_5 (see Section 2A). One might argue that having multiple paths to a given outcome is, in itself, a justification to select that outcome as "best".

Note that in all these cases, i_5's assignment stays the same; in other words, i_5 can be labeled a "hopeless student" analogous to the "hopeless man" in [Huang 2006]. Looking at the graph, we see that there is no path passing through i_5; there is no chance for his situation to be improved. We can simplify the graph by taking out the vertex corresponding to i_5.

3B. ***Properties of TADAP.*** We begin this section with an analysis of the performance of TADAP under strategic action. We first state a key result from Kesten:

Proposition 3.4 [Kesten 2010, Proposition 4]. *No Pareto efficient mechanism that can Pareto improve upon SOSM is fully immune to strategic action.*

Since TADAP produces Pareto improvements of SOSM, it follows then that it is not strategyproof. This is consistent with other improvements upon SOSM. However, lack of strategyproofness does not imply easy manipulability. The feasibility of manipulation decreases as the size of the market (school district) increases. This is analogous to our earlier assertion that substantial coalitions are hard to form naturally on their own in the context of the SCP. Students do not have complete information about preference profiles of other students, so potential profitable strategic behaviors are highly unlikely. Formulating an alternative ranked list which yields a better assignment, even with complete information on all other students will most likely not be feasible for individual students.

Making the above more precise in technical language, we first split the schools into categories in terms of perceived quality. Then we can prove the following (cf. [Kesten 2010, Theorem 2]):

Theorem 3.5. *Let the set of schools S be partitioned into categories of perceived quality*

$$S = S_1 \cup S_2 \cup \cdots \cup S_m \quad \textit{with} \quad S_i \cap S_j = \varnothing \textit{ if } i \neq j$$

such that for any $k, l \in \{1, \ldots, m\}$ with $k < l$, each student prefers any school in S_k to any school in S_l. Let each student's information be symmetric for any two schools in the same perceived quality category. Then for any student, the strategy of truth telling stochastically dominates any other strategy when other students behave truthfully. Thus truth telling is an ordinal Bayesian Nash equilibrium of the preference revelation game under TADAP.

A well-studied method of strategic action by students is truncation manipulation, one of the few tools available in such a largely incomplete information matching game [Ehlers 2008]. However it is easy to see that in TADAP, no student benefits from truncating her preference list; any such truncation results in fewer cliques and fewer opportunities for that student (and for others) to improve her lot.

Note also that there is no strategy that a group of students could employ resulting in an outcome that is not among those produced by some choice of clique using TADAP. This is because in considering all possible cliques, we obtain all possible Pareto improvements.

Another prominent feature of TADAP is the efficiency of all its outcomes. Each clique followed improves the efficiency of the outcome, neutralizing to an extent the inefficiency caused by SOSM. As each such improvement creates a Pareto domination of the previous matching, at the end of the algorithm, we stop at a Pareto efficient matching. In fact, TADAP produces all efficient matchings that Pareto dominate SOSM. We can actually prove a slightly stronger result. A straightforward proof yields the following:

Proposition 3.6. *If matching M (Pareto) dominates the SOSM matching M^*, then M is realizable by TADAP up to null cliques.*

Obviously, distinct Pareto efficient matchings are Pareto incomparable. At this point we might resort to another evaluative criterion. For instance, we may wish to then consider the matchings with minimal preference index, a criterion that considers the sum of each player's priority violation as a measure of "lost utility";[11] this can reduce our option size. And, if the mechanism itself includes a second stage procedure such as TADAP or EADAM with full consent assumed, the overlap of outcomes may be called upon to justify the subsequent modification of outcomes. Since the overall process includes adjustments made in a standard manner to an initial stable outcome, the "fairness" is built in. If the standard adjustments are selected based upon criteria that include a "multiple pathway" argument, then the EADAM or other identified outcome is strongly supported. That is, no priority must be "waived" as that priority is part of the input, but needn't be incorporated into the final output matching.

The above proposition easily yields the following:

Corollary 3.7. *All efficient outcomes of EADAM and CIP can be found by TADAP.*

Recall that both EADAM and CIP provide us with efficiency improvements to SOSM. However, TADAP can return all Pareto efficient matchings that dominate SOSM so that we can compare all choices and pick the most desirable matching.

The absolute efficiency of TADAP may appeal to a utilitarian. However, this efficiency is achieved at the expense of stability. By its very construction, TADAP is not stable. Obviously we need to make an effort to coordinate the tradeoff between stability and efficiency. In the school choice literature, "fairness", "stability", "justified envy", and "no priority violation" are often used interchangeably. Here we propose a more nuanced notion of fairness (originally due to Kesten).

Since TADAP starts with the SOSM outcome as input, we are starting at a point where student priorities are considered and respected. TADAP may then make changes to the assignments which cause instability, manifesting itself in terms of justified envy. However, if a student's assigned school could not get any better under any stable mechanism, we surmise that his "justified envy" for anybody's assignment should not be justified. To formalize this we make the following definition:

[11] See [Aksoy et al. 2013; Karaali et al. 2012] for more on the preference index. Readers interested in other efficiency metrics might also refer to [Boudreau and Knoblauch 2010].

Definition 3.8 (cf. [Kesten 2010]). A matching is *reasonably fair* if there is no stable matching that can improve the assignment of any student. A mechanism is *reasonably fair* if it always outputs reasonably fair matchings.

Then the following is a direct consequence:

Proposition 3.9. *Matchings produced by TADAP are reasonably fair.*

Finally we should note that cycle improvements are used in the literature in a variety of ways. For instance Kesten [2010] describes such a model. In [Erdil and Ergin 2008], a stable cycle improvement model is developed. In this sense, the point of our work is to devise a scheme which incorporates any Pareto improvement of the SOSM outcome in a cycle improvement model.[12]

4. Conclusion

In this paper, we introduce and investigate the properties of coalitions and cliques, two notions that can be incorporated into a school choice mechanism to improve the efficiency of SOSM. Our focus is on the examination of mathematical processes for producing improvements. Both approaches we examine, coalitions and cliques, allow us to consider opportunities for cooperation and collaboration among and between the players and designers. We also hope that the mathematical tenor of our approach amidst a crowded literature focusing on practical outcomes will be aesthetically appealing and valuable for some readers.

The theoretical framework we are interested in might even have practical implications. We argue that the concerns about fairness that are prevalent in the literature of practically implementable mechanisms for school choice may be alleviated by our theoretical framework which demonstrates multiple pathways to produce outcomes of mechanisms commonly in use.

Our work may also be viewed as a fresh examination of two well-known and widely used school choice mechanisms (SOSM and EADAM). Our utilization of the notion of "reasonably fair" (originally proposed, to the best of our knowledge, by Kesten [2010]) captures our focus on cooperation and collaboration as a means to address any perceived unfairness. The double meaning of reasonableness as "somewhat" as well as "what a reasonable person would accept" is especially apropos. The constructions here yield opportunities to improve upon SOSM while justifying resulting priority violations in new ways.

Clearly our two modifications work by Pareto improving the baseline outcome of SOSM. Considering a coalition or clique improvement to SOSM as part of the

[12]Alternatively, rather than starting with a stable outcome and then modifying, one can start instead with an efficient outcome (such as one obtained via the top trading cycles mechanism) and then modify it to reach a more stable matching. Just such a method is investigated in [Morrill 2013].

overall mechanism with an established way of selecting the best overall outcome would allow for implementation without the need to establish approval from certain families. While it was not our goal here to develop a practical replacement for the well-established mechanisms now in use, we argue that the improvements presented here can have genuine practical implications. This is in part because of their coincidental outcomes rather than despite them. We can justify the priority violations that result from coalition improvement and cliques by showing that the new assignments (Pareto) dominate the SOSM assignments and can be arrived at via multiple paths. Because many of the current school priorities in place are meant to create some certainty/security for families, once those have been taken into account in the initial assignment, and since we can demonstrate that no families are made worse off, neither schools nor families should have a reason to object.

We also note that indifferences in student preferences may be incorporated into our model. Both collaborative approaches presented (coalitions and cliques) can work when students submit lists with indifferences. Although a considerable amount of research has been done regarding indifferences within school priority classes, indifference in student preferences has not been studied in as much depth. As far as we know, this characteristic of cycle improvement models has not been investigated before, at least in the school choice context. This can be a good avenue to pursue further.

As a final note, we once again emphasize the fact the two notions introduced in this paper are related to one another as well as to SOSM and EADAM. More specifically, given a coalition $C = (K, A(K))$ in the notation of Section 2A, we can always construct a sequence of cliques that under TADAP yields the same outcome. In other words, coalitional outcomes can always be obtained via TADAP as well. Going the other way is also doable in the case of strict preference profiles: any clique in such a context corresponds to a cabal cycle and the accomplices may be determined afterwards by looking at the resulting priority violations. It is precisely these overlapping and interlocking relationships between disparate processes that intrigues us and motivates this work.

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